ANALYTICAL DEPOISSONIZATION AND ITS APPLICATIONS

May 3, 1997

Philippe Jacquet*
INRIA
Rocquencourt
78153 Le Chesnay Cedex
France
jacquet@blagny.inria.fr

Wojciech Szpankowski†
Department of Computer Science
Purdue University
W. Lafayette, IN 47907
U.S.A.
spa@cs.purdue.edu

Abstract

In combinatorics and analysis of algorithms a Poisson version of a problem (henceforth called *Poisson model* or *poissonization*) is often easier to solve than the original one, which we name here the *Bernoulli model*. Poissonization is a technique that replaces the original input (e.g., think of balls thrown into urns) by a Poisson process (e.g., think of balls arriving according to a Poisson process into urns). More precisely, analytical Poisson transform maps a sequence (e.g., characterizing the Bernoulli model) into a generating function of a complex variable. However, after poissonization one must *depoissonize* in order to translate the results of the Poisson model into the original (i.e., Bernoulli) model. We present in this paper several analytical depoissonization results that fall into the following general scheme: *If the Poisson transform has an appropriate growth in the complex plane, then an asymptotic expansion of the sequence can be expressed in terms of the Poisson transform and its derivatives evaluated on the real line.* Not surprisingly, actual formulations of depoissonization results depend on the nature of the growth of the Poisson transform, and thus we have *polynomial* and *exponential* depoissonization theorems. Normalization (e.g., as in the central limit theorem) introduces another twist that led us to formulate the so–called *diagonal* depoissonization theorems. Finally, we illustrate our results on numerous examples from combinatorics and the analysis of algorithms and data structures (e.g., combinatorial assemblies, digital trees, multiaccess protocols, probabilistic counting, selecting a leader, data compression, etc.).

Key Words: Poissonization, depoissonization, Cauchy integral formula, saddle point method, limiting distributions, Mellin transform of complex variable, analytical combinatorics, analysis of algorithms and data structures, digital trees.

*This work was partly supported by the ESPRIT Basic Research Action No. 7141 (ALCOM II).
†The work of this author was supported by NSF Grants NSF CCR-9201078, NCR-9206315 and NCR-9415491, and NATO Collaborative Grant CRG.950060.

Contents

1 Introduction

In some algorithms (e.g., sorting and hashing) n elements are placed randomly into m "bins". This paradigm is often called the "balls and urns" model. Questions arise such as how many urns are empty, how many balls are required to fill up all urns, etc. It is easy to see that the occupancies of urns are not independent (e.g., if all balls fall into one urn, then all the remaining urns are empty). To overcome this difficulty an interesting probabilistic technique called *poissonization* was suggested. Namely, it is assumed that balls are "generated" according to a Poisson process N with mean $z = n$. Due to some unique properties of the Poisson process, the streams of balls are now placed independently in every urn, thus overcoming the above–mentioned difficulty. Observe, however, that poissonization has its own problems since one must "extract" the original results from the Poisson model, that is, *depoissonize* the Poisson model.

To the best of our knowledge, poissonization was introduced by Marek Kac [34] who half a century ago investigated the deviations between theoretical and empirical distributions. Recently, poissonization was further popularized in the context of analysis of algorithms and combinatorial structures by Aldous [1], Arratia and Tavaré [2], Gonnet [20], Gonnet and Munro [21], Holst [23], Jacquet and Régnier [27], Jacquet and Szpankowski [28, 31], Rais *et al.* [47], Fill *et al.* [14], Kirschenhofer *et al.* [36], Poblete [44], and others.

Before we spell out more succinctly our depoissonization results, we first describe another scenario. In the analysis of algorithms and/or enumeration of combinatorial structures, the original problem – which we further call the Bernoulli model – is often represented by a recurrence equation or by a functional/differential equation. For example, this situation arises in a large class of algorithms involving a splitting process and/or digital trees (cf. [12, 14, 15, 26, 27, 28, 30, 31, 36, 37, 38, 41, 47, 48, 54, 55]), and hashing (cf. [20, 21]). Imbedding this splitting process into a Poisson process often leads to more tractable functional/differential equations. This was called by Gonnet and Munro [21] the *Poisson Transform.* We call this technique *analytical poissonization.*

More formally, let g_n be a characteristic of the Bernoulli model of size n. The Poisson transform is defined as $\widetilde{G}(z) = \mathbf{E}g_N = \sum_{n\geq 0} g_n \frac{z^n}{n!}e^{-z}$, that is, the input n becomes a Poisson variable N with mean z when $z \geq 0$. If $\widetilde{G}(z)$ is known, then one can extract the coefficient $g_n = n!\,[z^n](\widetilde{G}(z)e^z)$ (cf. [21, 22, 44]). However, in most interesting situations $\widetilde{G}(z)$ satisfies a complicated functional/differential equation that is difficult to solve exactly. Nevertheless, one can find an asymptotic expansion of $\widetilde{G}(z)$ for $z \to \infty$ on the real axis and can bound $\widetilde{G}(z)$ in the complex plane. Then, one aims at finding an asymptotic expansion of g_n from the asymptotics of $\widetilde{G}(z)$. This is called *analytical depoissonization* (which became one of the key tool for the analytical analysis of algorithms [14, 20, 21, 26, 27, 28, 36, 37, 41, 47, 48, 54, 55]) and analytical information theory [28, 31, 32, 38, 39, 40]). Analytical depoissonization is well understood and relatively easy when the Poisson transform $\widetilde{G}(z)$ is a meromorphic function (cf. [20, 21, 45]), that is, when the only singularities of $\widetilde{G}(z)$ in the complex plane are either poles or algebraic singularities. In this case, one can either apply Cauchy residue theorem or the singularities analysis of Flajolet and Odlyzko [18]. This well known technique was used by the authors of [20, 21, 44, 45].

A challenging situation in analytical depoissonization arises when the Poisson transform $\widetilde{G}(z)$ is entire (i.e., analytic in the whole complex plane), and consequently does not show any singularity in the complex plane. In this case, one can rely only on partial information such as asymptotic estimates of $\widetilde{G}(z)$ when z grows in the complex plane. The main contribution of this paper is in providing an asymptotic expansion of g_n when one knows only the asymptotic behavior of the Poisson transform $\widetilde{G}(z)$ for large z in the complex plane (so one can call it *asymptotic* analytical depoissonization). Surprisingly, these results can be also extended in a certain way to meromorphic Poisson transforms as is indicated in Remark 4 of Section 4.

One may ask *why the Poisson process*? Why not to imbed our Bernoulli model into another process? This seems to be a consequence of certain unique properties of the Poisson process, namely: It is the only process that has stationary and independent increments, and which is orderliness (no group arrives) [50]. In the analysis of some algorithms (e.g., splitting processes) two other properties are even more important, namely: a superposition of renewal processes is itself renewal if and only if it is Poisson; and a thinning process in which arrivals are accepted or rejected on an independent basis is Poisson if the original process is Poisson, too. For the ball and urns model another property is of some importance. It says that once we know that there are n arrivals from a Poisson process in a time interval of duration T, then these arrivals are distributed uniformly within $(0, T)$.

There is not much literature on analytical depoissonization, and most published results are tailored to specific problems. Exact analytical depoissonization for meromorphic functions (with simple singularities) was presented in [20, 21, 44]. Asymptotic analytical depoissonization for entire functions was initiated by Jacquet and Régnier in the analysis of limiting distributions of depth and size in tries [26, 27, 48]. They introduced the basic idea of the analytical depoissonization on an (implicit) solution of a specific non–linear functional equation, that is, the asymptotic analysis of a sequence of Cauchy integrals on the complex plane with saddle point-like estimates. Jacquet and Szpankowski [28, 31], and

Rais *et al.* [47] extended this result in order to obtain a first general (simple) version of the depoissonization result. The results of this paper have been partially motivated by our depoissonization results of [31], where we dealt with a nonlinear multiplicative differential/functional equation, and some sophisticated tools had to be applied to solve the problem. However, to the best of our knowledge none of the results presented in this paper has been published before in its current form.

The results of this paper can be summarized as follows: We start with a brief account on analytical poissonization (cf. Section 2). Then, after surveying some problems of combinatorics and analysis of algorithms where poissonization/depoissonization plays a significant role (cf. Section 3), we present a long list of depoissonization results (cf. Sections 4 and 5). We end up with some applications of our findings (cf. Section 6): In particular, we discuss conflict resolution algorithms, digital trees, randomized algorithm for electing a leader, probabilistic counting, as well as a general depoissonization result for a class of linear functional equations.

Our main results are presented in Sections 4 and 5, and all of them are new or at least presented in a stronger version (cf. Remark 1 after Theorem 1). While our goal is to prepare a readable account on depoissonization and its use in (precise) analysis of algorithms and analytical combinatoric, we aspire to present rigorous derivations of depoissonization results so that this paper can be used as a reliable reference on depoissonization. Actual formulations of the depoissonization statements depend on the growth of the Poisson transform in a cone around the real axis. In particular, we have *polynomial* and *exponential* depoissonization theorems (cf. Theorems 1, 2 and 3). Normalization (e.g., as in the central limit theorem) introduces another twist that led us to formulate the so called *diagonal* depoissonization theorems (cf. Theorems 4, 5 and 8). The most general depoissonization result is presented in Theorem 5 which is used to prove all our depoissonization findings that do not deal with limiting distributions. A corresponding general depoissonization tool for limiting distributions is proposed in Theorem 8, which is ultimately aimed at general central limit theorems that are also discussed in this paper (cf. Theorems 7 and 9).

2 Poisson Transform

We briefly discuss the poissonization technique: First, we review probabilistic poissonization and explain its success in solving some problems. Then, we briefly discuss analytical poissonization, and introduce some definitions that are used throughout the paper.

Consider a combinatorial structure in which n objects are randomly distributed into some locations, e.g., one can think of n balls thrown into urns. The objects are not necessary uniformly distributed among the locations (cf. the digital trees example below). We call such a setting the Bernoulli model. Let X_n be a characteristic of the model (e.g., the number of throws needed to fill up all urns, the moment generating function of the number of nonempty urns, etc.). Next, we define the Poisson model. Let N be a random variable distributed according to Poisson with parameter $z \geq 0$, that is, $\Pr\{N = k\} = e^{-z} z^k / k!$. Let X_N be the above characteristic defined in the Poisson model in which the *deterministic* input (i.e., n) is replaced by the Poisson variable N with parameter $z = n$. Then by definition

$$\begin{aligned} \widetilde{X}(z) := \mathbf{E} X_N &= \sum_{n \geq 0} \mathbf{E}\left(X_N \mid N = n\right) e^{-z} \frac{z^n}{n!} \\ &= \sum_{n \geq 0} \mathbf{E} X_n e^{-z} \frac{z^n}{n!}, \end{aligned} \tag{1}$$

and we use this formula to define $\widetilde{X}(z)$ for arbitrary complex z. Throughout, we make the following assumption (with an exception for our most general depoissonization finding, namely Theorem 5, where we slightly modify this assumption):

(A) The sum in (1) converges absolutely for every z, that is, $\widetilde{X}(z)$ is an entire function of the complex variable z.

Equation (1) can be viewed as a definition of the *Poisson transform* $\widetilde{X}(z)$ (cf. [21]) when it is analytically continued to *complex* z. Indeed, assuming that we know $\widetilde{X}(z)$, we can obtain $\mathbf{E} X_n$ as the coefficient at $z^n/n!$ of $X(z) = \widetilde{X}(z) e^z$. The reader is referred to [20, 22, 44] for some results on the exact or approximate "reverse" Poisson transform. Our goal is different: We aim at presenting easily verifiable conditions that allow one to extract *asymptotically* $\mathbf{E} X_n$ from $\widetilde{X}(z)$, provided one only knows $\widetilde{X}(z)$ *asymptotically* for $z \to \infty$ on the real axis and can bound the growth of $\widetilde{X}(z)$ in the complex plane.

Before we discuss some examples, a word about notation: We write g_n as a generic notation for a sequence characterizing the Bernoulli model (e.g., $g_n = \mathbf{E} X_n$ or, in general, $g_n = \mathbf{E} f(X_n)$ for some function f). Then, we denote by $\widetilde{G}(z)$ or $\mathcal{P}(g_n; z)$ its Poisson transform, that is,

$$\widetilde{G}(z) = \mathcal{P}(g_n; z) := \sum_{n \geq 0} g_n \frac{z^n}{n!} e^{-z} .$$

Section 3 below describes several Poisson transforms arising in combinatorics and the analysis for algorithms and data structures.

Table 1: Some Poisson Transforms and Their Properties

g_n	$\widetilde{G}(z)$
Const	Const
$(-1)^n$	e^{-2z}
α^n	$e^{(\alpha-1)z}$
$n(n-1)\cdots(n-k+1)$	z^k
additive (p,q)-splitting on F_n and G_n; $p+q=1$	$F(pz)+G(qz)$
multiplicative (p,q)-splitting on F_n and G_n	$F(pz)G(qz)$

Moreover, poissonization/depoissonization is also a useful tool for studying limiting distributions. In this case, we consider a random variable X_n, and its generating function $G_n(u) = \mathbf{E}u^{X_n}$ for some complex u. Then the Poisson transform of $G_n(u)$ is $\widetilde{G}(z,u) = \sum_{n\geq 0} G_n(u)\frac{z^n}{n!}e^{-z}$. In general, we may define a double-indexed sequence $g_{n,k}$ (e.g., think of $g_{n,k} = \Pr\{X_n = k\}$), and then the Poisson transform becomes

$$\widetilde{G}(z,u) = \mathcal{P}(G_n(u);z) := \sum_{n=0}^{\infty}\frac{z^n}{n!}e^{-z}\sum_{k=0}^{\infty} g_{n,k}u^k \ .$$

We discuss several examples of this kind below in Sections 3.2 – 3.5.

We now present some Poisson transforms and their properties. Let g_n be an arbitrary sequence. In the sequel, we say that a sequence g_n is Bernoulli additive (p,q)-splitting of sequences f_n and h_n if $g_n = \sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}(f_k + h_{n-k})$, where the pair (p,q) is the split probability vector with $p+q=1$. In this case, we have $\widetilde{G}(z) = \widetilde{F}(pz) + \widetilde{H}(qz)$ where $\widetilde{F}(z)$ and $\widetilde{H}(z)$ are Poisson transforms of sequences f_n and h_n, respectively. This property extends in a straightforward manner to $(p_1,\ldots,p_\ell)$-splittings of arbitrary ℓ sequences f_n^i with probability vector p_i, $p_1+\cdots+p_\ell = 1$. Indeed, $\widetilde{G}(z) = \widetilde{f}^1(p_1z)+\cdots+\widetilde{f}^\ell(p_\ell z)$ where $\widetilde{F}^i(z)$ is the Poisson transform of the sequence f_n^i. In the same spirit, one can define multiplicative (p,q)-splitting: $g_n = \sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}f_kh_{n-k}$. Then, $\widetilde{G}(z) = \widetilde{F}(pz)\widetilde{H}(qz)$. Table 1 summarizes these findings, however, a more complete list of properties can be found in [45].

An astute reader may have concluded that g_n is asymptotically equivalent to $\widetilde{G}(n)$ (i.e., $g_n \sim \widetilde{G}(n)$) when $n\to\infty$ in all cases in Table 1 except when $g_n = \alpha^n$ (think of $\alpha = -1$). This cannot be a coincidence, and we systematically explore this fact in this paper.

3 Poissonization in Combinatorics and Analysis of Algorithms

In this section, we discuss several examples from the fields of combinatorics and analysis of algorithms where poissonization and depoissonization turn out to be useful, if sometimes not the only technique, that may produce a result. In the sequel, we also try to show that the Poisson transform can be derived directly from the probabilistic analysis of the problem at hand with the omission of intermediate steps (and tedious algebraic manipulations) unavoidable when dealing with the Bernoulli model.

3.1 Conflict Resolution Algorithms

Imagine a (infinite) collection of (distributed) users trying to send messages among themselves. There is no coordination among them except that a transmission of a fixed length packet must start at the beginning of a time slot. When only one user sends a message, then a successful transmission takes place. Otherwise, there is a conflict, and all users involve in it must apply an algorithm to solve the conflict (i.e., conflict resolution algorithm). We consider two possible solution, namely: *tree-type* algorithm (cf. [4, 12, 11, 22, 42, 51, 54] and *interval searching algorithm* (cf. [5, 28, 56]).

In the **tree-type algorithm**, all colliding users, say n of them, flip a biased coin and only those who got *head* are allowed to transmit in the next slot, while the others must wait until the former group solved their subconflicts. Let L_n be the average length of the conflict resolution session provided n users are initially involved in the conflict. Let also p be the probability of flipping the head, thus $q = 1 - p$ flipping the tail. Clearly, L_n satisfies the following recurrence

$$L_n = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (L_k + L_{n-k}) \qquad n \geq 2 \ ,$$

with $L_0 = L_1 = 1$.

Actually, there are several modifications to the basic tree-type algorithm (cf. [42, 54]) that lead to a more general recurrence. Namely: given L_0 and L_1

$$L_n = a_n + \beta \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (L_k + L_{n-k}) \ , \qquad n \geq 2 \tag{3}$$

where a_n is a given sequence (also called the *additive term*), and $\beta > 0$ is constant. Let $L(z)$ and $A(z)$ be the exponential generating functions of L_n and a_n, respectively. An easy calculation reveals that

$$L(z) = \beta L(zp)e^{zq} + \beta L(zq)e^{zp} + A(z) - l_0 - l_1 z \ ,$$

where l_0, l_1 are constant depending on the initial condition. It is not clear how to solve this functional equation unless we rewrite it in terms of the Poisson generating function $\widetilde{L}(z) = L(z)e^{-z}$. Then:

$$\widetilde{L}(z) = \beta\widetilde{L}(zp) + \beta\widetilde{L}(zq) + \widetilde{A}(z) - l_0e^{-z} - l_1ze^{-z} \ . \tag{4}$$

which becomes a linear additive functional equation extensively studied in [17, 12, 36, 41, 54, 55].

Now, we briefly describe the **interval searching algorithm** (cf. [5, 28, 56]). This time access to the channel is controlled by a window based mechanism that we describe below. This window will be referred to as the *enabled interval* (EI). Let S_i denote the starting point for the i-th EI, and t_i is the corresponding starting point for the *conflict resolution interval* (CRI), where CRI represents the number of slots needed to resolve a collision. Roughly speaking, at each step of the algorithm, we compute the endpoints of the EI based on the outcome of the channel. Details can be found in [5, 28, 56]. The parameters of interest are: the length of the conflict resolution interval T_n, the fraction of resolved interval W_n, and the number of resolved packets C_n. To derive functional equations for these parameters, we immediately consider the Poisson model. For example, $\widetilde{C}(z)$ for C_n satisfies (cf. [5, 28, 56])

$$\widetilde{C}(z) = \left(1 + (1 + z/2)e^{-z/2}\right)\widetilde{C}(z/2) \ . \tag{5}$$

Indeed, if there is at most one packet in the first half of EI – which happens with probability $(1 + z/2)e^{-z/2}$ in the Poisson model – then we explore the second half of the interval, and this leads to $\widetilde{C}(z/2)$. Otherwise, we explore the first half of the interval, thus the first term in the above equation.

3.2 Combinatorial Assemblies of Labeled Structures

Let us first consider classical (labeled) combinatorial structures such as permutations decomposed into cycles, partitions of a finite set, mappings decomposed into connected components, and graphs decomposed into connected components. We adopt here Wilf's language [58] and call structures and set of components as "decks" and "hands", respectively. For example, in the permutation assembly a deck is represented by a cyclic permutation of, say n elements, while hands are cyclic decomposition of the permutation.

Let now $h_{k,n}$ denote the number of decompositions (hands) of weight n that have exactly k components (e.g., permutations of $[n] = \{1, \ldots, n\}$ into k cycles), and d_n is the number of different structures of weight n (e.g., cyclic permutations of $[n]$). Clearly, $h_n = \sum_{k\geq 0} h_{k,n}$ is the total number of hands of weight n (e.g., total number of permutations – not necessary cyclic permutations!). We write $H_n(z)$ and $D(z)$ for the exponential generating functions for hands and decks enumerations, respectively. Also,

$$H(z,u) = \sum_{n,k\geq 0} h_{n,k} \frac{z^n}{n!} u^k \ .$$

According to our notation, the Poisson transforms are respectively $\widetilde{H}(z,u) = e^{-z}H(z,u)$ and $\widetilde{H}_n(z) = e^{-z}H_n(z)$.

There is often a complicated relationship between the enumeration of hands $h_{n,k}$ and decks d_n, but their exponential generating functions (as well as the Poisson counterparts) are simply related as follows (cf. [57, 58])

$$H(z,u) = e^{uD(z)} \ , \tag{6}$$

and clearly $H(z) = e^{D(z)}$. The Poisson distribution is hidding a little bit behind these formulas, but there is one as noted in [2]. In passing, we mention that (6) is usually used to obtain exact enumeration of hands, however, we are more interested in asymptotic enumerations, that is, how to extract asymptotic behavior of h_n from $\widetilde{H}(z)$ (e.g., knowing $D(z)$).

3.3 Digital Trees and Algorithms on Words

Digital trees are data structures suitable for storing data (keys) represented by a sequence of symbols from a finite alphabet. We assume a binary alphabet, and we think of a key (string) as a (possibly infinite) sequence of zeros and ones (cf. [37, 41]).

There are three types of digital trees, namely: *trie*, *Patricia trie* (PAT), and *digital search tree* (DST). In **tries** and Patricia tries the keys are stored in external nodes while internal nodes are used only to branch out. More precisely, the branching policy at any level, say k, is based on the k-th symbol of a string. For example, for a binary alphabet $\Sigma = \{0, 1\}$, if the k-th symbol in a string is "0", then we branch-out left in the trie, otherwise we go to the right. This process terminates the first time we encounter a different symbol between a string that is currently being inserted into the trie and all other strings already in the trie. In other words, the access path from the root to an external node (a leaf of a trie) is the minimal prefix of the information contained in this external node; it is minimal in the sense that this prefix is not a prefix of any other strings (cf. [37, 41]). In **Patricia tries** all unary nodes are collapsed into one, that is, Patricia is a binary tree with all internal nodes having degree two. In a **digital search tree** keys (strings) are directly stored in nodes, and

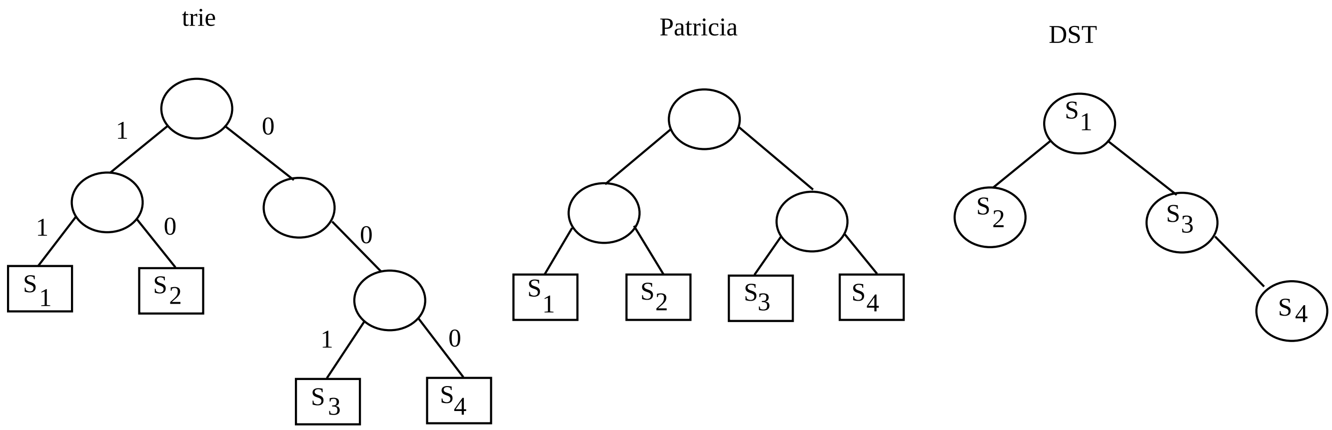

Figure 1: A trie, Patricia trie and a digital search tree (DST) built from the following four strings $S_1 = 11100\ldots$, $S_2 = 10111\ldots$, $S_3 = 00110\ldots$, and $S_4 = 00001\ldots$.

hence external nodes are eliminated. The branching policy is the same as in tries. Figure 1 illustrates these definitions.

In the Bernoulli model, one assumes that the number of keys is fixed and equal to n. The parameters of interests are:

- typical depth D_n, i.e., the length of a path from the root to a randomly selected (external) node,
- height H_n, i.e., maximum path from the root to a terminal node,
- total path length L_n, i.e., sum of all paths from the root to (external) nodes,
- size of the tree S_n, i.e., number of nodes.

We derive now some functional equations for the above parameters in the Poisson model. We start with tries. Let $D_n(u) = \mathbf{E}u^{D_n}$, $S_n(u) = \mathbf{E}u^{S_n}$, and $L_n(u) = \mathbf{E}u^{L_n}$ be the probability generating functions for the depth D_n, size S_n and the total path L_n in the Bernoulli model. We also define $H_n^k = \Pr\{H_n \le k\}$. The appropriate Poisson transforms satisfy the following equations:

$$\widetilde{D}(z,u) = u(p\widetilde{D}(zp,u) + q\widetilde{D}(zq,u)) + (1-u)e^{-z} \ , \tag{7}$$

$$\widetilde{H}^k(z) = \widetilde{H}^{k-1}(zp)\widetilde{H}^{k-1}(qz) \quad , \quad \widetilde{H}^0(z) = (1+z)e^{-z} \ , \tag{8}$$

$$\widetilde{S}(z,u) = u\widetilde{S}(zp,u)\widetilde{S}(zq,u) + (1-u)e^{-z} \ , \tag{9}$$

$$\widetilde{L}(z,u) = \widetilde{L}(zup,u)\widetilde{L}(zuq,u) + z(1-u)e^{-z} \ . \tag{10}$$

The above equations can be derived from their corresponding recurrences by (sometimes tedious) algebraic manipulations. But, a significant simplification is possible for the Poisson model. Indeed, let us consider for example the functional equation (7) for the Poisson generating function of the depth D_n. Since digital trees are recursive structures, and due to well–known properties of the Poisson process, we immediately observe that the Poisson transform of the depth of the left tree is $D(zp,u)$ while for the right subtree is $D(zq,u)$. Furthermore, since the subtrees are one level lower than the root, we have the factor u. Finally, we either compute the typical depth of the left subtree (and this happens with probability p) or the right subtree (with probability q). The initial conditions add $(1-u)e^{-z}$. Thus, the functional equation (7) follows. The other equations can be derived by the same (almost) semi-automatic method.

The functional equations for Patricia tries are harder to solve since often they contain additional unknown terms. In particular, the corresponding functional equation for the depth, height and total path length are (observe that the size is constant in the Bernoulli model, thus uninteresting):

$$
\begin{aligned}
\widetilde{D}(z,u) &= u(p\widetilde{D}(zp,u)+q\widetilde{D}(zq,u))+(1-u)(pD(zp,u)e^{-qz}+qD(zq,u)e^{-pz}) \ , && (11)\\
\widehat{H}^k(z) &= \widehat{H}^{k-1}(zp)\widehat{H}^{k-1}(qz)+\widehat{H}(zp)e^{-zq}+\widehat{H}(zq)e^{-zp} \quad , \quad \widehat{H}^0(z)=ze^{-z} \ , && (12)\\
\widehat{L}(z,u) &= \widehat{L}(zup,u)\widehat{L}(zuq,u)+\widehat{L}(zq,u)+\widehat{L}(zq,u) \ . && (13)
\end{aligned}
$$

where $\widehat{H}^k(z)=(H^k(z)-1)e^{-z}$ and $\widehat{L}(z,u)=(L(z,u)-1)e^{-z}$.

Finally, we discuss digital search trees. This time the functional equations become differential functional equations, thus making the problems much more challenging. We present them for the exponential generating functions instead of Poisson transforms since they have slightly simpler forms. With the definitions as above, we obtain the following differential-equations

$$
\begin{aligned}
\frac{\partial D(z,u)}{\partial z} &= u(pD(zp,u)+qD(zq,u))+1 \ , && (14)\\
\frac{\mathrm{d}H^k(z)}{\mathrm{d}z} &= H^{k-1}(zp)H^{k-1}(qz) \ , && (15)\\
\frac{\partial L(z,u)}{\partial z} &= L(zup,u)L(zuq,u) \ . && (16)
\end{aligned}
$$

with $H^0(z)=(1+z)$.

3.4 Leader Election Algorithm

The following elimination process (cf. [14, 46]) has several applications, such as the election of a leader in a distributed or parallel system (a practise exercised when a token is lost or when synchronization is lost in a token-passing ring-connected computer network). A group of n people (users) wishes to identify a leader by tossing fair coins. All n people who throw heads are losers and do not participate any more in the election, those who throw tails are candidate leaders and should flip their coins again. The process is repeated among candidate leaders until one leader is identified. If at any stage all remaining candidate leaders throw heads, the tosses are considered inconclusive and they all participate again as candidate leaders in the next round of coin tossing.

It is easy to notice that this election process can be represented by a special trie which was named by Prodinger [46] the *incomplete trie* since only one side of this tree is developed. Let H_n be the number of tosses until a leader is elected, that is, it is the path length in the incomplete trie from the root to the furthest terminal node on the left (assuming we move to the left when we toss a tail). If $G_n(u) = \mathbf{E}u^{H_n}$ denotes its probability generating function and $\widetilde{G}(z,u)$ its Poisson generating function, then the following functional equation can be derived (cf. [14, 46])

$$G_n(u) = \frac{u}{2^n}\sum_{k=0}^{n}\binom{n}{k}G_k(u) - \frac{u}{2^n} + \frac{uG_n(u)}{2^n}\,, \tag{17}$$

$$\widetilde{G}(z,u) = u(1+e^{-z/2})\widetilde{G}\left(\frac{z}{2},u\right) + e^{-z}[(1+z)(1-u) - ue^{z/2}]\,. \tag{18}$$

Furthermore, the Poisson mean $\widetilde{X}(z) = \mathbf{E}H_N$ and the Poisson second factorial moment $\widetilde{W}(z) = \mathbf{E}H_N(H_N-1))$ satisfy the following functional equations:

$$\widetilde{X}(z) = \widetilde{X}(z/2)(1+e^{-z/2}) + 1 - e^{-z} - ze^{-z} \tag{19}$$

$$\widetilde{W}(z) = \widetilde{W}(z/2)(1+e^{-z/2}) + 2\widetilde{X}(z/2)(1+e^{-z/2})\,, \tag{20}$$

and $\mathbf{Var}\ H_N = \widetilde{W}(z) + \widetilde{X}(z) - [\widetilde{X}(z)]^2$.

3.5 Generalized Probabilistic Counting

In some applications (cf. [15]) one needs to estimate quickly the cardinality of a large set. A probabilistic counting is a possible solution. To estimate the cardinality n of a set (with replications) every element of the set is hashed into a binary string. The bitwise OR-composition of modified hashed strings is used to build the so called `bitmap` and to obtain the estimate R_n of the cardinality n of a set. More precisely, the position of the leftmost zero in the `bitmap` string approximates $\log_2 n$ (for details see below and [15]).

We consider a generalized probabilistic counting as in [36] in which the `bitmap` is $(d+1)$-ary string instead of a binary one, where d is an integer parameter of the scheme. We can describe this scheme as follows: Let us consider an empty `bitmap` string, that is, with all positions filled by zeros. Assume that n objects (e.g., data, persons, etc.) can randomly insert (hit) a 1 at any position of the `bitmap`, however, the probability of hitting the $j \geq 0$ position is equal to 2^{-j-1}. In terms of probabilistic counting, this means that the probability of the occurrence of the pattern like $0^j 1 \cdots$ is equal to 2^{-j-1} since 0 and 1 are equally likely (0^j denotes j consecutive zeros). Every object can hit only one time. In addition, we count the number of hits in any position of the `bitmap`, but we count the number of hits only up to some value $d+1$, where d is a given parameter. In other words, the `bitmap` is a $d+1$-ary string. The parameter of interest is the length $R_{n,d}$ of the longest run of $d+1$ symbols in front of the `bitmap`. More precisely:

$$R_{n,d} = \min\{k : \ \texttt{bitmap}(k) < d+1 \text{ and for all } 0 \leq i < k \ \ \texttt{bitmap}(i) = d+1\} \ .$$

Let $G_n(u) = \mathbf{E} u^{R_{n,d}}$, and $\widetilde{G}(z,u)$ be its Poisson transform. As in [36] we observe that

$$G_n(u) = \sum_{k=0}^{d} \binom{n}{k} 2^{-n} + u \sum_{k=0}^{n-d-1} \binom{n}{k} 2^{-n} G_k(u) \ , \tag{21}$$

$$\widetilde{G}(z,u) = u f_d(z/2) \widetilde{G}(z/2,u) + (u-1)(f_d(z/2) - 1) \ , \tag{22}$$

where $f_d(z) = 1 - e_d(z) e^{-z}$ and

$$e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!} \tag{23}$$

is the truncated exponential function.

4 General Depoissonization Theorems

In this section, we present our main results. In particular, we derive conditions under which the Bernoulli model asymptotics can be inferred from its corresponding Poisson model. The plan for this section is as follows: We start with some definitions and notations followed by a presentation of our main depoissonization results (with an exception of distributional results that we move to the next section). Then, we discuss several technical lemmas needed to prove different version of our findings. The last subsection contains proofs. We first give a detailed proof of our basic depoissonization results (cf. Theorem 1), and then present a general depoissonization tool (cf. Theorem 5) from which all other results will follow.

4.1 Definitions and Notations

As discussed in Section 3, our goal is to study asymptotic expansion of a sequence g_n through its Poisson transform $\widetilde{G}(z) = \mathcal{P}(g_n; z)$. Extension to distributions requires to investigate

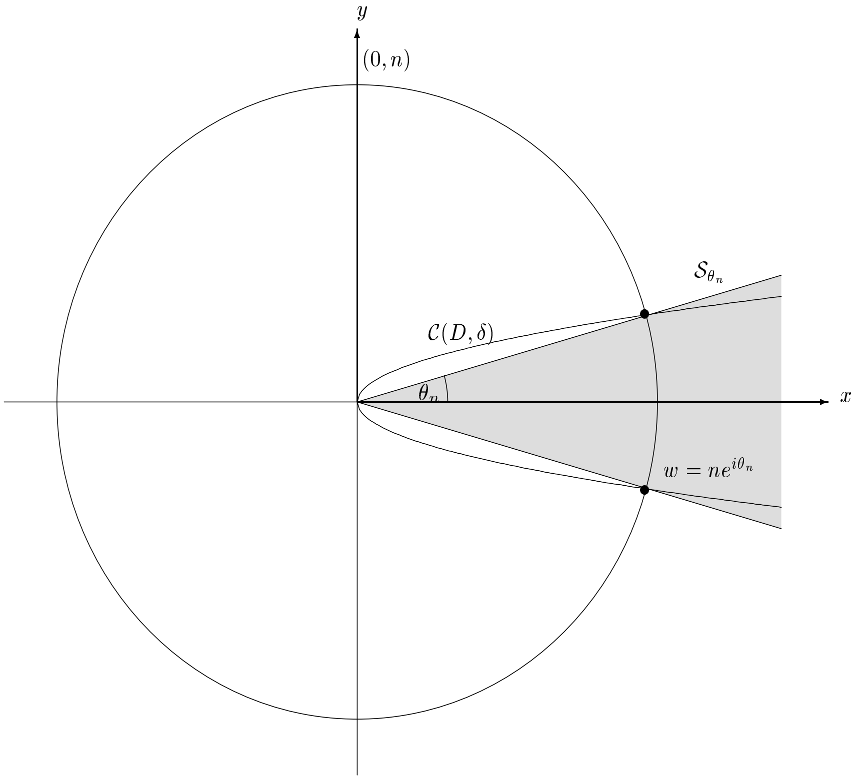

Figure 2: Illustration to polynomial cone definition

double-index sequence $g_{n,k}$ (e.g, $g_{n,k} = \Pr\{X_n = k\}$ or $g_{n,k} = \mathbf{E}e^{tX_n/\sqrt{V_k}}$ for some sequence V_k), and its Poisson transform is denoted as $\widetilde{G}(z,u) = \mathcal{P}(g_{n,k}; z, u)$. In the latter case, it is often more convenient to investigate the Poisson transform $\widetilde{G}_k(z) = \mathcal{P}(g_{n,k}; z)$ defined as

$$\widetilde{G}_k(z) = \sum_{n=1}^{\infty} g_{n,k} \frac{z^n}{n!} e^{-z} \ . \tag{24}$$

We also write $G(z) = \widetilde{G}(z)e^z$ for the standard exponential generating function of g_n.

As discussed above, our goal is to infer asymptotics of the Bernoulli model from the asymptotic behavior of the Poisson generating function as $z \to \infty$. However, in most cases the behavior of $\widetilde{G}(z)$ as $z \to \infty$ depends on the region in the complex plane in which $z \to \infty$. Usually, we must restrict the asymptotics of $\widetilde{G}(z)$ to a cone around the positive axis of $\Re(z)$. Two types of cones play a special role in our analysis, namely the so called *linear cones* and *polynomial cones*. They are defined below.

Definition 1 (i) *A linear $\mathcal{S}_\theta$ cone is defined as*

$$\mathcal{S}_\theta = \{z : \ |\arg z| \le \theta \ , \quad |\theta| < \pi/2\} \ . \tag{25}$$

(ii) *A polynomial cone* $C(D,\delta)$ *is defined as*

$$\mathcal{C}(D,\delta) = \{z = x + iy : \ |y| \le Dx^{\delta} \ , \qquad 0 < \delta \le 1 \ , \quad D > 0\} \ . \tag{26}$$

Observe that when $\delta = 1$ *the polynomial cone becomes a linear cone.*

It should be mentioned that there is another asymptotically equivalent definition of polynomial cones that is useful in some of our derivations (cf. proof of Theorem 5). Indeed, let $|z| > 1$, and observe that for $z \in \mathcal{C}(D,\delta)$ there exists a constant D' such that

$$D'\theta \le \tan\theta \le D|x|^{\delta-1} \le D|z|^{\delta-1}$$

where $\theta = \arg(z)$. Thus, we could define $\mathcal{C}(D,\delta)$ as

$$\mathcal{C}(D',\delta) = \{z : \ |\arg(z)| \le D'|z|^{\delta-1} \ , \qquad 0 < \delta \le 1 \ , \quad D' > 0\} \ , \tag{27}$$

and in some of our proofs we use this definition of the polynomial cone.

In Figure 2 we show an example of a linear cone $\mathcal{S}_{\theta_n}$ (shadow area) and a polynomial cone $\mathcal{C}(d,\delta)$. Observe that the circle of radius n and center O intersects the polynomial cone at the angle θ_n such that $\theta_n \sim Dn^{\delta-1}$ for $\delta < 1$ when $n \to \infty$.

4.2 Main Results

We consider now a sequence g_n and its Poisson transform $\widetilde{G}(z) = \mathcal{P}(g_n, z)$. Throughout, we assume that $\widetilde{G}(z)$ is an entire function. Our goal is to extract asymptotically g_n from $\widetilde{G}(z)$. By Cauchy's formula we have

$$g_n = \frac{n!}{2\pi i} \oint \frac{\widetilde{G}(z)e^z}{z^{n+1}} dz = \frac{n!}{n^n 2\pi} \int_{-\pi}^{\pi} \widetilde{G}(ne^{it}) \exp\left(ne^{it}\right) e^{-nit} dt \ . \tag{28}$$

All of our depoissonization results will follow from the above by a careful estimation of the integral using a saddle point method.

We now begin presenting our findings. We start with a basic depoissonization result that holds in a linear cone with a polynomial bound on $\widetilde{G}(z)$ (cf. Theorem 1). Then, we extend it to polynomial cones, more general bounds, and full asymptotic expansion of g_n (cf. Theorems 2, 3, 4) will be presented. The proof of Theorem 1 can be found in Subsection 4.4.1 where we strive to use only elementary tools. All remaining main results are proved in Subsection 4.4.2 by a simple application of a general depoissonization tool, namely: Theorem 5.

Theorem 1 Basic Depoissonization Lemma.
Let $\widetilde{G}(z)$ *be the Poisson transform of a sequence* g_n *that is assumed to be an entire function of* z. *We postulate that in a linear cone* $\mathcal{S}_\theta$ $(\theta < \pi/2)$ *the following two conditions simultaneously hold for some real numbers* $A, B, R > 0$, β, *and* $\alpha < 1$*:*
(I) *For* $z \in \mathcal{S}_\theta$

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)| \le B|z|^\beta \ , \tag{29}$$

(O) *For* $z \notin \mathcal{S}_\theta$

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \le A\exp(\alpha|z|) \ . \tag{30}$$

Then,

$$g_n = \widetilde{G}(n) + O(n^{\beta-1}) \tag{31}$$

for large n.

Remark 1. (i) As mentioned in the introduction, analytical depoissonization was initialized by Jacquet and Régnier on an (implicit) solution of a specific non–linear functional equation (cf. Sections 6.8). The first general depoissonization lemma was probably formulated in Rais *et al.* [47]: The authors of [47] gave (a non rigorous) proof of the following: *Under conditions (I) and (O) of Theorem 1 one concludes that* $g_n = \widetilde{G}(n) + O(n^{\beta-\frac{1}{2}})$. This early result should be compared to several new findings of this paper.

(ii) Recently, we proved in [32] that if the sequence g_n has a polynomial growth (more precisely: the analytical continuation $g(z)$ of g_n such that $g(n) = g_n$ satisfies $g(z) = O(z^\beta)$ in a cone), then $\widetilde{G}(z)$ automatically satisfies condition (O) outside the cone and condition (I) inside the cone (i.e., $\widetilde{G}(z) = O(z^\beta)$). $\square$

This basic depoissonization result can be extended in three directions. Namely:

1. we replace the linear cone by a polynomial cone;
2. we provide full asymptotic expansion for g_n in terms of $\widetilde{G}^{\langle k\rangle}(n)$ where $\widetilde{G}^{\langle k\rangle}(n)$ denotes the kth derivative of $\widetilde{G}(z)$ at $z = n$;
3. we generalize the polynomial bound on $\widetilde{G}(z)$ to certain exponential bounds (see also Section 5 for more about this kind of an extension).

Such extensions are needed in some applications, e.g., for the analysis of the number of phrases in the Lempel-Ziv parsing algorithm [31, 59, 61, 60] and/or redundancy of the Lempel-Ziv code as discussed in [39, 61].

The next theorem refers to the first and second extensions discussed above.

Theorem 2 GENERAL DEPOISSONIZATION LEMMA.
Consider a polynomial cone $\mathcal{C}(D,\delta)$ with $1/2 < \delta \le 1$. Let the following two conditions hold for some numbers A, B, $R > 0$ and $\alpha > 0$, β, and γ:
(I) *For $z \in \mathcal{C}(D,\delta)$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}(z)| \le B|z|^{\beta}\Psi(|z|) \ , \tag{32}$$

where $\Psi(x)$ is a slowly varying function , that is, such that for fixed t $\lim_{x\to\infty} \frac{\Psi(tx)}{\Psi(x)} = 1$ (e.g., $\Psi(x) = \log^d x$ for some $d > 0$);
(O) *For all $z = \rho e^{i\theta}$ with $\theta \le \pi$ such that $z \notin \mathcal{C}(D,\delta)$*

$$\rho = |z| > R \quad \Rightarrow \quad |\widetilde{G}(z)e^z| \le A\rho^{\gamma}\exp[(1-\alpha\theta^2)\rho] \ , \tag{33}$$

Then, for every nonnegative integer m

$$\begin{aligned} g_n &= \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}^{\langle j\rangle}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n)) \\ &= \widetilde{G}(n) + \sum_{k=1}^{m}\sum_{i=1}^{k} b_{i,k+i} n^i \widetilde{G}^{\langle k+i\rangle}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n)) \end{aligned} \tag{34}$$

where b_{ij} are as the coefficients of $\exp(x\log(1+y) - xy)$ at $x^i y^j$, that is:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij} x^i y^j = \exp(x\log(1+y) - xy) \ . \tag{35}$$

Note that $b_{ij} = 0$ for $j < 2i$.

Remark 2. (i) To visualize the expansion (34) we present it below for $m = 6$ and $\delta = 1$ (i.e., in a linear cone)

$$\begin{aligned} g_n \ = \ & \widetilde{G}(n) - \frac{1}{2}n\widetilde{G}^{\langle 2\rangle}(n) + \frac{1}{3}n\widetilde{G}^{\langle 3\rangle}(n) + \frac{1}{8}n^2\widetilde{G}^{\langle 4\rangle}(n) - \frac{1}{4}n\widetilde{G}^{\langle 4\rangle}(n) - \frac{1}{6}n^2\widetilde{G}^{\langle 5\rangle}(n) - \\ & -\frac{1}{48}n^3\widetilde{G}^{\langle 6\rangle}(n) + \frac{1}{5}n\widetilde{G}^{\langle 5\rangle}(n) + \frac{13}{72}n^2\widetilde{G}^{\langle 6\rangle}(n) + \frac{1}{24}n^3\widetilde{G}^{\langle 7\rangle}(n) + \frac{1}{384}n^4\widetilde{G}^{\langle 8\rangle}(n) - \\ & -\frac{1}{6}n\widetilde{G}^{\langle 6\rangle}(n) - \frac{11}{60}n^2\widetilde{G}^{\langle 7\rangle}(n) - \frac{17}{288}n^3\widetilde{G}^{\langle 8\rangle}(n) - \frac{1}{144}n^4\widetilde{G}^{\langle 9\rangle}(n) - \frac{1}{3840}n^5\widetilde{G}^{\langle 10\rangle}(n) \\ & +O(n^{\beta-6}) \ . \end{aligned}$$

(ii) It is not difficult to notice that indeed $b_{ij} = 0$ for $j < 2i$. Let $f(x,y) = \exp(x\log(1+y) - xy)$. Observe that $f(xy^{-2}, y)$ is analytical at $x = y = 0$, hence its Laurent expansion possesses only $x^i y^{j-2i}$ with nonnegative powers leading to $j \ge 2i$ for non zero coefficients b_{ij}, as desired.

(iii) The coefficients b_{ij} and its generating function $\exp(x\log(1+y)-xy)=(1+y)^x e^{-xy}$ are related to Poisson-Charlier polynomials and Laguerre polynomials (cf. [6]) as pointed to us by H-K. Hwang who also used them in a different setting (cf. [24]). Indeed, let us define $B_j(x)=\sum_{i=0}^{\infty} b_{ij}x^i$. Then, $B_j(x)=[y^j]\left(e^{-xy}(1+y)^x\right)$, and $B_j(x)$ is the Poisson-Charlier polynomial which are orthogonal with respect to the discrete measure that puts a weight $\mathrm{Po}(x)$ at the integers (cf. [6]). Thus, we have $B_j(x)=\sum_{k=0}^{n}\binom{x}{k}\frac{(-x)^{j-k}}{(j-k)!}$. Also, $B_j(x)=L_j^{(x-j)}(x)$ where $L_j^{(\alpha)}(x)$ is the Laguerre polynomial. Finally, as observed in Hwang [24], the following recurrence on $B_j(x)$ can be established

$$\begin{aligned} B_0(x) &= 1 \quad , \quad B_1(x)=0 \ , \\ (j+1)B_{j+1}(x) &= -jB_j(x)-xB_{j-1}(x) \quad j\geq 1 \ . \end{aligned}$$

Similarly, one can compute recursively $b_{ij}=[x^i]B_j(x)$ as follows: for all $i,j\geq 0$

$$(j+1)b_{i,j+1}=-jb_{i,j}-b_{i-1,j-1}$$

with $b_{00}=1$. □

In some applications (cf. Sections 6.8–6.9) polynomial growth of $|\widetilde{G}(z)|$ is a too severe restriction. This is remedied in the next theorem.

Theorem 3 EXPONENTIAL DEPOISSONIZATION LEMMA.
Let the conditions of Theorem 2 be satisfied with condition (I) replaced by

$$|\widetilde{G}(z)|\leq A\exp(B|z|^{\beta}) \tag{36}$$

for some $1-\delta\leq\beta\leq\frac{1}{2}$ and constants $A>0$ and B. Then, for every integer $m\geq 0$

$$g_n=\sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^i\widetilde{G}^{\langle j\rangle}(n)+O(n^{-(m+1)(1-2\beta)}\exp(Bn^{\beta})) \tag{37}$$

for large n.

Finally, when studying limiting distributions (in particular, the central limit theorem) one must study a double indexed sequence $g_{n,k}$ in order to obtain asymptotics of the "diagonal" sequence $g_{n,n}$. The next result deals with such a case. (For example, we can set $g_{n,k}=\mathbf{E}e^{tX_n/\sqrt{V_k}}$ where X_n is a sequence of random variable and V_k is a sequence representing the variance of X_k.) We recall that $\widetilde{G}_k(z)=\sum_{n=0}^{\infty} g_{n,k}\frac{z^n}{n!}e^{-z}$.

Theorem 4 DIAGONAL DEPOISSONIZATION LEMMA.

Let $\widetilde{G}_k(z)$ be a sequence of Poisson transforms of $g_{n,k}$ which are assumed to be a sequence of entire functions of z. Consider a polynomial cone $\mathcal{C}(D,\delta)$ with $1/2 < \delta \le 1$. Let the following two conditions hold for some $A > 0$, B, and $\alpha > 0$, β, and γ:

(I) *For $z \in \mathcal{C}(D,\delta)$ and*

$$|z| \in (n - Dn^{\delta}, n + Dn^{\delta}) \quad \Rightarrow \quad |\widetilde{G}_n(z)| \le Bn^{\beta}|\Psi(n)| \ , \tag{38}$$

where $\Psi(x)$ is a slowly varying function.

(O) *For z outside the polynomial cone:*

$$|z| = n \quad \Rightarrow \quad |\widetilde{G}_n(z)e^{z}| \le n^{\gamma}\exp(n - An^{\alpha}) \ , \tag{39}$$

Then, for large n

$$g_{n,n} = \widetilde{G}_n(n) + O(n^{\beta-(2\delta-1)}\Psi(n)) \ . \tag{40}$$

More generally, for every nonnegative integer m

$$g_{n,n} = \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^{i}\widetilde{G}_n^{\langle j\rangle}(n) + O(n^{\beta-(m+1)(2\delta-1)}\Psi(n)) \tag{41}$$

where $\widetilde{G}_n^{\langle j\rangle}(n)$ denotes the jth derivative of $\widetilde{G}_n(z)$ at $z = n$.

Finally, we address the issue of the necessity of conditions (I) and (O). Although we further generalize them below (cf. Section 5), we claim that some restrictions on the growth of $\widetilde{G}(z)$ are necessary in order to allow depoissonization as the following two examples show.

Example 1. *Condition (O) does not hold*

Let $g_n = (-1)^n$, thus $\widetilde{G}(z) = e^{-2z}$ and condition (I) is true for any β and for any $\theta < \pi/2$. But, in this case the condition (O) outside the cone $\mathcal{S}_\theta$ does not hold because $\widetilde{G}(z)e^{z} = e^{|z|}$ for $\arg(z) = \pi$. Clearly, g_n is not $O(n^{\beta})$ for any $\beta < 0$.

Example 2. *Condition (I) is violated*

If $g_n = (1+t)^n$ for $t > 0$, then $\widetilde{G}(z) = e^{tz}$. Condition (O) holds for some θ such that $(1+t)\cos\theta < 1$. But condition (I) inside the cone $\mathcal{S}_\theta$ does not hold because $\widetilde{G}(z)$ does not have a polynomial growth. And, as a matter of fact, g_n is not equivalent to $\widetilde{G}(n)$. □

4.3 Technical Lemmas

In the preparation for the proofs of the above results, we present in this section a series of technical lemmas that are of their own interest.

The lemma below is a well know result concerning the gaussian integral and can be found in textbooks (e.g., N. G. Bruijn [3]):

Lemma 1 *The following identities are true:*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^k e^{-tx^2} dx = \begin{cases} 0 & k = 1, 3, 5, \ldots \\ \frac{t^{-1/2-k/2}k!}{(k/2)!2^{k+1/2}} & k = 2, 4, 6, \ldots \end{cases} \tag{42}$$

and

$$\int_{\theta}^{\infty} x^k e^{-tx^2} dx = O\left(e^{-\frac{1}{2}t\theta^2}\right) \tag{43}$$

where θ is a positive number.

The next lemma is a simple extension of the Cauchy estimate on derivatives of an analytical function.

Lemma 2 *Suppose that f is an analytic function in a domain containing the disc $B(z_0, r) = \{z : |z - z_0| \le r\}$, for some $z_0 \in \mathbb{C}$ and $r > 0$. Suppose further that $|f(z)| \le M < \infty$ when $z \in B(z_0, r)$. Then,*

$$|f^{\langle k\rangle}(z_0)| \le k!Mr^{-k}, \qquad k \ge 0.$$

$f^{\langle k\rangle}(z)$ being the kth derivative of $f(z)$.

Proof. It follows directly from the Cauchy estimate:

$$f^{\langle k\rangle}(z_0) = \frac{k!}{2i\pi}\oint \frac{f(z)dz}{(z - z_0)^{k+1}} ,$$

where the integration being done along the circle of radius r and center z_0. ∎

The following lemma is crucial for our analysis. It bounds derivatives of $\widetilde{G}(z)$ inside a cone. We formulate it for linear cones as well as for polynomial cones.

Lemma 3 (i) Linear Cone. *Let $\theta_0 < \pi/2$ and $\xi > 0$ be such that for all $z \in \mathcal{S}_{\theta_0}$:*

$$|z| > \xi \qquad \Longrightarrow \qquad |G(z)| \le B|z|^{\beta} \tag{44}$$

for some real β. Then, for all $\theta < \theta_0$ there exists B' and $\xi' > \xi$ such that for all positive integers k the following holds

$$|z| > \xi' \qquad \Longrightarrow \qquad |G^{\langle k\rangle}(z)| \le k!(B')^k|z|^{\beta-k} . \tag{45}$$

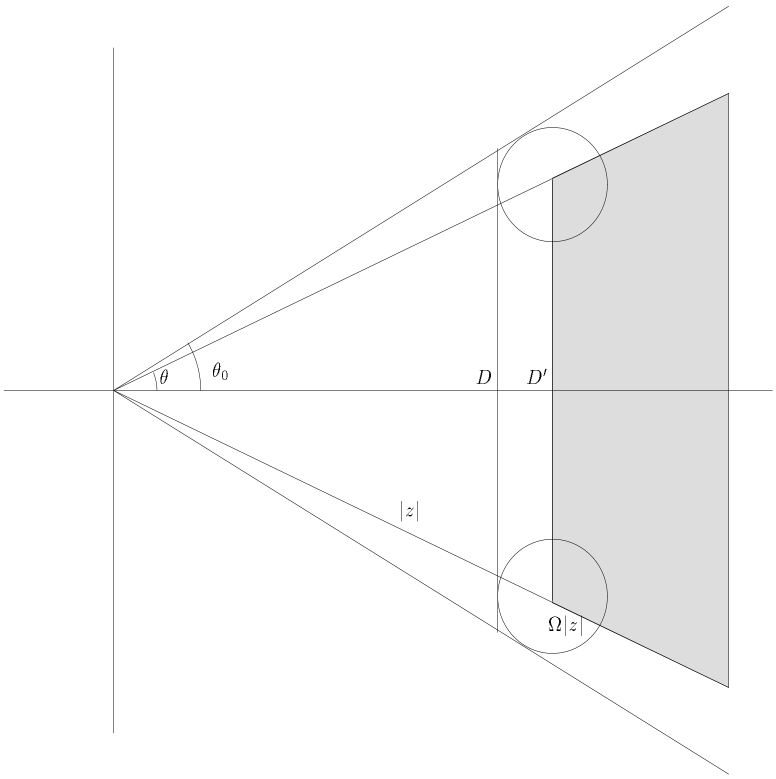

Figure 3: Illustration to Lemma 3 .

(ii) POLYNOMIAL CONE. *Let for all $z \in \mathcal{C}(D,\delta)$ there exist $\xi, B > 0$ such that $|G(z)| \le B|z|^\beta$ for $|z| > \xi$. Then, there exists a smaller cone $\mathcal{C}(D',\delta)$ with $D' < D$ such that for all positive integers k, some $\xi' > \xi$ and a constant B'*

$$|z| > \xi' \qquad \Longrightarrow \qquad |G^{\langle k \rangle}(z)| \le k!(B')^k |z|^{\beta - k\delta} \ . \tag{46}$$

Proof. It directly follows from the previous lemma. We place a circle of radius $\Omega|z|$ for some $\Omega > 0$ at the closest corner of the cone $\mathcal{S}_{\theta_0}$ just after $\Re(z) = \xi$ as shown in Figure 3. Then, by Lemma 2 with $M = \Omega$ we immediately prove (45). For part (ii) we set $\Omega = D'|z|^\delta$ for $z \in \mathcal{C}(D',\delta)$ such that the circle of radius Ω and center z is contained in $\mathcal{C}(D,\delta)$. ■

In the proof of our main results, we often will deal with derivatives of $\widetilde{G}(ne^{it/\sqrt{n}})$ for which we need a uniform bound. To handle it efficiently, we introduce below a special class of functions whose derivatives behave nicely on some intervals. We shall use the following throughout the paper:

Definition 2 *Let $f^{\langle k \rangle}(x)$ denote the kth derivative of a function $f(x)$. We say that a sequence of functions $f_n : I_n \to \mathbb{C}$ defined on subintervals I_n belongs to the class $\mathbb{D}_k(\omega)$*

for ω real if there exist $D > 0$ such that for all integers $j \leq k$ and $x \in I_n$ we have $|f_n^{\langle j \rangle}(x)| \leq D n^{-j\omega}$.

We illustrate the above definition in the example below:

Example 3. *Functions belonging to $\mathbb{D}_k(\omega)$.*

(i) Let f be infinitely differentiable on $[-1, 1]$, and define $f_n(x) = f(xn^{-\omega})$ on $I_n = [-n^\omega, n^\omega]$. Clearly, $f_n \in \mathbb{D}_k(\omega)$ for every $k \geq 0$.

(ii) Let F_n be an analytic function defined in $\{z : |z| \leq n^\omega\}$, $n \geq 1$, such that the sequence F_n is uniformly bounded. Then the restriction f_n of F_n to $I_n = [-\frac{1}{2}n^\omega, \frac{1}{2}n^\omega]$ belongs to $\mathbb{D}_k(\omega)$ for every $k \geq 0$. □

The next lemma presents some simple properties of the class $\mathbb{D}_k(\omega)$. Its proof is trivial, so we omit it.

Lemma 4 (i) *If f_n belongs to $\mathbb{D}_k(\omega)$, then f_n belongs to $\mathbb{D}_k(\omega')$ for all $\omega' \leq \omega$.*

(ii) *If f_n and g_n belong to class $\mathbb{D}_k(\omega)$, and if $H(x, y)$ is a function which is k times continuously differentiable, then $H(f_n, g_n)$ belongs to $\mathbb{D}_k(\omega)$. Consequently if f_n and $g_n \in \mathbb{D}_k(\omega)$, then $f_n + g_n$ and $f_n \times g_n \in \mathbb{D}_k(\omega)$.*

To prove our general asymptotic expansion like the one in Theorem 2, we must study bounds and Taylor's expansions of the following function

$$h_n(t) = \exp(n(e^{it/\sqrt{n}} - 1 - it/\sqrt{n})) \ . \tag{47}$$

One can interpret $h_n(t)$ as the kernel of the Cauchy integral (28) (cf. also (52)), thus not surprisingly this function often appears in our depoissonization theorems. Its property are discussed in the next lemma.

Lemma 5 *The following statements hold:*
(i) *For $t \in [-\pi\sqrt{n}, \pi\sqrt{n}]$ there exists $\mu > 0$ such that $|h_n(t)| \leq e^{-\mu t^2}$, where μ is a constant.*
(ii) *For complex t such that $|t| \leq Bn^{\frac{1}{6}}$ for some $B > 0$, the sequence of functions $F_n(t) = h_n(t)e^{-t^2/2}$ are bounded and belong to $\mathbb{D}_k(\frac{1}{6})$ for any integer k.*

(iii) *Let ξ_{ij} be defined as*

$$\sum_{ij} \xi_{ij} x^i y^j = \exp(y(e^{ix} - 1 - ix + \frac{1}{2}x^2)) \ . \tag{48}$$

Then, for all nonnegative integers k, there exists ν_k such that for all $t \in [-\log n, \log n]$,

$$h_n(t) = e^{-t^2/2}\left(1 + \sum_{i=3}^{2k}\sum_{j=1}^{k} \xi_{ij} t^i n^{j-i/2} + O(\nu_k \log^{3(k+1)} n^{-(k+1)/2})\right).$$

Proof: Part (i) is easy, and in fact, $\mu = 1/\pi^2$ suffices. Point (ii) is a little more intricate. According to Lemma 4, it suffices to prove that the sequence of function $\log h_n(t) \in \mathbb{D}_k(\frac{1}{6})$, and then refer to the fact that the sequence of exponential of $\log h_n(t)$ is still in $\mathbb{D}_k(\frac{1}{6})$. Denoting $e^{ix} - 1 - ix + x^2/2 = \ell(x)$, we observe that the sequence of functions $\ell(t/\sqrt{n})$ belongs to $\mathbb{D}_k(\frac{1}{2})$ for $x = O(\sqrt{n})$ and any integer $k \geq 1$. Therefore, the ith derivative for $3 \leq i \leq k$ of $n\ell(t/\sqrt{n})$ is $O(n^{1-i/2})$, which is $O(n^{-i/6})$ for all $i \geq 3$. In particular, the third derivative is $O(n^{-1/2})$. But, we observe also that by successive integrations the first derivative of $n\ell(t/\sqrt{n})$ is $O(n^{-1/6})$, the second derivative is $O(n^{-1/3})$, and $n\ell(t/\sqrt{n})$ is $O(1)$, because the first two derivatives of $\ell(t)$ and $\ell(0)$ are zero at $t = 0$ by the construction. Hence, part (ii) follows.

Finally, to prove part (iii), we proceed as follows: Let, as before, $e^{ix}-1-ix+x^2/2 = \ell(x)$. There exists μ_3 such that $|\ell(x)| \leq \mu_3 |x|^3$ holds for all real x, and therefore the absolute value of $n\ell(t/\sqrt{n})$ is smaller than $\mu_3 (\log n)^3 n^{-1/2}$ for $t = O(\log n)$. Clearly, we can write $n\ell(t/\sqrt{n}) = \log h_n(t) + t^2/2$, and thus due to the fact that $n\ell(t/\sqrt{n})$ remains uniformly bounded for $t = O(\log n)$, the following Taylor expansion of $h_n(t)\exp(t^2/2)$ with respect to $n\ell(t/\sqrt{n})$ is valid:

$$h_n(t)\exp(t^2/2) = 1 + n\ell(\frac{t}{\sqrt{n}}) + \frac{n^2}{2}(\ell(\frac{t}{\sqrt{n}}))^2 \cdots + \frac{n^{k-1}}{(k-1)!}(\ell(\frac{t}{\sqrt{n}}))^{k-1} + O\left((\ell(\frac{t}{\sqrt{n}}))^k n^k\right) \tag{49}$$

We now expand with respect to t each function $\ell^i(\cdot)$ up to degree $k + 2i$. Replacing each $\ell^i(t/\sqrt{n})$ in (49) by the corresponding Taylor expansion with respect to t leads to the remainder $O(t^{2i+k} n^{-i-k/2})$. Note that n^{-i} cancels the factor n^i in front of $\ell^i(t/\sqrt{n})$ in (49). Regrouping the rests yields the desired expansion for some ξ_{ij}. To identify the expression (48), we just observe that we have $\sum_{ij} \xi_{ij} x^i y^j = \exp(y\ell(x))$ by a formal identification $y = n$ and $x = t/\sqrt{n}$ for any t. ■

Furthermore, when proving Theorems 2, we need an extension of Lemma 1 which is presented next.

Lemma 6 *For nonnegative H and β, let a sequence of complex functions $F_n(x)$ defined on $x \in [-H\log n, H\log n]$ belong to the class $\mathbb{D}_k(\beta)$ for any $k \geq 1$. Then,*

$$\frac{1}{\sqrt{2\pi}}\int_{-H\log n}^{H\log n} F_n(x) e^{-x^2/2} dx = \sum_{i=0}^{k-1} \frac{1}{2^i i!} F_n^{\langle 2i \rangle}(0) + O(n^{-2k\beta}) .$$

Proof: By Taylor's expansion $F_n(x) = \sum_{i=0}^{k-1} \frac{x^i}{i!} F^{\langle i \rangle}(0) + \Delta_n(x) x^k$ where $\Delta_n(x) = O(n^{-k\beta})$ due to $F_n \in \mathbb{D}_k(\beta)$ for any k. Thus,

$$\frac{1}{\sqrt{2\pi}} \int_{-H \log n}^{H \log n} F_n(x) e^{-x^2/2} dx = \sum_{i=0}^{k-1} \frac{F_n^{\langle i \rangle}(0)}{i! \sqrt{2\pi}} \int_{-H \log n}^{H \log n} x^i e^{-x^2/2} dx + O(n^{-k\beta})$$

Observe that $F_n^{\langle i \rangle}(0) = O(n^{-i\beta})$ since $F_n \in \mathbb{D}_k(\beta)$ and $i \leq k$. Furthermore, by Lemma 1 changing the limit of integration in the above to $\pm\infty$, one introduces an error of order $O(e^{-(H \log n)^2/4})$ which decreases faster than any polynomial. This completes the proof after some algebra. ■

Finally, for our exponential depoissonization results (cf. Theorem 3, and Theorems 8 and 9 of Section 5), we replace the above lemma by the following one which is much harder to prove:

Lemma 7 *Let $\beta_1 \geq 0 \geq \beta_2$ be such that $2\beta_1 + \beta_2 < 0$. Let $D > 0$ and $\gamma > \beta_1$. Let $F_n(z)$ be a sequence of complex analytical functions defined for $z \in J_n$ where J_n is the set of complex z such that both $\Re(z)$ and $\Im(z)$ belong to $[-Dn^\gamma, Dn^\gamma]$. We assume that $F_n(0) = 0$ and that there exists a real number $B > 0$ such that $F_n(0) = 0$, $|F_n'(0)| \leq Bn^{\beta_1}$ and the second derivatives $|F_n^{\langle 2 \rangle}(z)| \leq Bn^{\beta_2}$ uniformly in $z \in J_n$. Then:*

$$\frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(x)] e^{-\frac{1}{2}x^2} dx = \exp\left(\frac{1}{2}(F_n^{\langle 1 \rangle}(0))^2\right) \left(1 + O(n^{2\beta_1 + \beta_2})\right) .$$

Proof: Using the first order Taylor's expansion of $F_n(z)$ around $z = 0$, for all $z \in J_n$, we have the following estimate:

$$F_n(z) = F_n'(0) z + \Delta_n(z) z^2 .$$

where $\Delta_n(z)$ is the remainder that can be computed as:

$$\Delta_n(z) = \int_0^1 F_n^{\langle 2 \rangle}(zt)(1-t) dt .$$

Multiplying the function $\exp[F_n'(x)] e^{-\frac{1}{2}x^2}$ by $\exp[-\frac{1}{2}(F_n(0))^2]$ yields $\exp[-\frac{1}{2}(x - F_n'(0))^2 + \Delta_n(x) x^2]$. Thus:

$$\frac{\exp[-\frac{1}{2}(F_n^{\langle 1 \rangle}(0))^2]}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(x)] e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp\left(-(x - F_n'(0))^2 + \Delta_n(x) x^2\right) dx$$

Moving the contour of integration to a horizontal line containing $F_n'(0)$ we obtain the following expression:

$$\int_{-Dn^\gamma}^{Dn^\gamma} \exp[-\frac{1}{2}(x \quad - \quad F_n'(0))^2 + \Delta_n(x) x^2] dx \tag{50}$$

$$
\begin{aligned}
&= \int_{i\Im(F_n'(0))-Dn^\gamma}^{i\Im(F_n'(0))+Dn^\gamma} \exp[-\frac{1}{2}(z-F_n'(0))^2+\Delta_n(z)z^2]dz + \\
&+ \int_{-Dn^\gamma}^{i\Im(F_n'(0))-Dn^\gamma} \exp[-\frac{1}{2}(z-F_n'(0))^2+\Delta_n(z)z^2]dz + \\
&+ \int_{i\Im(F_n'(0))+Dn^\gamma}^{Dn^\gamma} \exp[-\frac{1}{2}(z-F_n'(0))^2+\Delta_n(z)z^2]dz \ .
\end{aligned}
$$

By our hypotheses $|\Delta_n(z)| \le Bn^{\beta_2}$ for $z \in J_n$, and $|F_n'(0)| < Bn^{\beta_1}$. Therefore, the two last integrals on the right hand side of (50) are of order

$$
Bn^{\beta_1} \exp\left(-\frac{1}{2}(Dn^\gamma)^2 + (Bn^{\beta_1})^2 + BD^2 n^{\beta_2+2\gamma} + B^3 n^{\beta_2+2\beta_1}\right)
$$

which decreases to zero faster than any polynomial. But, after substitution $z - F_n'(0) = x$, we can now rewrite the first integral of the right hand side of (50) as

$$
\int_{-\Re(F_n'(0))-Dn^\gamma}^{-\Re(F_n'(0)+Dn^\gamma} \exp[-\frac{1}{2}x^2 + R_n(x)(x+F_n'(0))^2]dx
$$

with $R_n(z) = \Delta_n(z + F_n'(0))$. Let us take $\gamma' \in]\beta_1, -\beta_2/2[$ such that $\gamma' < \gamma$. Notice that $\gamma' > \beta_1$ and $2\gamma' + \beta_2 < 0$. We can again split the integral into three parts as follows:

$$
\begin{aligned}
\int_{-\Re(F_n'(0))-Dn^\gamma}^{-\Re(F_n'(0))+Dn^\gamma} \exp\left(-\frac{1}{2}x^2\right. &+ \left. R_n(x)(x+F_n'(0))^2\right) dx \qquad (51) \\
&= \int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2 + R_n(x)(x+F_n'(0))^2\right) dx \\
&+ \int_{-\Re(F_n'(0))-Dn^\gamma}^{-Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2 + R_n(x)(x+F_n'(0))^2\right) dx \\
&+ \int_{Dn^{\gamma'}}^{-\Re(F_n'(0))+Dn^\gamma} \exp\left(-\frac{1}{2}x^2 + R_n(x)(x+F_n'(0))^2\right) dx
\end{aligned}
$$

Observe that $|R_n(x)| \le Bn^{\beta_2}$ since $x + F_n'(0) \in J_n$, thus in the two last integrals of the above quantity $R_n(x)(x+F_n'(0))^2$ can be estimated as follows (for $x > Dn^{\gamma'}$ when n is large enough)

$$
|R_n(x)(x+F_n'(0))^2| \le Bn^{\beta_2}(|x| + Bn^{\beta_1})^2 \le 4Bn^{\beta_2}x^2 \le \frac{1}{4}x^2,
$$

which by Lemma 1 leads to an estimate $O(e^{-D^2 n^{2\gamma'}/8})$ of the integrals.

Coming back to the first integral of the right-hand side of (51), namely:

$$
\int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp[-\frac{1}{2}x^2 + R_n(x)(x+F_n'(0))^2]dx \ ,
$$

we proceed as follows: The quantity $|R_n(x)(x+F_n'(0))^2| \le B(B+D)^2 n^{\beta_2+2\gamma'}$ tends to zero as $n \to \infty$ due to $\beta_2 + 2\gamma' < 0$, thus $\exp[R_n(x)(x+F_n'(0))^2] = 1 + O(R_n(x)(x+F_n'(0))^2)$. The above expression becomes:

$$
\begin{aligned}
\int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\Big(-\frac{1}{2}x^2 \quad &+ \quad R_n(x)(x+F_n'(0))^2\ \Big)\,dx \\
&= \int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2\right)\left(1+O(R_n(x)(x+F_n'(0))^2)\right)dx \\
&= \int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2\right)dx \\
&+ \int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2\right) O(R_n(x)(x+F_n'(0))^2)2dx
\end{aligned}
$$

If we expand the limit of the integral $\int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} \exp\left(-\frac{1}{2}x^2\right)dx$ to $\pm\infty$, then by Lemma 1 we only introduce an error of order $\exp\left(-(Dn^{\gamma'})^2/4\right)$ that decays to zero faster than any polynomial. Then, such an integral has classical value $\sqrt{2\pi}$.

To complete the proof, it suffices to estimate

$$
\int_{-Dn^{\gamma'}}^{Dn^{\gamma'}} O(R_n(x)(x+F_n'(0))^2)\exp\left(-\frac{1}{2}x^2\right)dx\ .
$$

But this can be bounded from the above by

$$
\int_{-\infty}^{\infty} O(Bn^{\beta_2})(x^2+2|F_n'(0)||x|+|F_n'(0)|^2\exp\left(-\frac{1}{2}x^2\right)dx\ .
$$

which is $O(n^{\beta_2})\left(4|F_n'(0)|+(1+|F_n'(0)|^2)\sqrt{2\pi}\right)$ and this yields $O(n^{\beta_2+2\beta_1})$. The lemma is proved. ■

Finally, when dealing with limiting distributions (cf. Theorem 9 of Section 5), we need a slight extension of Levy's continuity theorem. The lemma below seems to be well known but since we could not find a reference, we present it with a short proof.

Lemma 8 (Analytical Levy Theorem) *Let Y_n and Y be respectively a sequence of real random variables and a real random variable such that $g_n(t) = \mathbf{E}[e^{tY_n}]$ and $g(t) = \mathbf{E}[e^{tY}]$ are their moment generating functions defined in a real neighbourhood of $t = 0$. Suppose that $\lim_{n\to\infty} g_n(t) = g(t)$ for t belonging to such a real neighbourhood of 0. Then, Y_n converges to Y both in distribution and in moments.*

Proof: In order to show the convergence in both distribution and moments, it suffices to prove that $g_n(t)$ and $g(t)$ are defined in a complex neighbourhood of 0, and $g_n(t)$ converge to $g(t)$ in such a complex neighbourhood. Indeed in this case $g_n(t)$ and $g(t)$ are analytical, therefore due to *Levy's Continuity Theorem*, the convergence in distribution holds since we have convergence for imaginary t, and $g(t)$ is continuous at 0. The convergence in moments is easier since the kth moment of Y_n is $g^{\langle k\rangle}(0)$ and the convergence of analytical functions implies the convergence of their derivatives.

Let us first extend the definition of $g_n(t)$ to complex neighbourhood of $t = 0$ defined by t such $\Re(t)$ belongs to a real neighbourhood. Then,

$$|e^{tY_n}| \le e^{\Re(t)Y_n} \ .$$

Therefore, $g_n(t)$ exists and is bounded in a complex neighbourhood of 0. Clearly, $g_n(t)$ and $g(t)$ are analytical functions. Let us take a compact complex neighbourhood of 0. We know that $|g_n(t)|$ is uniformly bounded in this neighbourhood by, say, a number $A > 0$. We redefine this neighbourhood by removing all point with distance smaller than $\varepsilon > 0$ of the boundary of the former neighbourhood, where ε is arbitrarily small. Due to Cauchy estimate the derivatives $|g_n'(t)|$ are also uniformly bounded by $A\varepsilon^{-1}$, and therefore $g_n(t)$ are bounded and uniformly continuous. By *Ascoli's Theorem*, from every sequence of $g_n(t)$ we can extract a convergent subsequence. This limit function can only be the unique analytical continuation of $g(t)$, thus $g_n(t)$ converges to $g(t)$ in this complex neighbourhood of 0. ■

4.4 Proofs of Main Results

In this subsection, we present proofs of the main results. For the reader's convenience we first present a detailed proof of Theorem 1 to illustrate some techniques used in the proof of our other findings. Then, in the next subsection we present a general depoissonization tool theorem (cf. Theorem 5) that is used to prove all of the remaining theorems in a uniform manner.

4.4.1 Proof of Basic Depoissonization Theorem

We now prove Theorem 1. The proof relies on the Cauchy formula (28). By Stirling's approximation $n! = n^n e^{-n}\sqrt{2\pi n}(1 + O(1/n))$, thus (28) becomes

$$\begin{aligned} g_n &= (1 + O(1/n))\sqrt{\frac{n}{2\pi}} \int_{-\pi}^{\pi} \widetilde{G}(ne^{it}) \exp\left(n\left(e^{it} - 1 - it\right)\right) dt \\ &= (1 + O(1/n))(I_n + E_n) \end{aligned}$$

where

$$I_n = \sqrt{\frac{n}{2\pi}} \int_{-\theta}^{\theta} \widetilde{G}(ne^{it}) \exp\left(n\left(e^{it} - 1 - it\right)\right) dt$$

$$
\begin{aligned}
E_n &= \sqrt{\frac{n}{2\pi}} \int_{|t|\in[\theta,\pi]} \widetilde{G}(ne^{it}) \exp\left(n\left(e^{it}-1-it\right)\right) dt \\
&= \frac{n^n e^{-n}\sqrt{2\pi n}}{2\pi i} \int_{|t|\in[\theta,\pi]} \frac{\widetilde{G}(z)e^z}{z^{n+1}} dz \; .
\end{aligned}
$$

We estimate the above two integrals. We begin with the latter. Observe that by condition (O) (cf. (30)), we obtain

$$
|E_n| \le A'\sqrt{2\pi n} e^{-(1-\alpha)n}
$$

where A' depends only on A and ξ, thus E_n exponentially decays to zero.

Having E_n "under hat", we can turn our attention to integral I_n which is more intricate to handle. First of all, we substitute $t = t'/\sqrt{n}$ to get

$$
I_n = \frac{1}{\sqrt{2\pi}} \int_{-\theta\sqrt{n}}^{\theta\sqrt{n}} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right) dt \; . \tag{52}
$$

Let $h_n(t) = \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)$. We need to estimate $h_n(t)$ in the interval $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$, and find the Taylor expansion of it in a smaller interval, say for $t \in [-\log n, \log n]$. The latter restriction is necessary since $\frac{t}{\sqrt{n}} = O(1)$ for $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$. Thus, we can split the integral I_n into two terms I_n' and I_n'', that is, $I_n = I_n' + I_n''$ where

$$
\begin{aligned}
I_n' &= \frac{1}{\sqrt{2\pi}} \int_{-\log n}^{\log n} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right) dt \; , \\
I_n'' &= \frac{1}{\sqrt{2\pi}} \int_{t\in[-\theta\sqrt{n},-\log n]} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right) dt + \\
&+ \frac{1}{\sqrt{2\pi}} \int_{t\in[\log n,\theta\sqrt{n}]} \widetilde{G}(ne^{it/\sqrt{n}}) \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right) dt \; .
\end{aligned}
$$

To estimate the second integral I_n'' we observe that $|h_n(t)| \le e^{-\mu t^2}$ for $t \in [-\theta\sqrt{n}, \theta\sqrt{n}]$, where μ is a constant (in fact, $\mu = 1/\pi^2$ suffices). Thus, by Lemma 1 and condition (I) we immediately show that $I_n'' = O(n^\beta e^{-\mu \log^2 n/2})$ which decays faster than any polynomial.

Now, we estimate I_n'. Observe first that for $t \in [-\log n, \log n]$ we can write

$$
\begin{aligned}
h_n(t) &= \exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right) \\
&= e^{-t^2/2}\left(1 - \frac{it^3}{6\sqrt{n}} + \frac{t^4}{24n} - \frac{t^6}{72n} + O\left(\frac{\log^9 n}{n\sqrt{n}}\right)\right)
\end{aligned}
$$

Furthermore, using (I) and Lemma 3 for $|z| > (1+\Omega)\xi$ and $z \in \mathcal{S}_{\theta'}$ for $\theta' < \theta$ we have $|\widetilde{G}'(z)| \le B_1 |z|^{\beta-1}$ and $|\widetilde{G}''(z)| \le B_2 |z|^{\beta-2}$, for some constants B_1 and B_2. Thus, we can expand $\widetilde{G}(ne^{it/\sqrt{n}})$ around $t = 0$ as

$$
\widetilde{G}(ne^{it/\sqrt{n}}) = \widetilde{G}(n) + it\sqrt{n}\widetilde{G}'(n) + \Delta_n(t)t^2
$$

where $|\Delta_n(t)| \leq (B_1+B_2)n^{\beta-1}$. In summary, the integral I_n' becomes

$$
\begin{aligned}
I_n' &= \frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n)+\widetilde{G}'(n)it\sqrt{n}\right)\left(1-\frac{it^3}{6\sqrt{n}}+\frac{t^4}{24n}-\frac{t^6}{72n}\right)\mathrm{d}t + \\
&+\frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\Delta_n(t)t^2h_n(t)\mathrm{d}t + \\
&+\frac{1}{\sqrt{2\pi}}\int_{-\log n}^{\log n} e^{-t^2/2}\left(\widetilde{G}(n)+\widetilde{G}'(n)it\sqrt{n}\right)O\left(\frac{\log^9 n}{n\sqrt{n}}\right)\mathrm{d}t\ .
\end{aligned}
$$

To complete the proof we must estimate the above three integrals. From Lemmas 1 and 3 we assess that the first integral is equal to $\widetilde{G}(n)+O(n^{\beta-1})$. Thus it suffices to show that the last two integrals are $O(n^{\beta-1})$. Indeed, using an estimate on $\Delta_n(t)$ we observe that the absolute value of second integral is smaller than $(B_1+B_2)n^{\beta-1}$. Finally, the last integral is $O(n^{\beta-\frac{3}{2}}\log^9 n)$. All together the error becomes $O(n^{\beta-1})$. Theorem 1 is proved. ■

4.4.2 Basic Depoissonization Tool Theorem

In this subsection, we prove all remaining theorems by an application of a general depoissonization result that we formulate next:

Theorem 5 Basic Diagonal depoissonization tool
Let $\widetilde{G}_k(z)$ be the Poisson transform of a sequence $g_{n,k}$ that is assumed to be a sequence of analytical functions for $|z|\leq n$. We postulate that the following three conditions simultaneously hold for some $H>0$, γ, $\delta\leq 1$ and integer $m\geq 0$:

(I) *For $z\in\mathcal{C}(D,\delta)$, such that $|z|=n$: $|\widetilde{G}_n(z)|\leq F(n)$,*

(L) *The sequence of function $f_n(t)=\widetilde{G}_n(ne^{it/\sqrt{n}})/F(n)$ defined for $t\in[-H\log n, H\log n]$ belongs to the class $\mathbb{D}_m(\gamma)$,*

(O) *For $z\notin\mathcal{C}(D,\delta)$ and $|z|=n$: $|\widetilde{G}_n(z)e^z|\leq p(n)e^nF(n)$ where $p(n)$ decays faster than $n^{-\frac{1}{2}-(m+1)\min\{\gamma,\frac{1}{6}\}}$, i.e., $p(n)=o(n^{-\frac{1}{2}-(m+1)\min\{\gamma,\frac{1}{6}\}})$. Then,*

$$
g_{n,n}=\widetilde{G}_n(n)+O(n^{-\min\{\gamma,\frac{1}{6}\}})F(n)\ , \tag{53}
$$

and more generally for any integer $m\geq 0$

$$
g_{n,n}=\sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^i\widetilde{G}_n^{\langle j\rangle}(n)+O(n^{-(m+1)\min\{\gamma,1/6\}})F(n) \tag{54}
$$

where b_{ij} are defined in (35).

Proof. The proof again relies on the Cauchy integral that we split into I_n and E_n as before, where I_n is the integral over t such that ne^{it} is inside $\mathcal{C}(D,\delta)$, and E_n is the integral for the ne^{it} outside the cone, that is:

$$\begin{aligned} I_n &= \omega_n\sqrt{\frac{n}{2\pi}}\int_{-Dn^{\delta-1}}^{Dn^{\delta-1}} \widetilde{G}_n(ne^{it})\exp\left(n\left(e^{it}-1-it\right)\right)dt \\ E_n &= \frac{n!}{2\pi i}\int_{|t|\in[Dn^{\delta-1},\pi]} \frac{\widetilde{G}_n(z)e^z}{z^{n+1}}dz \\ &= \omega_n\sqrt{\frac{n}{2\pi}}\int_{|t|\in[Dn^{\delta-1},\pi]} \widetilde{G}_n(ne^{it})\exp\left(n\left(e^{it}-1-it\right)\right)dt\ , \end{aligned}$$

where $\omega_n = n!n^{-n}e^n(2\pi n)^{-1} = 1+O(1/n)$ by the Stirling formula. We estimate the above two integrals. We begin with the latter. Observe that by condition (O) we have $|E_n|/F(n) \le \sqrt{2\pi n}p(n) = o(n^{-k\min\{\gamma,\frac{1}{6}\}})$ which is negligible compared to the error term.

The evaluation of I_n is more intricate. First of all, we substitute $t = t'/\sqrt{n}$ to get

$$\begin{aligned} I_n &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-Dn^{\delta-1/2}}^{Dn^{\delta-1/2}} \widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt \\ &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-Dn^{\delta-1/2}}^{Dn^{\delta-1/2}} \widetilde{G}_n(ne^{it/\sqrt{n}})h_n(t)dt \end{aligned}$$

where $h_n(t)$ is defined in (47) and analyzed in Lemma 5. We further split the integral I_n into I'_n and I''_n as follows (below, we use the alternative definition of the polynomial cone as discussed after Definition 1):

$$\begin{aligned} I'_n &= \frac{\omega_n}{\sqrt{2\pi}}\int_{-H\log n}^{H\log n} \widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt\ , \\ I''_n &= \frac{\omega_n}{\sqrt{2\pi}}\int_{t\in[-Dn^{\delta-1/2},-H\log n]} \widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt \\ &+ \frac{\omega_n}{\sqrt{2\pi}}\int_{t\in[H\log n,Dn^{\delta-1/2}]} \widetilde{G}_n(ne^{it/\sqrt{n}})\exp\left(n\left(e^{it/\sqrt{n}}-1-it/\sqrt{n}\right)\right)dt\ . \end{aligned}$$

To estimate the second integral I''_n we use the fact that $|h_n(t)| \le e^{-\mu t^2}$ with $\mu > 0$ for $t \in [-Dn^{\delta-1/2}, Dn^{\delta-1/2}]$ as discussed in Lemma 5. Thus, by Lemma 1 and condition (I) we immediately obtain that $I''_n/F(n) = O(e^{-\mu H^2\log^2 n/2})$ which decays faster than any polynomial.

Now, we estimate I'_n. Observe first that since $h_n(t)e^{t^2/2} \in \mathbb{D}_m(\frac{1}{6})$ for any m and $\widetilde{G}_n(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma)$, therefore their product $R_n(t) = h_n(t)e^{t^2/2}\widetilde{G}_n(ne^{it/\sqrt{n}})/F(n) \in \mathbb{D}_m(\gamma_2)$ where $\gamma_2 = \min\{\gamma,\frac{1}{6}\}$. In view of this, by Lemma 6 we arrive at

$$I'_n/F(n) = \frac{\omega_n}{\sqrt{2\pi}}\int_{H\log n}^{H\log n} R_n(t)e^{-t^2/2}dt = \omega_n(F_n(0)+O(n^{-\gamma_2}))\ ,$$

and consequently $g_n = \widetilde{G}_n(n) + O(n^{-\gamma_2})F(n)$ (observing that $\omega_n = 1 + O(n^{-1})$) which proves (53).

To prove our general result (54), we apply Lemma 6 for any k to obtain

$$g_{n,n}/F(n) = \omega_n \sum_{i=0}^{k-1} \frac{1}{2^i i!} R_n^{\langle 2i \rangle}(0) + O(n^{-2k\gamma_2})$$

replacing the derivatives of $R_n(0)$ by their actual values (that involve the derivatives of $\widetilde{G}(z)$ at $z = n$), and noting that all odd powers of $n^{\frac{1}{2}}$ disappear (since we are considering only derivatives of $F_n(t)$ of even order), multiplying by $\omega_n F(n)$, replacing ω_n by the Stirling expansion, we finally obtain

$$g_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}_n^{\langle j \rangle}(n) + O(n^{-(m+1)\min\{\gamma, \frac{1}{6}\}})F(n)$$

where we formally changes $m + 1 = 2k$. Notice that the terms in the expansion do not involve m since the Stirling expansion and the expansion of Lemma 6 do not contain m (cf. below Remark 3(ii)). ■

Remark 3. (i) We have already explained in the proof of Lemma 5 a way to identify the coefficients b_{ij} in our general depoissonization statement, but here we provide another more detailed derivation. We can write $g_{n,n} = \frac{n!}{2i\pi} \oint \widetilde{G}_n(z) e^z \frac{dz}{z^{n+1}}$. Finding Taylor's expansion of $\widetilde{G}_n(z) = \sum_k \widetilde{G}_n^{\langle k \rangle}(n)(z-n)^k/k!$ and formally replacing each $\widetilde{G}_n^{\langle k \rangle}(n)$ by y^k we obtain the identity:

$$\widetilde{G}_n(z) = \sum_{k \geq 0} \frac{(z-n)^k}{k!} y^k = \exp(y(z-n)) \ .$$

Therefore

$$g_{n,n} = n![z^n](e^z \widetilde{G}_n(z)) = n![z^n](e^{-yn} e^{(1+y)z}) = (1+y)^n e^{-ny} \ .$$

On the other hand, $g_{n,n} = \sum_{i,j} b_{ij} n^i y^j$ and this provides the desired identification of the coefficients b_{ij}. There is a more classical proof of the above based on Stirling numbers which is presented in Appendix A.

(ii) A careful reader will notice that there is a minor problem in the above proof which is more pedagogical than mathematical. When we apply Lemma 6 for some given k and expand the derivatives, we obtain an expansion of the indicated type, but not all of the coefficients b_{ij} are the right ones. For example, with $k = 2$ (expanding to F_n''), there appears a term $-\frac{1}{2}\widetilde{G}_n'(n)$ (which is absent from the final result since $b_{01} = 0$); this term is cancelled when expanding one step further, but then other coefficients are wrong. The explanation is, of course, that you get the right coefficients for all terms that matter; the discrepancies only appear in terms which anyhow can be absorbed in the error term, and there is nothing wrong with the final result. □

Having the depoissonization tool of Theorem 5, we can now finally prove our remaining theorems by identifying the function $F(n)$ (see condition (I)) and finding the right γ in condition (L) of Theorem 5.

We start with the **proof of Theorem 4** which immediately implies **Theorem 2** after setting $g_{n,n} = g_n$. First of all, observe that conditions (I) and (O) of Theorem 4 imply conditions (I) and (O) of Theorem 5 when we set $F(n) = Bn^{\beta}\Psi(n)$. Thus, to complete the proof we must verify condition (L) of Theorem 5 and find γ such that $f_n(t) = \widetilde{G}_n(ne^{it/\sqrt{n}})n^{-\beta}/\Psi(n) \in \mathbb{D}_m(\gamma)$ for any integer $m \geq 1$. But, by Lemma 3(ii) for all integer m and for all z belonging to a smaller polynomial cone $\mathcal{C}(D', \delta)$ with $D' < D$ there exists B_m such that $|\widetilde{G}_n^{\langle m\rangle}(z)| \leq B_m|z|^{\beta-m\delta}\Psi(|z|)$. Thus, after setting $z = ne^{it/\sqrt{n}}$ we immediately prove that the sequence of functions $f_n(t)$ belongs to $\mathbb{D}_m(\delta - \frac{1}{2})$ for all integer m. Set now $\gamma = \min\{\delta - \frac{1}{2}, \frac{1}{6}\}$. Then, by Theorem 5 there is some $m' > m$ such that

$$g_{n,n} = \sum_{i=0}^{m'} \sum_{j=0}^{i+m'} b_{ij} n^i \widetilde{G}_n^{\langle j\rangle}(n) + O(n^{\beta - m'\gamma}) \ .$$

The last delicate point is to obtain the correct error term of Theorem 4. But this can be achieved by setting $m' = \lceil (m+1)(2\delta - 1)/\gamma \rceil$ for given value m. This will lead to the correct error term equal to $O(n^{\beta-(m+1)(2\delta-1)})$. Indeed, the additional terms obtained are those with $i > m$ and $j > i+m$. The corresponding coefficients are $b_{ij}n^i\widetilde{G}_n^{\langle j\rangle}(n)$ with $b_{ij} \neq 0$ for $j \geq 2i$. But, their orders are:

$$n^{\beta+i-j\delta} \leq n^{\beta+i-2i\delta} = n^{\beta-i(2\delta-1)} \leq n^{\beta-(m+1)(2\delta-1)} \ ,$$

and this proves Theorems 4 and 2.

Finally, we **prove Theorem 3**. As before, the proof relies on Theorem 5 with $F(n) = \exp(Bn^{\beta})$. Conditions (I) and (O) are again easy to verify, so we only need to check condition (L) of Theorem 5, that is, we must decide about the growth of $f_n(t) = \widetilde{G}(ne^{it/\sqrt{n}})\exp(-Bn^{\beta})$. Actually, we prove that $f_n(t) \in \mathbb{D}(\frac{1}{2} - \beta)$. For this we extend Lemma 3, and will show that $|\widetilde{G}^{\langle m\rangle}(z)| \leq B_m|z|^{m(\beta-1)}\exp(B|z|^{\beta})$. We use again the Cauchy formula to obtain for some constant Ω

$$|\widetilde{G}^{\langle k\rangle}(z)| \leq \frac{\max_{|\omega|=\Omega|z|^{1-\beta}} |\widetilde{G}(z+\omega)|}{\Omega|z|^{m(1-\beta)}} \ .$$

Notice that $z + \omega$ still belongs to the polynomial cone since $1 - \beta \leq \delta$, and therefore exponential estimate $|\widetilde{G}(z+\omega)| < A\exp(B|z+\omega|^{\beta})$ is still valid. Due to the fact that the function z^β has derivative $O(z^{\beta-1})$, we can derive as follows:

$$\begin{aligned}\max_{|\omega|=\Omega|z|^{1-\beta}} |\widetilde{G}(z+\omega)| &\leq A\exp(|B||z|^\beta + O(z^{\beta-1})|z|^{1-\beta}\Omega) \\ &\leq A'\exp(|B||z|^\beta)\end{aligned}$$

for some constant A'. Thus, $|\widetilde{G}^{\langle k\rangle}(z)| \leq A'\exp(B|z|^\beta)$. To complete the proof, we must establish the error term. Let $\gamma = \min\{\frac{1}{2}-\beta, \frac{1}{6}\}$. An application of Theorem 5 leads to the error term $O\left(n^{-m'\gamma}\exp(Bn^\beta)\right)$ for some integer $m' \geq 0$. To establish the right error term $O(n^{-(m+1)(1-2\beta)}\exp(Bn^\beta))$ we follow the same approach as before: That is, we set $m' = \lfloor (m+1)(1-2\beta)/\gamma \rfloor$ which introduces additional terms of the value $b_{ij}n^i\widetilde{G}^{\langle j\rangle}(n)$ that contribute $n^{i-j(1-\beta)}\exp(Bn^\beta)$. But, since $b_{ij} \neq 0$ for $j \geq 2i$,

$$n^{i-j(1-\beta)}\exp(Bn^\beta) \leq n^{i-2i(1-\beta)}\exp(Bn^\beta) \leq n^{-i(1-2\beta)}\exp(Bn^\beta) \leq n^{-m(1-2\beta)}\exp(Bn^\beta)\ ,$$

and this completes the proof.

Remark 4. One should notice that in condition (I) of Theorem 5 we bound $\widetilde{G}_n(z)$ only for $|z| = n$, not for $|z-n| \leq Dn^\delta$. Because of this we need additional assumption (L). This is the strongest depoissonization theorem that we could prove. But, as pointed to us by S. Janson, if we replace our condition (I) by a weaker one, namely:

(I') *For* $z \in \mathcal{C}(D,\delta)$, *such that* $|z-n| \leq Dn^\delta$: $|\widetilde{G}_n(z)| \leq AF(n)$,

then, condition (L) follows directly form the above as an easy exercise. The following example shows that there are functions for which Theorem 5 works while a weaker one with (I) replaced by (I') does not: Let $g_{n,k} = 1 + n!/(k+\log k)^n$, so that $\widetilde{G}_k(z) = 1 + e^{-z}(1 - \frac{z}{k+\log k})^{-1}$ (observe that $\widetilde{G}_k(z)$ is meromorphic and it has a pole at $z = k\log k$). If we take a linear cone $\mathcal{C}_\theta$ with $|\theta| < \frac{\pi}{2}$, clearly for $|z| = n$ and $z \in \mathcal{C}_\theta$, we have $\widetilde{G}_k(z) = O(1)$ and for $z \notin \mathcal{C}_\theta$ we find $|e^z\widetilde{G}_k(z)| = O(e^{n\cos\theta})$. Condition (L) holds since for every $m \geq 1$ and for $|z| = n$ such that $z \in \mathcal{C}_\theta$:

$$\widetilde{G}_n^{<m>}(z) = O(e^{-n\cos\theta}(n+\log n)\log^{-m-1} n)$$

Hence $\widetilde{G}_n(ne^{it/\sqrt{n}}) \in \mathbb{D}_m(\gamma)$ for any γ real. Thus, we can apply Theorem 5, and indeed $g_{n,n} = \widetilde{G}(n) + O(1/n) = 1 + O(1/n)$. On the other hand, observe that the above condition (I') does not hold for our choice of $\widetilde{G}_k(z)$. □

5 Limiting Distribution

Hereafter, we use the depoissonization idea to establish limiting distributions for some dependent random variables. Let X_n be a sequence of integer random variable and X_N its corresponding Poisson driven sequence, where N is a Poisson random variable with mean z. Let $\widetilde{G}(z,u) = \mathbf{E}u^{X_N} = \sum_{n=0}^{\infty} \mathbf{E}u^{X_n} \frac{z^n}{n!} e^{-z}$ be its Poisson transform. We introduce also the Poisson mean $\widetilde{X}(z)$ and the Poisson variance $\widetilde{V}(z)$ as

$$\begin{aligned} \widetilde{X}(z) &= \widetilde{G}'_u(z,1) \ , \\ \widetilde{V}(z) &= \widetilde{G}''_u(z,1) + \widetilde{X}(z) - \left(\widetilde{X}(z)\right)^2 \ , \end{aligned}$$

where $\widetilde{G}'_u(z,1)$ and $\widetilde{G}''_u(z,1)$ denote respectively the first and the second derivative of $\widetilde{G}(z,u)$ with respect to u at $u=1$.

Remark 5. For historical reasons, in the analysis of algorithms and combinatorics we restrict X_N to (non-negative) integer values. But, of course, X_N can be extended to non-integer real values, and then $\widetilde{G}(z,t) = \mathbf{E}e^{tX_N}$ and one would redefine $\widetilde{X}(z) = \widetilde{G}'_t(z,0)$ and $\widetilde{V}(z) = \widetilde{G}''_t(z,0) - (X(z))^2$. In the sequel, we assume that X_N is integer valued for simplicity of the presentation. Furthermore, multidimensional random variables can be handled in a similar fashion, but due to the length of this paper we shall not pursue this topic in the current paper. □

5.1 Relationships Between Poisson and Bernoulli Mean and Variance

Before we proceed, we must understand a relationship between the Poisson mean $\widetilde{X}(z)$ and variance $\widetilde{V}(z)$ of X_N, and the Bernoulli mean $\mathbf{E}X_n$ and variance $\mathbf{Var}\ X_n$. The next result describes such a relationship. It is actually a corollary to Theorem 2. We restrict our considerations only to linear cones, but extension to polynomial cones is easy and left to the interested reader.

Theorem 6 *Let $\widetilde{X}(z)$ and $\widetilde{V}(z) + \widetilde{X}^2(z)$ satisfy condition (O), and $\widetilde{X}(z)$ and $\widetilde{V}(z)$ satisfy condition (I) of Theorem 2 with $\beta \le 1$, e.g., $\widetilde{X}(z) = O(z^\beta \Psi(z))$, and $\widetilde{V}(z) = O(z^\beta \Psi(z))$ in a linear cone $\mathcal{S}_\theta$ and appropriate conditions (O) outside the cone, where $\Psi(z)$ is a slowly varying function. Then, the following holds*

$$\begin{aligned} \mathbf{E}X_n &= \widetilde{X}(n) - \frac{1}{2} n \widetilde{X}^{\langle 2 \rangle}(n) + O(n^{\beta-2}\Psi(n)) & (55) \\ \mathbf{Var}\ X_n &= \widetilde{V}(n) - n[\widetilde{X}'(n)]^2 + O(\max\{n^{\beta-1}\Psi(n); n^{2\beta-2}\Psi^2(n)\}) & (56) \end{aligned}$$

for large n.

Proof. The asymptotic expansion (55) follows directly from Theorem 2 for $m=1$. To derive (56) observe that the Poisson transform of $\mathbf{E}X_n^2$ is $\widetilde{V}(z) + (\widetilde{X}(z))^2$, thus again by

Theorem 2

$$\begin{aligned}
\mathbf{E}X_n^2 &= \widetilde{V}(n) + (\widetilde{X}(n))^2 - \frac{1}{2}n\widetilde{V}^{\langle 2\rangle}(n) - n[\widetilde{X}'(n)]^2 - n\widetilde{X}^{\langle 2\rangle}(n)\widetilde{X}(n) + O(n^{\beta-2}\Psi(n)) \\
&= \widetilde{V}(n) + \Big((\widetilde{X}(n))^2 - n\widetilde{X}^{\langle 2\rangle}(n)\widetilde{X}(n) + O(n^{2\beta-2}\Psi(n))\Big) - n[\widetilde{X}'(n)]^2 + O(n^{\beta-1}\Psi(n))
\end{aligned}$$

where the last error term is a consequence of $n\widetilde{V}^{\langle 2\rangle}(n) = O(n^{\beta-1}\Psi(n))$. Since $\mathbf{Var}\ X_n = \mathbf{E}X_n^2 - \mathbf{E}^2X_n$, the result follows. ■

We illustrate the above theorem on a simple example.

Example 4. *I.I.D. Random Variables*

Let $Z_1, \ldots, Z_n$ be a sequence of independently identically distributed random variables with generating function $P(u) = \mathbf{E}u^{Z_1}$ and mean μ and variance v. The generating function of $X_n = Z_1 + \ldots + Z_n$ is $G_n(u) = (P(u))^n$. Observe that $\mathbf{E}X_n = n\mu$ and $\mathbf{Var}\ X_n = nv$. If we consider the Poisson transform of X_n we obtain $\widetilde{G}(z,u) = \exp(z(P(u)-1))$, and $\widetilde{X}(z) = z\mu$, $\widetilde{V}(z) = (\mu^2+v)z$. Thus, $\mathbf{Var}\ X_n = \widetilde{V}(n) - n[\widetilde{X}'(n)]^2$, as predicted by Theorem 6. □

5.2 Limiting Distribution Results with Polynomial Bounds

When dealing with distributions, one must estimate $G_n(u)$ from the Poisson transform $\widetilde{G}(z,u)$. If u belongs to a compact set, then our previous depoissonization results can be directly applied. For example, we have the following two corollaries that are quite useful in some analysis (cf. [14, 33, 47], and Sections 6.5–6.7). We again only consider linear cones.

Corollary 1 *Suppose $\widetilde{G}_k(z) = \sum_{n=0}^{\infty} g_{n,k}\frac{z^n}{n!}e^{-z}$, for k belonging to some set K, are entire functions of z. Let for some constants A, B, $R > 0$, β and $\alpha < 1$ the following two conditions holds uniformly in $k \in K$:*

(I) *For $z \in \mathcal{S}_\theta$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}_k(z)| \le B|z|^\beta \ , \tag{57}$$

(O) *For $z \notin \mathcal{S}_\theta$*

$$|z| > R \quad \Rightarrow \quad |\widetilde{G}_k(z)e^z| \le A\exp(\alpha|z|) \ . \tag{58}$$

Then, uniformly in $k \in K$

$$g_{n,k} = \widetilde{G}_k(n) + O(n^{\beta-1}) \tag{59}$$

and the error estimate does not depend on K.

Proof. In fact, the above corollary is a direct consequence of our previous proofs. Nevertheless, we show below how it can be concluded from Theorem 5. Indeed, let us assume contrary that the thesis of the corollary, that is (59), does not hold. In other words, there is a subsequence (n_i, k_i) such that

$$\lim_{i\to\infty}\left|\frac{g_{n_i,k_i}-\widetilde{G}_{k_i}(n_i)}{n_i^{\beta-1}}\right| = \infty \ . \tag{60}$$

Observe that n_i cannot be bounded. Indeed, if the subsequence n_i would be bounded, then in this case the uniform boundness of $G_k(z)$ (by our assumption (I)) in any compact set yields that the $g_{n,k}$ are uniformly bounded if n_i are uniformly bounded (it suffice to bound the integrand in $g_{n,k} = \frac{n!}{2i\pi}\oint G_k(z)e^z z^{-n-1}dz$), which contradicts (60). So assume now that n_i is unbounded and strictly increasing, and define for a nonnegative integer m

$$\widetilde{H}_m(z) = \begin{cases} 0 & m \neq n_i \\ \widetilde{G}_{k_i}(z) & m = n_i \ . \end{cases}$$

Then, $h_{n_i,n_i} = g_{n_i,k_i}$ for all i. Clearly, $\widetilde{H}_m(z)$ satisfies assumptions of Theorem 5 (since it satisfies condition (I) and (O) of the corollary which – as we already pointed out in our Remark 4 – imply conditions (I), (O) and (L) of Theorem 5). Thus,

$$h_{n_i,n_i} = g_{n_i,k_i} = \widetilde{G}_{k_i}(n_i) + O(n_i^{\beta-1})$$

which is the desired contradiction. ■

The next corollary is a direct consequence of Corollary 1 and it already found several applications in the analysis of algorithms (cf. Sections 6.5–6.7).

Corollary 2 *Let $\widetilde{G}(z,u)$ satisfy hypothesis of Theorem 1, i.e. for some numbers $\theta < \pi/2$, A, B, $\xi > 0$, β, and $\alpha < 1$ (I) and (O) hold for all u in a set $\mathcal{U}$. Then*

$$G_n(u) = \widetilde{G}(n,u) + O(n^{\beta-1})$$

uniformly in $u \in \mathcal{U}$.

Proof. Follows directly from Corollary 1 with $\widetilde{G}_u(z) = \widetilde{G}(z,u)$ and $K = \mathcal{U}$.

While the above corollaries are quite useful in deriving limiting distributions (cf. some examples discussed in the next section), there are cases when condition (I) is not applicable. In fact, for many problems encountered in the analysis of algorithms $\log \widetilde{G}(z,u)$ increases in a polynomial manner (cf. [25, 26, 27, 31], and Sections 6.8–6.9), thus $\widetilde{G}(z,u)$ grows exponentially fast , and none of the theorems of previous sections can handle such a case. We need, therefore, another depoissonization result to handle these cases. This is discussed below.

5.3 Central Limit Theorem for the Poisson Model

First of all, we present below a central limit theorem for the Poisson model. It is later extended to the Bernoulli model in Theorem 9 which is our strongest result.

Theorem 7 *Let X_N be a characteristic of the Poisson model with $\widetilde{G}(z,u) = \mathbf{E}u^{X_N}$, mean $\widetilde{X}(z)$ and variance $\widetilde{V}(z)$. Let the following hold for $z \to \infty$ in a polynomial cone $\mathcal{C}(D,\delta)$ with $\delta > 1/2$ and for u belonging to a neighbourhood $\mathcal{U}$ of $u=1$ of the complex plane*

$$\log \widetilde{G}(z,u) = O(A(z)) \ , \tag{61}$$

where $A(z)$ is such that

$$\lim_{z\to\infty} \frac{A(z)}{\widetilde{V}^{3/2}(z)} = 0 \tag{62}$$

and further $\lim_{z\to\infty} \widetilde{V}(z) = \infty$. Then for complex τ

$$\widetilde{G}\left(z, e^{\tau/\sqrt{\widetilde{V}(z)}}\right) e^{-\tau \widetilde{X}(z)/\sqrt{\widetilde{V}(z)}} = e^{\frac{\tau^2}{2}} \left(1 + O\left(\frac{A(z)}{\widetilde{V}^{3/2}(z)}\right)\right) \ , \tag{63}$$

that is, $\bar{X}_N = (X_N - \widetilde{X}(z))/\sqrt{\widetilde{V}(z)}$ converges in distribution and in moments to the standard normal distribution.

Proof. Developing $\log \widetilde{G}(z,u)$ into Taylor's expansion around $u=1$, we obtain

$$\log \widetilde{G}(z,u) = (u-1)\frac{\partial}{\partial u}\log \widetilde{G}(z,1) + \frac{(u-1)^2}{2}\frac{\partial^2}{\partial u^2}\log \widetilde{G}(z,1) + R(z,u)$$

where $R(z,u)$ is the remainder in the Taylor's expansion. Observe that

$$\begin{aligned} \frac{\partial}{\partial u}\log \widetilde{G}(z,1) &= \widetilde{X}(z) \ , \\ \frac{\partial^2}{\partial u^2}\log \widetilde{G}(z,1) &= \widetilde{V}(z) - \widetilde{X}(z) \ . \end{aligned}$$

The remainder can be expressed as (cf. [49])

$$R(z,u) = \frac{(u-1)^3}{2\pi i}\oint \frac{\log \widetilde{G}(z,v)}{(v-u)(v-1)^3} dv \ .$$

Now, we apply (61) to the above to find

$$R(z,u) = O((u-1)^3 A(z))$$

which, after substitution $u = e^{\tau/\sigma(z)}$, finally leads to

$$\widetilde{G}\left(z, e^{\tau/\sigma(z)}\right) e^{-\tau\widetilde{X}(z)/\sigma(z)} = e^{\tau^2/2}\left(1 + O\left(\tau^3 A(z)/\sigma^3(z)\right)\right)$$

where $\sigma(z) = \sqrt{\widetilde{V}(z)}$. This completes the proof. ■

5.4 Limiting Distribution Results for the Bernoulli Model

Depoissonization of the case discussed in Theorem 7 does not follow yet from any of the results presented in Section 4. This is due to the fact that in Theorem 7 we normalized $u = e^{t/\widetilde{V}(z)}$. We need another tool that resembles our idea of diagonalization of Theorem 5. Thus, we present one more depoissonization result that is used for establishing limiting distributions when some kind of normalization is used.

Theorem 8 DIAGONAL EXPONENTIAL DEPOISSONIZATION TOOL

Let $\widetilde{G}_k(z)$ be a sequence of Poisson transforms of $g_{n,k}$, and each $\widetilde{G}_k(z)$ is assumed to be an entire function of z. Let $\log \widetilde{G}_k(z)$ exist in a a polynomial cone $\mathcal{C}(D,\delta)$ with $1/2 < \delta \leq 1$. We suppose that there exists $\beta \in [\delta-1/2, 4/3(\delta-1/2)[$ such that the following two conditions hold:

(I) for all $z \in \mathcal{C}(D,\delta)$ such that for $|z| \in [n - Dn^\delta, n + Dn^\delta]$

$$|\log \widetilde{G}_n(z)| \leq Bn^\beta \tag{64}$$

for some constant $B > 0$.

(O) for all $z \notin \mathcal{C}(D,\delta)$ such that $|z| = n$:

$$|\widetilde{G}_n(z)e^z| \leq \exp(n - An^\alpha) \tag{65}$$

for some $\alpha > \beta$. Then, for all $\varepsilon > 0$:

$$g_{n,n} = \widetilde{G}_n(n)\exp[-\frac{n}{2}(L_n'(n))^2]\left(1 + O(n^{3\beta-4(\delta-1/2)+\varepsilon})\right) \tag{66}$$

where $L_n(z) = \log \widetilde{G}_n(z)$ and $L_n'(z) = \widetilde{G}_n'(z)/\widetilde{G}_n(z)$ is the first derivative of $L_n(z)$.

Proof: We use Lemma 7 with $F_n(t) = \log(h_n(t)) + t^2/2 + L_n(ne^{it/\sqrt{n}}) - L_n(n)$. Under condition (I), the sequence of functions $L_n(ne^{it/\sqrt{n}})n^{-\beta}$ defined on $\{t : \Re(t) \,\&\, \Im(t) \in [-D'n^{\delta-1/2}, D'n^{\delta-1/2}]\}$ belong to $\mathbb{D}_m(\delta - \frac{1}{2})$ (more precisely: it belongs to the natural extension of $\mathbb{D}_m(\delta - 1/2)$ to functions of a complex variable) for any integer $m \geq 0$. Let $\gamma < \delta - \frac{1}{2}$. As before, using Cauchy's formula, we obtain

$$\begin{aligned} g_{n,n} &= \frac{\widetilde{G}_n(n)\omega_n}{\sqrt{2\pi}}\int_{-Dn^\gamma}^{Dn^\gamma}\exp[F_n(x)]e^{-x^2/2}dx + \\ &+ \frac{\omega_n}{\sqrt{2\pi}}\int_{|x|\in[Dn^\gamma, Dn^{\delta-\frac{1}{2}}]}\widetilde{G}_n(ne^{ix/\sqrt{n}})h_n(x)dx + \\ &+ O(n^{1/2}\exp(-An^\alpha)) \ , \end{aligned}$$

with $\omega_n = n!n^{-n}e^n(2\pi n)^{-1} = 1 + O(1/n)$. Our aim is to apply Lemma 7 to the first integral above. Denoting $F_n'(x)$ and $F_n''(x)$ as the first and second derivative of $F_n(x)$, we must estimate $F_n(0)$, $F_n'(0)$ and $F_n''(x)$ for $\{\Re(x) \ \& \ \Im(x) \in [-Dn^{\delta-1/2}, Dn^{\delta-\frac{1}{2}}]\}$ in order to identify the constant β_1 and β_2. Observe that by the definition of $F_n(t)$, we have $F_n(0) = 0$ and $F_n'(0) = i\sqrt{n}L_n'(n)$. But, by the same argument as in the proof of Lemma 3, we deduce that $L'(n) = O(n^{\beta-\delta})$, and hence $F_n'(0) = O(n^{\beta_1})$ with $\beta_1 = \beta - \delta + 1/2$.

In order to estimate the second derivative $F_n''(x)$ of $F_n(x)$, we observe that it is the sum of the second derivative of $\log\left(h_n(x)e^{x^2/2}\right)$ and second derivative of $L_n(ne^{ix/\sqrt{n}})$. The second derivative of $\log\left(h_n(x)e^{x^2/2}\right)$ is exactly $1 - e^{ix/\sqrt{n}}$ which is $O(n^{\gamma-1/2})$ for $x = O(n^\gamma)$. Since the second derivative of $L_n(ne^{ix/\sqrt{n}})$ is $O(n^{\beta-2\delta+1})$, we identify $\beta_2 = \max\{\gamma-1/2, \beta-2\delta+1\}$. Let us now fix $\gamma = \beta_1 + \varepsilon$ with $0 < \varepsilon < 4(\delta - 1/2) - 3\beta$. We still have $\gamma \leq \delta - 1/2$ since $\gamma < \beta_1 + 4(\delta - 1/2) - 3\beta = 3(\delta - 1/2) - 2\beta \leq \delta - 1/2$.

To apply Lemma 7, we must check that the conditions on γ, β_1 and β_2 required in Lemma 7 are actually satisfied. In particular:

- $\beta_1 \geq 0$: by hypothesis $\beta \geq \delta - 1/2$;
- $\gamma > \beta_1$: by hypothesis about ε;
- Moreover we claim that $2\beta_1 + \beta_2 \leq 3\beta - 4(\delta - 1/2) + \varepsilon$ (we already know that $3\beta - 4(\delta - 1/2) + \varepsilon < 0$): we must check both case $\beta_2 = \beta_1 + \varepsilon - 1/2$ and $\beta_2 = \beta - 2(\delta - 1/2)$:
    1. case $\beta_2 = \beta_1 + \varepsilon - 1/2$: $2\beta_1 + \beta_2$ becomes $3\beta - 3(\delta - 1/2) - 1/2 + \varepsilon$ which is equal to $3\beta - 4(\delta - 1/2) + \varepsilon + \delta - 1$, and since $\delta \leq 1$ we obtained the desired result;
    2. case $\beta_2 = \beta - 2(\delta - 1/2)$: $2\beta_1 + \beta_2$ becomes $3\beta - 4(\delta - 1/2)$, as needed.

In summary: all conditions of Lemma 7 hold and we can apply Lemma 7 to yield:

$$\frac{1}{\sqrt{2\pi}} \int_{-Dn^\gamma}^{Dn^\gamma} \exp[F_n(x)]e^{-x^2/2}dx = \exp[\frac{1}{2}(F_n'(0))^2](1 + O(n^{2\beta_1+\beta_2})).$$

By identifying $\frac{1}{2}(F_n'(0))^2$ with $-\frac{n}{2}(L_n'(n))^2$, we prove that

$$\begin{aligned} g_{n,n} &= \omega_n \widetilde{G}_n(n) \exp[-\frac{n}{2}(L_n'(n))^2](1 + O(n^{2\beta_1+\beta_2}) + \\ &+ \frac{\omega_n \widetilde{G}_n(n)}{\sqrt{2\pi}} \int_{|x| \in [Dn^\gamma, Dn^{\delta-\frac{1}{2}}]} \exp[L_n(ne^{ix/\sqrt{n}}) - L_n(n)]h_n(x)dx + O(\sqrt{n}\exp(-An^\alpha)) \end{aligned}$$

The last step consists of observing that the above integral together with the error term decay faster than any polynomial in n even when multiplied by $\widetilde{G}(n)\exp[-\frac{n}{2}(L_n'(n))^2]$. Indeed:

the logarithm of $\widetilde{G}_n(n)\exp[-\frac{n}{2}(L'_n(n))^2]$ is of order n^β (the order of $-\frac{n}{2}(L'_n(n))^2$ is $n^{2\beta-2\delta+1}$ which is smaller). The order of the last integral above is negligible since $|L_n(ne^{ix/\sqrt{n}})-L_n(n)|=O(n^{\beta_1}x)$ and this leads to

$$\left|\exp\left(L_n(ne^{ix/\sqrt{n}})-L_n(n)\right)h_n(x)\right|\leq\exp(O(n^{\beta_1})x-\mu x^2)\ ,$$

and therefore the integral multiplied by $\exp[\frac{n}{2}(L'_n(n))^2]$ is of order $n^{\delta-\frac{1}{2}}\exp[O(n^{2\beta_1}+Dn^\gamma(O(n^{\beta_1})-\mu Dn^\gamma)]$ which decays faster than any polynomial since $\beta_1<\gamma$. Also, $O(\sqrt{n}\exp(-An^\alpha)(\widetilde{G}_n(n))^{-1}\exp[n/2(L'_n(n))^2]$ decreases faster than any polynomial in n since $\alpha>\beta$, and can also be neglected. Finally, the error term $O(1/n)$ introduced by the Stirling approximation (i.e., $\omega_n=1+O(1/n)$) is swallowed by the order $O(n^{3\beta-4(\delta-1/2)+\varepsilon})$ which is larger than $O(n^{-1+\varepsilon})$. This completes the proof. ■

With the help of the above depoissonization lemma we can now invert Theorem 7 to derive the normal limiting distribution for the Bernoulli model, that is, for X_n and its generating function $G_n(u)=\mathbf{E}u^{X_n}$. We recall that the mean $\mathbf{E}X_n$ and the variance $\mathbf{Var}\ X_n$ of the Bernoulli model can be obtained from the mean $\widetilde{X}(z)$ and the variance $\widetilde{V}(z)$ of the Poisson model as discussed in Theorem 6.

Let us consider the Poisson generating function $\widetilde{G}(z,u)$ such that inside a polynomial cone $\mathcal{C}(D,\delta)$ the logarithm of $\widetilde{G}(z,u)$ exists, and we denote $L(z,t)=\log\widetilde{G}(z,e^t)$. We have the standard expansion:

$$L(z,t)=\widetilde{X}(z)t+\widetilde{V}(z)\frac{t^2}{2}+R(z,t)t^3 \tag{67}$$

where $R(z,t)$ the remainder in Taylor's expansion of $L(z,t)$. We denote $V_n=\widetilde{V}(n)-n\left(\widetilde{X}'(n)\right)^2$ which by Theorem 6 is asymptotically equal to the variance in the Bernoulli model.

Theorem 9 *Let χ and ν such that $3\nu/2>\chi\geq\nu>0$ and $\beta=\chi-\nu/2\in[\delta-1/2,4/3(\delta-1/2)[$ with $\frac{1}{2}<\delta\leq 1$. Suppose that there exist $D>0$, $B>0$ and $F>0$ such that for all real $t\in[-F,F]$ the following three conditions hold:*

(I) $z\in\mathcal{C}(D,\delta)$:

$$\begin{aligned}|L(z,t)| &\leq B|z|^\chi\ , &(68)\\ |R(z,t)| &\leq B|z|^\chi &(69)\end{aligned}$$

(O) For all $z \notin \mathcal{C}(D,\delta)$ such that $|z| = n$:

$$|\widetilde{G}_n(z)e^z| \le \exp(n - An^\alpha) \tag{70}$$

for some $\alpha > \chi - \nu/2$.

(V) For some $B' > 0$

$$V_n > B'n^\nu \ .$$

Then, for all real t and any $\varepsilon > 0$:

$$G_n(e^{t/\sqrt{V_n}}) \exp\left(-\widetilde{X}(n)\frac{t}{\sqrt{V_n}}\right) = e^{\frac{1}{2}t^2}\left(1 + O(tn^{\gamma+\varepsilon})\right)$$

with $\gamma = \max\{\chi - 3\nu/2, 3\chi - 3\nu/2 - 4(\delta - 1/2)\}$. In words: $Y_n = (X_n - \widetilde{X}(n))/\sqrt{V_n}$ converges in distribution and in moments to the standard normal distribution.

Proof. Since $L(z,t)$ and $R(z,t)$ are both $O(z^\chi)$, it is clear that for $z \in \mathcal{C}(D,\delta)$ the mean $\widetilde{X}(z)$ and the variance $\widetilde{V}(z)$ are also $O(z^\chi)$. Consequently, the derivative with respect to z of these quantities are of order $O(z^{\chi-\delta})$ for z in a smaller cone $\mathcal{C}(D',\delta)$ with $D' < D$. Let us now fix t. We write $\widetilde{G}_k(z) = \widetilde{G}(z, e^{t/\sqrt{V_k}})$ and $L_k(z) = L(z, t/\sqrt{V_k})$. Notice that $t/\sqrt{V_k} \in [-F, F]$ when $k \to \infty$ since in the sequel we suppose that k is large enough so that the above holds. When $z = O(n)$ in the cone we have $L_n(z) = O(tn^{\chi-\nu/2})$. Therefore, conditions of Theorem 8 are fulfilled: with $\beta = \chi - \nu/2$. We have $3\beta - 4(\delta - 1/2) < 0$ and for all $\varepsilon > 0$:

$$G_n(e^{t/\sqrt{V_n}}) = \widetilde{G}_n(n) \exp\left(-\frac{n}{2}(L_n'(n))^2\right) \cdot \left(1 + O(tn^{3\beta-4(\delta-1/2)+\varepsilon})\right)$$

In order to complete the proof, we use the expansion

$$\begin{aligned} \log \widetilde{G}_n(n) &= L(n, t/\sqrt{V_n}) \\ &= \widetilde{X}(n)\frac{t}{\sqrt{V_n}} + \widetilde{V}(n)\frac{t^2}{2V_n} + t^3\frac{R(n,t)}{V_n^{3/2}} \end{aligned}$$

According to our estimate on $R(n,t)$ and V_n the last term is $n^{\chi-3\nu/2}$ which tends to zero. We also have the estimate

$$L_n'(n) = \widetilde{X}'(n)\frac{t}{\sqrt{V_n}} + t^2 O(n^{\chi-\delta-\nu})$$

which leads to

$$n(L_n'(n))^2 = n(\widetilde{X}'(n))^2\frac{t^2}{V_n} + O(n^{2\chi-2\delta-3\nu/2}) \ .$$

Since $2\chi - 2\delta - 3\nu/2 < 0$ the error term converges to zero. In other words, we obtain

$$G_n(e^{t/\sqrt{V_n}}) = \exp\left[\widetilde{X}(n)\frac{t}{\sqrt{V_n}} + \frac{\widetilde{V}(n) - n(\widetilde{X}'(n))^2}{V_n}\frac{t^2}{2} + O(tn^{3\beta-4(\delta-1/2)+\varepsilon})\right]$$

with $3\beta - 4(\delta - 1/2) < 0$.

Thus, we proved that for a real $t \in (-F, F)$ the function $g_n(t) = \mathbf{E}e^{tY_n}$ where $Y_n = (X_n - \widetilde{X}(n))/\sqrt{V_n}$ converges to $e^{\frac{1}{2}t^2}$. Convergence of Y_n in moments follows directly from Lemma 8. This completes the proof. ■

6 Applications of Depoissonization Results

In this section, we apply our depoissonization theorems to various problems arising in combinatorics and analysis of algorithms and data structures. We start with a general result (cf. Theorem 10) that provides a simple tool to depoissonize a class of linear functional equations often appearing in the analysis of digital data structures and algorithms (cf. Section 6.1). Then, we deal with problems where depoissonization of moments is required (cf. Sections 6.2–6.4), and finally we present some depoissonization results for limiting distributions (cf. Sections 6.5–6.9).

6.1 Depoissonization of a Linear Functional Equation

Several problems discussed in Section 3 can be reduced to the following general linear functional equation

$$\widetilde{G}(z,u) = \beta a(z/2,u)\widetilde{G}(z/2,u) + b(z,u) \ , \tag{71}$$

where $|\beta| \leq 1$, and u is either fixed (cf. (5), (19), (20), etc.) or u belongs to a compact neighborhood $\mathcal{U}(u_0)$ of u_0 (cf. (7), (11), (18), (22), etc.). Iterating (71), we obtain a general solution of the above equation (cf. [28, 36, 54])

$$\widetilde{G}(z,u) = \sum_{n=0}^{\infty} \beta^n b(z2^{-n},u) \prod_{k=1}^{n} a(z2^{-k},u) \tag{72}$$

provided $\widetilde{G}(0,u) = 0$, and all the series above converge. Define

$$\varphi(z,u) = \prod_{j=0}^{\infty} a(z2^j,u) \ , \tag{73}$$

if the infinite product in (73) converges. Then, the general solution (72) can be rewritten as

$$\widetilde{G}(z,u)\varphi(z,u) = \sum_{n=0}^{\infty} \beta^n b(z2^{-n},u)\varphi(z2^{-n},u) \ . \tag{74}$$

Solution (74) is often used to derive asymptotics of the functional equation (72), and in particular we used it to solve some problems discussed in Section 3 (e.g., (5) and (19)-(20)) as already presented in [19, 28, 36]. This is possible since the sum in (74) falls under the so called *harmonic sum* that can be handled by the Mellin transform technique (cf. [16]). In general, this allows to derive an asymptotic solution to the above equation for $z \to \infty$, and then "depoissonize" to recover the original sequence.

To apply our depoissonization findings of the previous sections, one must establish conditions (I) and (O), that is, some upper bounds for the Poisson transform inside and outside a cone, respectively. A technique we shall use to demonstrate such bounds for equation (72) was proposed for the first time in [25, 27], and in a more formal way in [31, 47]. It is based on the mathematical induction over the so called *increasing domains* that we define next. In fact, we can treat a more general functional equation, namely (for simplicity we drop the variable u)

$$\widetilde{G}(z) = \gamma_1(z)\widetilde{G}(zp) + \gamma_2(z)\widetilde{G}(zq) + t(z) \ , \tag{75}$$

where $\gamma_1(z)$, $\gamma_2(z)$ and $t(z)$ are functions of z such that the above equation has a solution. The reader is referred to [12, 51] for conditions on these functions under which a solution of (75) exists. We further assume that $p + q = 1$ (cf. (4)), but in fact zp and zq could be replaced by more general functions that form a semigroup of substitutions under the operation of composition of functions (cf. [12, 51]). In passing, we observe that the above more general equation arises in non-blocking conflict resolution algorithms (cf. [12, 51]) and asymmetric leader election algorithm (cf. [33]).

Let us define for integers $m = 0, 1, \ldots$ and a constant λ such that $0 < \max\{p, q\} \leq \lambda^{-1} < 1$, a sequence of increasing domains (cf. Figure 4) $\mathcal{D}_m$ as

$$\mathcal{D}_m = \{z : \ \xi\delta \leq |z| \leq \xi\lambda^{m+1}\}$$

for some constant $\xi > 0$ and $\delta \leq \min\{p, q\}$. Observe that

$$z \in \mathcal{D}_{m+1} \qquad \Longrightarrow \qquad pz, \ qz \in \mathcal{D}_m \ . \tag{76}$$

The last property is crucial to apply mathematical induction over m in order to establish appropriate bounds on $\widetilde{G}(z)$ over a whole complex plane, so we can apply Theorem 2.

The next result presents a "depoissonization lemma" for the functional equation (75) in the case $p + q = 1$.

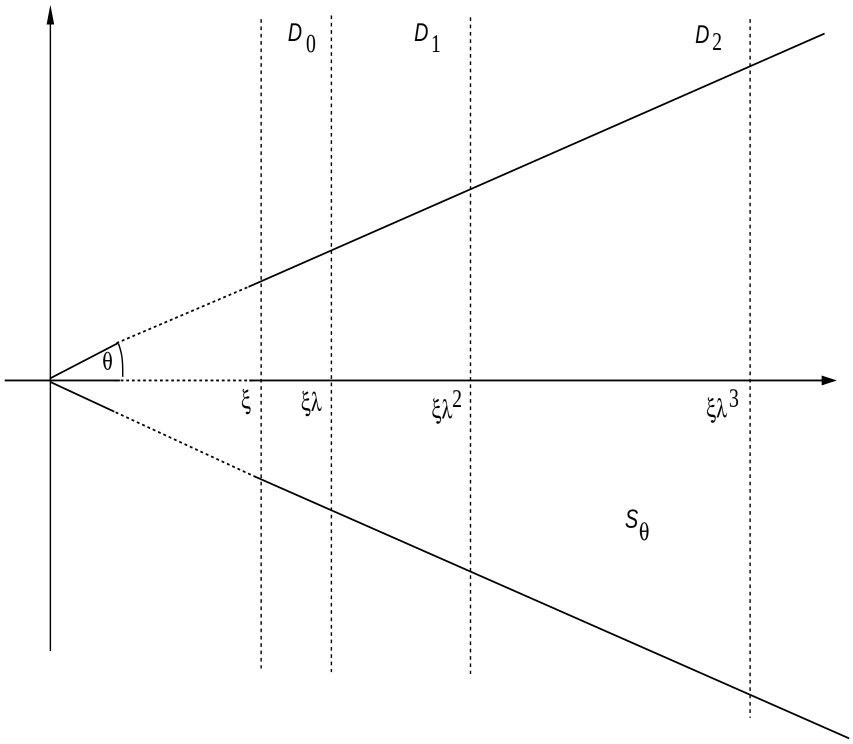

Figure 4: The linear cone $\mathcal{S}_\theta$ and the increasing domains $\mathcal{D}_m$.

Theorem 10 *Consider the functional equation (75) with $p+q=1$, that is,*

$$\widetilde{G}(z) = \gamma_1(z)\widetilde{G}(zp) + \gamma_2(z)\widetilde{G}(zq) + t(z)$$

which is postulated to have an entire solution. Let also for some positive A, β, $0<\theta<\pi/2$, ξ and $0<\eta<1$ the following conditions hold for $|z|>\xi$:

(I) For $z \in \mathcal{S}_\theta$

$$|\gamma_1(z)|p^\beta + |\gamma_2(z)|q^\beta \leq 1-\eta, \tag{77}$$

$$|t(z)| \leq A\eta|z|^\beta \; ; \tag{78}$$

(O) For $z \notin \mathcal{S}_\theta$ we request that for some $\alpha < 1$ the following three inequalities are true

$$|\gamma_1(z)|e^{q\Re(z)} \leq \frac{1}{3}e^{\alpha|z|q} \; , \tag{79}$$

$$|\gamma_2(z)|e^{p\Re(z)} \leq \frac{1}{3}e^{\alpha|z|p} \; , \tag{80}$$

$$|t(z)|e^{\Re(z)} \leq \frac{1}{3}e^{\alpha|z|} \; . \tag{81}$$

Then,

$$g_n = \widetilde{G}(n) + O(n^{\beta-1}) \tag{82}$$

or more generally, for all nonnegative integer m

$$g_n = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \widetilde{G}^{\langle j \rangle}(n) + O(n^{\beta-m-1}) \tag{83}$$

where $g_n = [z^n]\left(n! e^z \widetilde{G}(z)\right)$.

Proof. We apply Theorem 2 (with $\delta = 1$). For this, we need to establish the following bounds

$$|\widetilde{G}(z)| \leq B|z|^\beta \tag{84}$$

for $z \in \mathcal{S}_\theta$, and

$$|\widetilde{G}(z)e^z| \leq Be^{\alpha|z|} \tag{85}$$

for $z \notin \mathcal{S}_\theta$ with $\alpha < 1$. It suffices to prove that under (77)-(78) and (79)- (81) the above two conditions take place.

The proof is by induction over the increasing domains $\mathcal{D}_m$. Let us first consider $z \in \mathcal{S}_\theta$. Define $\hat{\mathcal{D}}_m = \mathcal{D}_m \cap \mathcal{S}_\theta$. We take B large enough and greater than A such that for $m = 0$ the above inequalities hold for $|z| \leq \xi$. We also take $\alpha \geq \cos\theta$. Let us assume now that inequality (84) is satisfied in $\hat{\mathcal{D}}_m$, and we prove that they also hold in a larger region, namely $\hat{\mathcal{D}}_{m+1}$, thus proving (84) in $\mathcal{S}_\theta$. Assume now that $z \in \hat{\mathcal{D}}_{m+1} - \hat{\mathcal{D}}_m$. But by property (76) we have $zp, zq \in \hat{\mathcal{D}}_m$, and we can invoke our induction hypothesis. Hence, taking into account (77)-(78) we conclude from equation (75) that

$$\begin{aligned}
|\widetilde{G}(z)| &\leq |\gamma_1(z)||\widetilde{G}(zp)| + |\gamma_2(z)||\widetilde{G}(zq)| + |t(z)| \ , \\
&\leq B|\gamma_1(z)||z|^\beta p^\beta + B|\gamma_2(z)||z|^\beta q^\beta + |t(z)| \ , \\
&\leq B(1-\eta)|z|^\beta + B\eta|z|^\beta = B|z|^\beta
\end{aligned}$$

which is the desired result.

Assume now that $z \notin \mathcal{S}_\theta$ and we aim at proving (85). We first observe that $|e^z| = e^{\Re(z)} \leq e^{\alpha|z|}$ where $\alpha \geq \cos\theta \geq \cos(\arg(z))$. Now, the induction over the increasing domains can be applied as before, however, this time we consider $\overline{\mathcal{D}}_m = \mathcal{D}_m \cap \overline{\mathcal{S}}_\theta$ where $\overline{\mathcal{S}}_\theta$ is the complementary set to $\mathcal{S}_\theta$. Observe that

$$\begin{aligned}
|\widetilde{G}(z)e^z| &\leq |\gamma_1(z)||\widetilde{G}(zp)e^{zp}||e^{zq}| + |\gamma_2(z)||\widetilde{G}(zq)e^{zq}||e^{zp}| + |t(z)e^z| \ , \\
&\leq B|\gamma_1(z)||\widetilde{G}(zp)|e^{\alpha|z|p}e^{q\Re(z)} + B|\gamma_2(z)||\widetilde{G}(zq)|e^{\alpha|z|q}e^{p\Re(z)} + |t(z)|e^{\Re(z)} \ , \\
&\leq Be^{\alpha|z|} \ ,
\end{aligned}$$

where the last inequality follows from (79)-(81) and $\Re(z) = |z|\cos(\arg(z)) \leq |z|\cos\theta \leq \alpha|z|$. This completes the proof. ■

Below, we discuss some examples that illustrate Theorem 10. The first set of examples deals with moments (cf. Sections 6.2–6.4). Next, we turn our attention to limiting distributions (cf. Sections 6.5–6.9).

6.2 Conflict Resolution algorithm

In Section 3.1 we described several conflict resolution algorithms. While certain functional equations arising in these applications can be handled without Poisson transforms, equation (5) and others require poissonization. Let us, as an example, consider (5) which is repeated below

$$\widetilde{C}(z) = \left(1 + (1 + z/2)e^{-z/2}\right)\widetilde{C}(z/2) \ .$$

This equation is of the form of (71), thus has (72) as a solution. Jacquet and Szpankowski [28] proved that for large $z \in \mathcal{S}_\theta$ the above equation attains asymptotically the following solution $\widetilde{C}(z) = D + P(\log z) + O(1/z)$ where D is a constant computed in [28], and $P(\log z)$ is a fluctuating function with a small amplitude.

Can we depoissonize this solution? One must check conditions (I) and (O) of Theorem 1. But, $\widetilde{C}(z) = O(1)$ in a cone $\mathcal{S}_\theta$ by the results of [28] just cited, thus we need only to verify (O) outside the cone $\mathcal{S}_\theta$. By Theorem 10 (e.g., see (79)) we need to show the existence of ξ such that for $|z| > \xi$ the following holds

$$(1 + (1 + |z|/2))e^{\Re(z)/2} \leq e^{\alpha|z|/2}$$

which is clearly true for large enough ξ since $\alpha \geq \cos\theta$. Thus, by Theorem 10 we conclude that $C_n = D + P(\log n) + O(n^{-1})$ (which is stronger than in [28] where only $O(1/\sqrt{n})$ error term was established).

6.3 Leader Election Algorithm or Incomplete Tries

In Section 3.4 we discussed a leader election algorithm due to Prodinger [46] (cf. also [14]). In [14] it was shown that the first moment $\widetilde{X}(z)$ and the second *factorial* moment $\widetilde{W}(z)$ of the number of steps required to elect a leader in the Poisson model satisfy the following functional equations (cf. (17)-(19))

$$\begin{aligned}
\widetilde{X}(z) &= \widetilde{X}(z/2)(1 + e^{-z/2}) + 1 - e^{-z} - ze^{-z}, \\
\widetilde{W}(z) &= \widetilde{W}(z/2)(1 + e^{-z/2}) + 2\widetilde{X}(z/2)(1 + e^{-z/2}) \ .
\end{aligned}$$

These equations fall under (71), and therefore depoissonization can be obtained via Theorem 10. Indeed, in a cone $\mathcal{S}_\theta$ the authors of [14] proved that for large z and any $R > 0$

$$\begin{aligned}
\widetilde{X}(z) &= \log_2 z + \frac{1}{2} - \delta_1(\lg z) + O(|z|^{-R}), \\
\widetilde{W}(z) &= \log_2^2 z + \frac{\pi^2}{6\log^2 2} - \frac{1}{6} - \frac{2\gamma_1}{\log^2 2} - \frac{\gamma^2}{\log^2 2} + \tilde{\delta}_2(\log z) + O(|z|^{-R}),
\end{aligned}$$

with $\delta_1(\cdot)$ and $\delta_2(\cdot)$ being fluctuating functions with small amplitudes. Here the constants $(-1)^k\gamma_k/k!$, $k \geq 0$, are the so-called Stieltjes constants, with

$$\gamma_k := \lim_{m\to\infty}\left(\sum_{i=1}^{m}\frac{\ln^k i}{i} - \frac{\ln^{k+1} m}{k+1}\right);$$

in particular, $\gamma_0 = \gamma = 0.577215\ldots$ is Euler's constant and $\gamma_1 = -0.072815\ldots$. Since both Poisson transforms grow in a poly-log fashion, condition (I) of Theorem 1 is automatically satisfied. As a matter of fact, the above asymptotics were derived through Mellin transform, thus the above formally holds only for z real (since Mellin results were originally defined only for z real). In Appendix B we introduce a complex Mellin transform, and show that all results valid for real z can be easily translated into complex z, as needed for most of our depoissonization results. In passing, we should also mention that alternatively we can use analytical continuation to extend the above asymptotics to complex z. However, using complex Mellin approach is more powerful, as we shall see in Section 6.6.

To complete depoissonization of $\widetilde{X}(z)$ and $\widetilde{W}(z)$ we must verify condition (O) outside the cone $\mathcal{S}_\theta$. This can be done along the steps shown in the example above, but for completeness we deal here with $\widetilde{W}(z)$ provided we already proved that outside a cone $\mathcal{S}_\theta$ we have $|\widetilde{X}(z)e^z| \leq e^{\alpha|z|}$ for some $\alpha > \cos\theta$. By Theorem 10 it suffices to show that for sufficiently large ξ the following holds

$$2(1 + e^{-\Re(z)/2})e^{\Re(z)/2} \leq \frac{1}{2}e^{\alpha|z|/2} .$$

But, the above trivially holds since $\Re(z) = |z|\cos(\arg(z)) < \alpha|z|$ for some $\alpha > 0$. This completes the depoissonization and leads to:

Proposition 1 (Prodinger [46], Fill et al. [14]) *Define* $L := \ln 2$ *and* $\chi_k := 2\pi ik/L$. *Then:*

$$\begin{aligned}
\mathbf{E}[H_n] &= \log_2 n + \frac{1}{2} - \delta_1(\lg n) + O\left(\frac{1}{n}\right), \\
\mathbf{Var}[H_n] &= \frac{\pi^2}{6L^2} + \frac{1}{12} - \frac{2\gamma_1}{L^2} - \frac{\gamma^2}{L^2} + \delta_2(\lg n) + O\left(\frac{\ln n}{n}\right) \\
&= 3.116695\ldots + \delta_2(\lg n) + O\left(\frac{\ln n}{n}\right).
\end{aligned}$$

where $\delta_1(\cdot)$ *is a periodic function of magnitude* $\le 2\times 10^{-5}$ *given by*

$$\delta_1(x) := \frac{1}{L}\sum_{\substack{-\infty<k<\infty\\ k\neq 0}} \zeta(1-\chi_k)\,\Gamma(1-\chi_k)\,e^{2\pi i k x},$$

and $\zeta(\cdot)$ *and* $\Gamma(\cdot)$ *denote Riemann's zeta function and Euler's gamma function, respectively. The periodic function* $\delta_2(\cdot)$ *has magnitude* $\le 2\times 10^{-4}$.

6.4 Path Length in Digital Search Trees

This example is to illustrate an application of Theorem 6 in which the leading term of the Bernoulli variance is effected by the correction term of the depoissonization (cf. (56)). Let us consider the path length L_n in a digital search tree (cf. Section 3.3). The Poisson transform of $\mathbf{E}u^{L_n}$ satisfies the differential-functional equation (16). One shows that the Poisson mean $\widetilde{X}(z) = \widetilde{G}'(z,1)$ and the Poisson second factorial moment $\widetilde{W}(z) = \widetilde{G}''(z,1)$ become (cf. [31])

$$\begin{aligned} \widetilde{X}(z) + \widetilde{X}'(z) &= \widetilde{X}(zp) + \widetilde{X}(zq) + z\ , && (86)\\ \widetilde{W}(z) + \widetilde{W}'(z) &= \widetilde{W}(zp) + \widetilde{W}(zq) + 2zp\widetilde{X}'(zp) + 2zq\widetilde{X}'(zq) + (\widetilde{X}'(z))^2\ . && (87)\end{aligned}$$

Using Mellin transform approach Jacquet and Szpankowski [31] proved that in a polynomial cone $\mathcal{C}(D,\delta)$ $(\delta > \frac{1}{2})$

$$\begin{aligned} \widetilde{X}(z) &= \frac{z}{h}\log z + \frac{z}{h}\left(\gamma - 1 + \frac{h_2}{2h} - \alpha - \delta_1(\log z)\right) + O(1)\ , && (88)\\ \widetilde{V}(z) &= \frac{z\log^2 z}{h^2} + \frac{2z\log z}{h^3}\left(\gamma h + h_2 - \frac{h^2}{2} - \alpha h - h\delta_1(\log z)\right) + O(z)\ , && (89)\end{aligned}$$

where $\widetilde{V}(z) = \widetilde{W}(z) + \widetilde{X}(z)$, and $h = -p\log p - q\log q$, $h_2 = p\log^2 p + q\log^2 q$, and finally

$$\alpha = -\sum_{k=1}^{\infty} \frac{p^{k+1}\log p + q^{k+1}\log q}{1 - p^{k+1} - q^{k+1}}\ .$$

In the above $\delta_1(z)$ is a fluctuating function with a small amplitude.

In order to depoissonize $\widetilde{X}(z)$ and $\widetilde{W}(z)$ we first observe that (86) and (87) are *differential*-functional equations which do not fall into our general functional equation treated in Theorem 10. But, fortunately, the method of increasing domains (presented in the proof of Theorem 10) still works. Since we deal with Poisson mean and Poisson variance we are in the framework of Theorem 6. Nevertheless, one must verify conditions (I) and (O) for $\widetilde{X}(z)$ and $\widetilde{W}(z)$. The condition (I) inside the cone follows directly from our asymptotics (88)–(89) and either analytical continuation or an application of the complex Mellin transform (cf.

Appendix B). Thus, we must prove a bound for $\widetilde{X}(z)e^z$ and $\widetilde{W}(z)e^z$ outside a polynomial cone $\mathcal{C}(D.\delta)$, that is, to verify condition (O) of Theorem 2.

Let us analyze only the Poisson mean $\widetilde{X}(z)$, however, we consider a more general differential-functional equation (arising in the analysis of the depth in a generalization of digital search trees known as b-digital search trees [19, 40]), namely:

$$X^{\langle b\rangle}(z) = X(zp)e^{zq} + X(zq)e^{zp} + ze^z$$

where $X^{\langle b\rangle}(z)$ denotes the bth derivative of $X(z) = \widetilde{X}(z)e^z$. Observe that the above equation can be alternatively represented as

$$X(z) = \underbrace{\int_0^z \int_0^{w_1} \cdots \int_0^{w_{b-1}}}_{b \text{ times}} (X(w_1p))e^{w_1q} + X(w_1q)e^{w_1p} + w_1e^{w_1})\, dw_b \cdots dw_2 dw_1$$

provided $X^{\langle k\rangle}(z) = 0$ for all $k < b$ which we assume to hold. We must prove $|X(z)| \le e^{\alpha|z|}$ for $z \notin \mathcal{S}_\theta$ for $\alpha < 1$. As in the proof of Theorem 10 we apply mathematical induction over increasing domains $\mathcal{D}_m$. To carry out the induction, we first define $\overline{\mathcal{D}}_m = \mathcal{D}_m \cap \bar{S}_\theta$ where $\bar{S}_\theta$ denotes points in the complex plane outside S_θ. Since $X(z)$ is bounded for $z \in \overline{\mathcal{D}}_0$, thus the initial step of induction holds. Let us now assume that for $z \in \overline{\mathcal{D}}_m$ we have $|X(z)| \le e^{\alpha|z|}$ for some m and $\alpha < 1$. We intend to prove that $|X(z)| \le e^{\alpha|z|}$ for $z \in \overline{\mathcal{D}}_{m+1}$. Indeed, let $z \in \overline{\mathcal{D}}_{m+1}$. If also $z \in \mathcal{D}_m$, then the proof is completed. So let us now assume that $z \in \overline{\mathcal{D}}_{m+1} - \overline{\mathcal{D}}_m$. Since then $zp, zq \in \overline{\mathcal{D}}_m$, we can use our induction hypothesis together with the above integral equation to obtain the following estimate for $|z| > \xi$ where ξ is large enough

$$|X(z)| \le |z|^b \left(e^{|z|(p\alpha+q\cos\theta)} + e^{|z|(q\alpha+p\cos\theta)} + e^{|z|\cos\theta}\right) .$$

Let us now define $\alpha > \cos\theta$ such that the following three inequalities are simultaneously fulfilled

$$\begin{aligned} |z|^b e^{|z|(p\alpha+q\cos\theta)} &\le \frac{1}{3}e^{\alpha|z|} \\ |z|^b e^{|z|(q\alpha+p\cos\theta)} &\le \frac{1}{3}e^{\alpha|z|} \\ |z|^{b+1} e^{|z|\cos\theta} &\le \frac{1}{3}e^{\alpha|z|} . \end{aligned}$$

Then, for $z \in \overline{\mathcal{D}}_{m+1}$ we have $|X(z)| \le e^{\alpha|z|}$, as needed to verify condition (O) of Theorem 1. In a similar manner we can handle $\widetilde{W}(z)$.

Now, we are in a position to apply Theorem 6. Interestingly enough, the leading term in the variance of the Poisson model is cancelled resulting in $\mathbf{Var}\, L_n = O(n \log n)$ (instead of $O(z \log^2 z)$ as in the Poisson model). More precisely:

Proposition 2 (Jacquet and Szpankowski [31]) *With the notation as above, we obtain*

$$\mathbf{E}L_n = \frac{n}{h}\left(\log n + \frac{h_2}{2h} + \gamma - 1 - \alpha + \delta_0(\log m)\right) \tag{90}$$

$$+ \frac{1}{h}\left(\log n + \frac{h_2}{2h} - \gamma - \log p - \log q + \alpha\right) + O(1) \tag{91}$$

$$\mathbf{Var}\ L_n = \frac{h^2 - h_2}{h^3} n \log n + O(n) \tag{92}$$

for $p \neq q$, where $h = -p\log p - q\log q$ and $h_2 = p\log^2 p + q\log^2 q$.

In passing, we should point out that that in the case $p = q = \frac{1}{2}$ the variance is $O(n)$ as proved in [35], and in fact the term in front of n contains a fluctuating function. We should add that the techniques of [35] and [31] are different even if both use Mellin transforms.

6.5 Leader Election Algorithm Revisited

We refer to Section 3.4 or Section 6.3 above for a description of the problem. Hereafter, we are interested in the limiting distribution of the height H_n which represents the number of tosses before a leader is elected (cf. [14, 46]). The generating function of H_n is given by (17), and its Poisson transform by (18) which we repeat below for the reader's convenience

$$\widetilde{G}(z,u) = u(1+e^{-z/2})\widetilde{G}\left(\frac{z}{2},u\right) + e^{-z}[(1+z)(1-u) - ue^{z/2}] \ .$$

Actually, using the Mellin transform approach, we can solve asymptotically the above for large z to obtain (cf. [14])

$$\widetilde{G}(z,u) = -\frac{z^{\log_2 u}}{\ln 2}\left((1-u)\Gamma(1-\log_2 u)\zeta(1-\log_2 u) + \delta(z,u)\right)(1-e^{-z}) + O(z^{-M}) \tag{93}$$

where $|u| < 1/2$ and M is a large positive number, and $\delta(z,u)$ is a fluctuating function with a small amplitude. The above can be used to obtain exact and asymptotic formulæ for the distribution of H_n. We concentrate here on the limiting distribution, if it exists. First, however, we must verify depoissonization conditions (I) and (O) to assure that we can appeal to Corollary 2. Since the functional equation falls under the general functional equation treated in Theorem 10, an easy verification lead us to a conclusion that all conditions for depoissonization are in place. In particular, $\widetilde{G}(z,u) = O(1)$ since $\frac{1}{2} < |u| < 1$, and therefore by Theorem 10 we conclude that $G(u) = \widetilde{G}(n,u) + O(n^{-1})$ (for details see [14]).

Knowing that depoissonization of $\widetilde{G}(z,u)$ works, we may ask about the limiting distribution of H_n. From (93) and the Cauchy formula we have for an integer k

$$\Pr\{H_n \leq \log_2 n + k\} = \Pr\{H_n \leq \lfloor \log_2 n + k \rfloor\}$$

$$\begin{aligned} &= -\frac{1}{2\pi i \ln 2}\oint z^{\{\log_2 n\}-k-1}\Gamma(1-\log_2 z)\zeta(1-\log_2 z)dz \\ &+ \oint \frac{O(n^{-1})}{(1-z)z^{\lfloor \log_2 n+k\rfloor}}dz \end{aligned}$$

where $\{\log_2 n\} = \log_2 n - \lfloor \log_2 n \rfloor$. Observe that the last integral is $O((n^{-1})$. Indeed, it suffices to enlarge the circle of integrating to $z = 1$ (more precisely: we can take $z = 2^{1/(\log_2 n+k)}$). Furthermore, the integrand in the first integral has simple poles at $z_j = 2^j$ $j = 1, 2, \ldots$, which are contributed by the $\Gamma(\cdot)$ function, and one additional pole at $z_0 = 1$ coming from Riemann's zeta function. The residue at z_0 is $-\ln 2$. The residue at z_j, for $j = 1, 2, \ldots$, is

$$\frac{(-1)^j \ln 2}{(j-1)!} 2^{j(\{\log_2 n\}-k)}\zeta(1-j) \ .$$

The special values of Riemann's zeta function appearing in the last expression are related to the Bernoulli numbers B_r, $r = 0, 1, 2\ldots$, as follows: $\zeta(0) = -1/2$, $\zeta(1-2k) = -B_{2k}/(2k)$, for $k = 1, 2, 3, \ldots$, and $B_{-2k} = 0$, for $k = 1, 2, \ldots$. Subsequently we arrive at

$$\Pr\{H_n \le \log_2 n + k\} = 1 - 2^{\{\log_2 n\}-k-1} + \sum_{j=1}^{\infty} 2^{j(\{\log_2 n\}-k)}\frac{B_{2j}}{(2j)!} + O\left(\frac{1}{n}\right) \ .$$

The remaining sum is expressible in terms of the classical generating function of the Bernoulli numbers

$$\frac{y}{e^y - 1} = \sum_{j=0}^{\infty} \frac{B_j y^j}{j!},$$

and this yields (with a new improved error term compared to [14])

Proposition 3 (Fill et al. [14]) *Uniformly over all integers k,*

$$\Pr\{H_n \le \lfloor \log_2 n \rfloor + k\} = \frac{2^{\{\log_2 n\}-k}}{\exp(2^{\{\log_2 n\}-k}) - 1} + O\left(\frac{1}{n}\right) \tag{94}$$

as $n \to \infty$, where $\{\log_2 n\} = \log_2 n - \lfloor \log_2 n \rfloor$.

Finally, we observe that the function $\{\log_2 n\}$ is dense on the interval $[0, 1)$ but not uniformly dense, hence a limiting distribution of H_n does not exist (but formula above presents the asymptotic distribution). The reader is referred to [14] for more details where the above result was derived for the first time (but, with a weaker error term, namely $O(1/\sqrt{n})$).

An interesting extension was recently considered by Janson and Szpankowski [33] who analyzed the asymmetric leader election process in which p is the probability of survival at each round ($p \neq \frac{1}{2}$). In this case, the Poisson transform $\widetilde{G}(z,u)$ satisfies the following functional equation

$$\widetilde{G}(z,u) = u\widetilde{G}(pz,u) + u\widetilde{G}(qz,u)e^{-pz} + (1-u)ze^{-z} .$$

Asymptotic solution of this equation is much more intricate. We observe that the Poisson transform $\widetilde{G}_k(z) = \sum_{n\geq 0} \Pr\{H_n = k\}\frac{z^n}{n!}e^{-z}$ fulfills

$$\begin{aligned} \widetilde{G}_0(z) &= ze^{-z} , \\ \widetilde{G}_{k+1}(z) &= \widetilde{G}_k(pz) + e^{-pz}\widetilde{G}_k(qz) , \qquad k \geq 1 . \end{aligned} \tag{95}$$

This yields the following solution as proved in [33]

$$\widetilde{G}_k(z) = p^k z \int_0^{p^{-k}} e^{-p^k zt} d\mu(t), \tag{96}$$

where $\mu(t)$ is defined as follows: Partition the positive real axis into an infinite sequence of *consecutive* intervals $I_0, I_1, ...$ such that I_k has length $(q/p)^{s(k)}$, where $s(k)$ is the number of 1's in the binary expansion of k. Thus $I_0 = [0,1], I_1 = [1, 1+q/p], \ldots$. Note that the total length of the first 2^m intervals $(I_0, ..., I_{2^m-1})$ is p^{-m}, and that these 2^m intervals are obtained by repeated subdivisions of $[0, p^{-m}]$, each time dividing each interval in the proportions $p : q$. Given these intervals, define μ by putting a point mass $|I_k|$ at the right endpoint of I_k, for each $k = 0, 1, ...$

To obtain an asymptotic distribution for H_n, one must depoissonize $\widetilde{G}_k(z)$ and then apply Corollary 1. Observe that only a slight generalization of Theorem 10 is required to handle the functional equation (95). Conditions of Theorem 10 are easy to check. However, since we have an explicit formula for $\widetilde{G}_k(z)$ (cf . (96)), we may check condition (I) and (O) directly. For the reader's convenience, we prove now that inside a cone $\mathcal{S}_\theta$ we have $\widetilde{G}_k(z) = O(1)$ uniformly over all natural numbers k. Indeed, we first evaluate the function $F(x)$ defined for positive x as

$$F(x) = x\int_0^\infty e^{-xt} d\mu(t) . \tag{97}$$

Let, without losing generality, $q \geq p$, and set $\alpha = q/p \geq 1$. Noting that the interval I_j satisfy: $\alpha \leq |I_j| \leq \alpha^j$, we can estimate

$$\begin{aligned} F(x) &:= x\int_0^\infty e^{xt} d\mu(t) = x\sum_{j=1}^\infty e^{-x(|I_1|+\ldots+|I_j|)}|I_j| \\ &\leq x\sum_{j=1}^\infty e^{-xj\alpha}\alpha^j = \frac{x}{e^{\alpha x - \log\alpha} - 1} . \end{aligned}$$

Clearly, the above is bounded when we replace x either by $|z|p^k$ or $\Re(z)$ in a cone S_θ. This establishes condition (I). Conditions (O) can be proved in a similar manner (cf. [33] for details).

Now, we are ready to apply Corollary 1 to arrive at, as already proved in [33]:

Proposition 4 (Janson and Szpankowski [33]) *The following holds, uniformly for all integers k,*

$$\Pr\{(H_n \le k\} = F(p^k n) + O(n^{-1}),$$

where $F(x)$ is defined in (97). In particular, when $k = \lfloor \log_P n \rfloor + \kappa$ where κ is an integer and $P = 1/p$, then for large n the following asymptotic formula is true uniformly over κ

$$\Pr\{H_n \le \lfloor \log_P n \rfloor + \kappa\} = p^{\kappa - \{\log_P n\}} \int_0^\infty e^{-tp^{\kappa-\{\log_P n\}}} d\mu(t) + O\left(\frac{1}{n}\right),$$

where $\{\log_P n\} = \log_P n - \lfloor \log_P n \rfloor$.

Observe that for $p = q = \frac{1}{2}$ we obtain

$$F(x) = x \sum_{j=1}^{\infty} e^{-jx} = \frac{x}{e^x - 1} ,$$

and our results coincide with those of [14] and our (94).

6.6 Probabilistic Counting

In Section 3.5 we described a generalized probabilistic counting algorithm. We argued that to estimate the cardinality of set, one can measure a quantity $R_{n,d}$ whose generating function and Poisson generating function satisfy functional equations (21) and (22), respectively. In this example, we show how to depoissonize $\widetilde{G}(z,u)$ to obtain an asymptotic distribution (actually, we shall see that the limiting distribution of $R_{n,d}$ does not exist).

Let $G_n(u) = \mathbf{E}u^{R_{n,d}}$, and $\widetilde{G}(z,u) = \mathbf{E}u^{R_{N,d}}$ be the probability generating function and its Poisson transform, respectively. As in [36] we observe that

$$\widetilde{G}(z,u) = uf_d(z/2)\widetilde{G}(z/2,u) + (u-1)(f_d(z/2) - 1) ,$$

where $f_d(z) = 1 - e_d(z)e^{-z}$ and $e_d(z) = 1 + \frac{z^1}{1!} + \cdots + \frac{z^d}{d!}$ is the truncated exponential function.

Our goal is to estimate the limiting distribution through depoissonization. First of all, we consider $\widetilde{H}(z,u) = \widetilde{G}(z,u) - 1$ instead of $\widetilde{G}(z,u)$ which satisfies the following functional equation:

$$\widetilde{H}(z,u) = uf_d(z/2)\widetilde{H}(z/2,u) + (u-1)f_d(z/2) .$$

This is a linear functional equation of type (71), hence (72) is its solution which becomes in this case:

$$\varphi(x)\widetilde{H}(z,x) = (z-1)\sum_{n=0}^{\infty} z^n \varphi(x2^{-n-1}) ,$$

where

$$\varphi(z) = \prod_{j=0}^{\infty} f_d(z2^j) = \prod_{j=0}^{\infty}\left(1 - e_d(z2^j)e^{-z2^j}\right) .$$

This implies

$$\frac{1-\widetilde{G}(z,u)}{1-u} = \sum_{k=0}^{\infty} u^k \frac{\varphi(z2^{-k-1})}{\varphi(z)} ,$$

and finally

$$\widetilde{h}_k(z) = [u^k]\left(\frac{\widetilde{H}(z,u)}{1-u}\right) = e^{-z}\sum_{N=0}^{\infty} \Pr\{R_{n,d} > k\}\frac{z^N}{N!} = \frac{\varphi(z2^{-k-1})}{\varphi(z)} .$$

To evaluate the limiting distribution we apply Corollary 1 to $\widetilde{G}_k(z)$. We must verify conditions (I) and (O). The condition (O) outside the cone $\mathcal{S}_\theta$ where $\theta < \pi/2$ is easy to check, and it follows directly from the functional equation, as we already seen in previous sections. The difficulty is to prove a bound on $\widetilde{G}_k(z)$ inside the cone $\mathcal{S}_\theta$. We use the *complex* Mellin transform as discussed in Appendix B to establish the following bound:

Lemma 9 *For all $z \in \mathcal{S}_\theta$ with $z \neq 0$ and $|\theta| < \pi/2$ we have uniformly*

$$|\varphi(z)| < A \tag{98}$$

where A is a constant. Also, for large z we have $\varphi(z) = 1 + O(z^{-M})$ for any $M > 0$.

Proof. First of all, we observe that for large z the series in $\varphi(z)$ converges exponentially fast, and one immediately proves that $\varphi(z) = 1 + O(z^{-M})$ for any $M > 0$. Thus, the only difficult part is to prove (98) for z close to zero, which we consider next. For simplicity of presentation we only assume $d = 0$. We use complex Mellin transform. Let $\ell(z) = \log\varphi(z)$. Observe that (for $d = 0$)

$$\ell(z) = \log(1-e^{-z}) + \ell(2z) .$$

Hence, the Mellin $\ell^*(s)$ of $\ell(z)$ exists in $\Re(s) \in (0,\infty)$ and it becomes (cf. [16])

$$\ell^*(s) = -\frac{\Gamma(s)\zeta(s+1)}{(1-2^{-s})} .$$

There is a triple pole at $s = 0$ which gives us for $z \to 0$ (ignoring the fluctuating term)

$$\ell(z) = -\frac{1}{2}(\log^2 z + \log z) + O(1) .$$

Thus $\varphi(z) = \exp(\ell(z)) = \exp(-\frac{1}{2}\log^2 z - \frac{1}{2}\log z + O(1))$. To show a uniform bound around $z = 0$, let $z = \rho e^{i\theta}$. Then,

$$|\varphi(z)| = \exp(\Re(\ell(z)) = \exp(-0.5(\log^2 \rho - \theta^2 + \log \rho)) \ .$$

Observe that for $\rho < \rho_c$ where

$$\log^2 \rho_c - \pi^2/4 + \log \rho_c > 0$$

the function $\varphi(z)$ is bounded for ρ small, and this completes the proof. ■

Having the above result, we easily show that

$$\frac{|\varphi(z2^{-k-1})|}{|\varphi(z)|} \leq A \ .$$

which completes the proof of the depoissonization inequality (I). Thus, by Corollary 1 we prove that $\Pr\{R_{n,d} > k\} = \frac{\varphi(n2^{-k-1})}{\varphi(n)} + O(1/n)$, or more precisely we obtain the following result of [36] with a better error term (in [36] only $O(n^{-\frac{1}{2}+\varepsilon})$ error was established):

Proposition 5 (Kirschenhofer et al. [36]) *For any integer* m

$$\Pr\{R_{n,d} \leq \log_2 n + m - 1\} = 1 - \varphi\left(2^{-m-\{\log_2 n\}}\right) + O(n^{-1})$$

where $\{\log n\} = \log n - \lfloor \log n \rfloor$.

6.7 Depth in a PATRICIA Trie

We illustrate in this subsection a usage of the "diagonal depoissonization" with a polynomial bound inside a cone, that is, Theorem 4, while in the next two subsections we apply "diagonal exponential depoissonization", that is, Theorems 8 and 9.

Let us consider the depth in a PATRICIA trie (cf. Section 3.3), that is, we study $\widetilde{D}(z,u)$ which satisfies the functional equation (11) that we repeat here

$$\widetilde{D}(z,u) = u(p\widetilde{D}(zp,u) + q\widetilde{D}(zq,u)) + (1-u)(p\widetilde{D}(zp,u)e^{-qz} + q\widetilde{D}(zq,u)e^{-pz}) \ .$$

Observe that the unknown function $\widetilde{D}(z,u)$ is involved in a product with e^{-qz} and e^{-pz} which makes the equation complicated, and in fact it does not have a simple explicit solution. Nevertheless, it falls under Theorem 10 with $\gamma_1(z) = up + (1-u)pe^{-qz}$, $\gamma_2(z) = uq + (1-u)qe^{-pz}$ and $t(z) = 0$. Furthermore, for $p \neq q$ the asymptotic behavior of $\widetilde{D}(z,u)$ inside a cone $\mathcal{S}_\theta$ ($z \to \infty$) is determined by the first two terms of the above equation. Using Mellin transform Rais *et al.* [47] proved that for large z the Poisson depth is normal, that is, for z large in a cone $\mathcal{S}_\theta$, and $u = e^t$ where t is complex we have:

$$e^{-t\widetilde{X}(z)/\sigma(z)}\widetilde{D}(z, e^{t/\sigma(z)}) = e^{t^2/2}(1 + O(1/\sigma(z))) \ , \tag{99}$$

where $\widetilde{X}(z) = O(\log z)$ is the Poisson mean, and $\sigma^2(z) = \widetilde{V}(z) = O(\log z)$ is the Poisson variance. This result extends to PATRICIA trie similar results of Jacquet and Regnier [26], and Jacquet and Szpankowski [29] (no poissonization/depoissonization was used) established for tries.

One now must depoissonize (99) in order to establish the central limit law for the Bernoulli model. First all, an easy application of Theorem 6 leads us to $\mathbf{E}D_n \sim \widetilde{X}(n)$ and $\mathbf{Var}\ D_n \sim \widetilde{V}(n)$. In fact, from Szpankowski [55] we know that $\mathbf{E}D_n = \frac{1}{h}\log n + O(1)$ and $\mathbf{Var}\ D_n = \frac{h^2 - h_2}{h^3}\log n + O(1)$ where $h = -p\log p - q\log q$ is the entropy, and $h_2 = p\log^2 p + q\log^2 q$. Thus, we only need to depoissonize (99) in order to obtain the limiting distribution. We first observe that the above functional equation falls under Theorem 10 as mentioned above (with $\gamma_1(z) = up + (1-u)pe^{-qz}$, $\gamma_2(z) = uq + (1-u)qe^{-pz}$ and $t(z) = 0$). Thus, the mathematical induction over increasing domains works in this case, too, and we can conclude – as the authors of [47] – that $\widetilde{D}(z,u) = O(z^\varepsilon)$ for any $\varepsilon > 0$ inside a cone $\mathcal{S}_\theta$ for z large. This actually completes the depoissonization, and by Corollary 2 we arrive at (cf. [47]):

Proposition 6 (Rais et al. [47]) *For complex t*

$$e^{-t\mathbf{E}D_n/\sqrt{\mathbf{Var}\ D_n}}\mathbf{E}e^{tD_n/\sqrt{\mathbf{Var}\ D_n}} = e^{t^2/2}(1 + O(1/\sqrt{\log n})) \ ,$$

that is, $(D_n - \mathbf{E}D_n)/\sqrt{\mathbf{Var}\ D_n}$ converges in distribution and in moments to the standard normal distribution.

6.8 Size of a Tries

Let us now consider the size S_n of a trie (i.e., number of internal nodes in a trie built from n strings). In section 3.3 we show that its Poisson transform $\widetilde{S}(z,u)$ satisfies *multiplicative* functional equation (9), which we repeat below:

$$\widetilde{S}(z,u) = u\widetilde{S}(zp,u)\widetilde{S}(zq,u) + (1-u)e^{-z} \ .$$

Due to the multiplicative nature of the above functional equation, one should expect larger rate of growth inside a cone, hence our depoissonization results with polynomial growth could not work anymore, and we have to appeal to our exponential depoissonization (cf.

Theorems 3 and 8). Indeed, Jacquet and Régnier [27] (cf. also [48]) proved that $\log \widetilde{S}(z,u)$ exists and $\log \widetilde{S}(z,u) = O(z)$ inside a linear cone $\mathcal{S}_\theta$ for some $0 < |\theta| < \pi/2$ and u complex in a neighbourhood of 1 (equivalently, by setting $u = e^t$ we consider t in a complex neighbourhood of 0). Furthermore, Régnier and Jacquet [48] proved that Poisson mean $\widetilde{X}(z) = O(z)$, Poisson variance $\widetilde{V}(z) = O(z \log z)$, and the remainder $R(z,t) = O(z)$, thus $\log \widetilde{S}(z, e^t) = t\widetilde{X}(z) + t^2\widetilde{V}(z)/2 + t^3 R(z,t)$. A simple application of Theorem 6 shows that the Bernoulli variance $\mathbf{Var}\ S_n = O(n)$. By Theorem 7 the size in the Poisson model (i.e., S_N where N is a Poisson distributed random variable) is asymptotically normally distributed. More formally:

$$e^{-t\widetilde{X}(z)/\sqrt{\widetilde{V}(z)}}\widetilde{S}\left(z, e^{t/\sqrt{\widetilde{V}(z)}}\right) = e^{t^2/2}\left(1 + O\left(\frac{1}{\sqrt{\widetilde{V}(z)}}\right)\right) .$$

To translate the above into the Bernoulli model, we must apply Theorem 9. We first observe that all assumptions of this theorem hold with $\chi = 1$, $\nu = 1$ and $\delta = 1$: $\beta = 1/2 \in [1/2, 2/3[$. Thus:

Proposition 7 (Jacquet and Regnier [27]) *The normalized size of a trie that is,*

$$\frac{(S_n - \mathbf{E}S_n)}{\sqrt{\mathbf{Var}S_n}} \xrightarrow{d} N(0,1)$$

where $N(0,1)$ is the standard normal distribution. In fact, $(S_n - \mathbf{E}S_n)/\sqrt{\mathbf{Var}S_n}$ also converges in moments, and the convergence rate is $O(n^{\varepsilon - 1/2})$ for $\varepsilon > 0$ arbitrarily small.

6.9 Path Length in a Digital Search Tree Revisited

Finally, we study one of the most challenging problems of this paper, namely, the limiting distribution of the path length L_n in digital search tree (cf. Section 3.3). Its exponential generating function $L(z,u) = \widetilde{L}(z,u)e^z$ satisfies differential-functional multiplicative equation (16) that we repeat below

$$\frac{\partial L(z,u)}{\partial z} = L(puz,u)L(quz,u) .$$

This equation was studied in Jacquet and Szpankowski [31] in order to determine the limiting distribution of the number of phrases in the Lempel-Ziv parsing scheme [61]. Interestingly enough, the growth of $L(z,u)$ is quite complicated. It was proved in [31] that $\log L(z,u)$ exists and $\log L(z,u) = O(z^{\kappa(u)+\varepsilon})$ for u in a real neighbourhood of 1 and z in a cone $\mathcal{C}(D,\delta)$ with $\delta < 1$ arbitrary close to 1. Actually, $\kappa(u)$ satisfies the relation $(pu)^{\kappa(u)} + (qu)^{\kappa(u)} = 1$, and one can show that $\kappa(u) = 1 + h(u-1) + O((u-1)^2)$ where h is the entropy. We should also add that that this problem motivated us to introduce polynomial cones. In such a cone $\mathcal{C}(D, \delta)$ with $\delta < 1$ (but arbitrary close to $\delta = 1$ when u approaching 1), we shown in [31] that any derivative of $\log L(z, u)$ grows like $O(z^{\kappa(u)+\varepsilon})$ for any $\varepsilon > 0$. Furthermore, in Section 6.4 we studied the Poisson mean $\widetilde{X}(z)$ and the Poisson variance $\widetilde{V}(z)$ and proved that $\widetilde{X}(z) = O(z \log z)$ and $\widetilde{V}(z) = O(z \log^2 z)$. Thus, by Theorem 7 we immediately see that L_N in the Poisson model is normally distributed with the rate of convergence $O(z^{-\frac{1}{2}+\varepsilon})$.

Extension of the above result to the Bernoulli model requires substantial work as demonstrated in [31]. Fortunately, we can significantly simplify the proof by applying our Theorem 9. We first observe that (cf. Section 6.4) the Bernoulli variance $V_n = \mathbf{Var}\ L_n = O(n \log n)$, hence all conditions of Theorem 9 hold with $\chi = 1 + \varepsilon$ and $\nu = 1$, and $\delta = 1 - \varepsilon$ and $\varepsilon > 0$ arbitrarily small. Thus $\beta = 1/2 + \varepsilon$ which belongs to $[1/2 - \varepsilon, 2/3 - 4/3\varepsilon[$ (it suffices to choose ε small enough). Consequently, the external length in digital search trees is asymptotically normal in distribution and in moments. More formally:

Proposition 8 (Jacquet and Szpankowski [31]) *For arbitrary $\varepsilon > 0$ and real t*

$$e^{-t\mathbf{E}L_n/\sqrt{V_n}}\mathbf{E}\left(e^{tL_n/\sqrt{V_n}}\right) = e^{\frac{1}{2}t^2}\left(1 + O\left(\frac{1}{n^{\frac{1}{2}-\varepsilon}}\right)\right)$$

where $\mathbf{E}L_n$ is given by (90), and $V_n = \mathbf{Var}\ L_n$ is computed according to (92).

ACKNOWLEDGEMENT

This paper has come into existence over several last years during our frequent visits to INRIA, Rocquencourt and Purdue University, West Lafayette. We thank our sponsors (INRIA, NSF, NATO, Purdue University) for funding this research. Our deepest gratitude, however, goes to *le bon docteur* Philippe Flajolet whose continuous encouragement and inspiration were crucial for our research work over last decade. Philippe also contributed to some parts of the paper (e.g., Appendix B).

We also thank numerous of our colleagues who worked with us on depoissonization results. We mention here: A. Fill, H-K. Hwang, S. Janson, P. Kirschenhofer, H. Mahmoud, H. Prodinger, B. Rais, and M. Régnier. We are particularly obliged to Svante Janson for a very careful reading countless versions of the paper, numerous enlighting discussions, and several suggestions (we have received 20 pages of invaluable comments from Svante!) that led to important improvements of the text and our derivations. *Tusen tack, Svante*! Finally, but not last, we thank J. Diaz for inviting this paper to the "Fundamental Studies" of *Theoretical Computer Science*.

A A Formal Proof of (35)

We now formally prove the general expression (34) by showing that the coefficients b_{ij} satisfy (35). To identify these coefficients, we are allowed to choose any function g_n that satisfies our hypothesis. For simplicity, we now assume that $g_n = (n)_\ell$, where $(x)_\ell = x(x-1)\cdots(x-\ell+1)$. In this case $\widetilde{G}(z) = z^\ell$, so we are in the framework of Theorem 2. We use the identity

$$(n)_\ell = \sum_{i=0}^{i=\ell} n^{\ell-i} s(\ell, \ell-i) \ ,$$

where the $s(n,k)$'s are the Stirling numbers of first kind (cf. [7]). It is known (cf. [7], Ex. 16, pp. 227) that

$$s(\ell, \ell-i) = \sum_{j=i+1}^{2i} \binom{\ell}{j} s_2(j, j-i) \ ,$$

where $s_2(n,k)$ are defined as (cf. [7] Ex. 20 pp. 295)

$$\sum_{n,k} s_2(n,k) t^n u^k / n! = e^{-tu}(1+t)^u \ . \tag{100}$$

Noticing that $\frac{G^{\langle j\rangle}(n)}{j!} = \binom{\ell}{j} n^{\ell-j}$, we finally obtain

$$\begin{aligned}
(n)_\ell &= \sum_{i=0}^{i=\ell} \sum_{j=i+1}^{2i} s_2(j, j-i) n^{\ell-i} \binom{\ell}{j} \\
&= \sum_{i=0}^{i=\ell} \sum_{j=i+1}^{2i} G^{\langle j\rangle}(n) \frac{s_2(j, j-i)}{j!} n^{j-i} \\
&= \sum_{j=0}^{2\ell} \sum_{i=0}^{i=\ell} G^{\langle j\rangle}(n) \frac{s_2(j,i)}{j!} n^i \ .
\end{aligned}$$

Thus, $c_{ij} = s_2(j,i)/j! = b_{ij}$, and our result follows from (100).

B Mellin Transform of a Complex Variable

In various applications of poissonization/depoissonization (and also singularity analysis), one needs to deal with the Mellin transform of a complex analytical function. Since we did not find a proper reference to such an extension (except for some remarks in [8, 10]; see also [16, 26]), we add this appendix for completeness of the presentation.

In the theorem below, we list basic facts about Mellin transform of a function $f(x)$ of a real variable x. Its proof is a classical one, and can be found in many places. e.g., [10, 16].

Theorem 11 *Let $f(x)$ be a function defined for a non-negative real x such that $f(x) = O(x^{-c})$ for $x \to 0$ and $f(x) = O(x^{-d})$ for $x \to \infty$. We assume that $f(x)$ is twice differentiable, and $f''(x) = O(x^{-c-2})$ for $x \to 0$ while $f''(x) = O(x^{-d-2})$ for $x \to \infty$ (or more generally: there is an interval (c, d) such that for all $b \in (c, d)$: $f''(x) = O(x^{-b-2})$ as $x \to 0$ and $x \to \infty$). Then:*

(i) *The Mellin transform $\mathcal{M}(f, s) = f^*(s) = \int_0^\infty f(x)x^{s-1}dx$ exists in the strip $\Re(s) \in (c, d)$.*
(ii) *The Mellin transform $f^*(b + it)$ is absolutely integrable with respect to variable t from $-\infty$ to $+\infty$ as long as $b \in (c, d)$, that is,*

$$\int_{-\infty}^{\infty} |f^*(b + it)|dt < \infty \ .$$

(iii) *The inverse Mellin transform exists and*

$$f(x) = \frac{1}{2i\pi} \int_{\Re(s)=b} x^{-s} f^*(s)ds$$

for $b \in (c, d)$.

We now extend the Mellin transform to a complex function $F(z)$ such that $f(x) = F(z) \mid_{z=x>0}$ for z complex.

Theorem 12 *Let (θ_1, θ_2) be an interval containing 0. Let $F(z)$ be an analytical function defined for z in a cone $S_{\theta_1,\theta_2} = \{z : \ \theta_1 < \arg(z) < \theta_2\}$ such that $F(z) = O(z^{-c})$ for $z \to 0$, and $F(z) = O(z^{-d})$ for $z \to \infty$ with z confined to the cone (or more generally: there is an interval (c, d) such that for all $b \in (c, d)$: $F(z) = O(z^{-b})$ as $z \to 0$ and $z \to \infty$ in the cone). Then:*
(i) *Let $\mathcal{L}_0^\infty$ be a curve that starts at $z = 0$ and goes to ∞ inside the cone $\mathcal{S}_{\theta_1,\theta_2}$. For $\Re(s) \in (c, d)$ the integral*

$$\mathcal{M}(F, s) = F^*(s) := \int_{\mathcal{L}_0^\infty} F(z)z^{s-1}dz$$

exists and does not depend on the curve $\mathcal{L}_0^\infty$. We call it the **complex Mellin transform**.

(ii) *The complex Mellin transform $F^*(s)$ is identical to the real Mellin transform $f^*(s)$, that is,*

$$\int_0^\infty F(x)x^{s-1}dx = f^*(s) = F^*(s) = \int_{\mathcal{L}_0^\infty} F(z)z^{s-1}dz \ .$$

Furthermore, $\mathcal{M}(F(za), s) = a^{-s}f^*(s)$ defined on $S_{\theta_1-\arg(a),\theta_2-\arg(a)}$ with $a \in \mathcal{S}_{\theta_1,\theta_2}$.

(iii) *For $b \in (c,d)$ and $\theta_1 < \theta < \theta_2$, $f^*(b+it)e^{\theta t}$ is absolutely integrable with respect to variable t from $-\infty$ to $+\infty$. The inverse Mellin exists and becomes*

$$F(z) = \frac{1}{2i\pi}\int_{\Re(s)=b} z^{-s} f^*(s)ds$$

for $b \in (c,d)$.

Proof: For part (i), observe that the function $F(z)z^{s-1}$ is analytical, thus by Cauchy formula

$$\oint F(z)z^{s-1}dz = 0$$

where the integration is over any arbitrary closed curve inside the cone $\mathcal{S}_{\theta_1,\theta_2}$. Let z_1 and z_2 be two points on $\mathcal{L}_0^\infty$, let $r = |z_1|$ and $R = |z_2|$ and let $\mathcal{L}_r^R$ be the part of $\mathcal{L}_0^\infty$ between z_1 and z_2. Let further A_r and A_R be the circular arcs connecting z_1 with r and z_2 with R. We integrate along the closed curve formed by $\mathcal{L}_0^R$, the subinterval (r,R) of the real line and the two arcs A_r and A_R. Let us first consider the contribution of A_R. Then:

$$\int_{\mathcal{L}_0^R} F(z)z^{s-1}dz = \int_0^R F(z)z^{s-1}dz + \int_{A_R} F(z)z^{s-1}dz \ . \qquad (101)$$

Passing to the limit in the above, and noticing that $\int_{A_R} F(z)z^{s-1}dz = O(R^{s-b})$ for any $b \in (\Re(s), d)$, we conclude that the contribution of the arc vanishes when $R \to \infty$. In a similar manner, we consider the contribution of A_r for $r \to 0$. This proves part (i) and the first part of (ii).

For part (ii), we infer immediately from (101), that is,

$$F^*(s) = \lim_{R\to\infty}\int_{\mathcal{L}_0^R} F(z)z^{s-1}dz = \lim_{R\to\infty}\int_0^R F(z)z^{s-1}dz = f^*(s) \ .$$

To prove the identity $\mathcal{M}(F(za),s) = a^{-s}f^*(s)$ one only needs to make the change of variable $za = z'$ in the integration. Notice that function $F(za)$ is now defined in cone $S_{\theta_1-\arg(a),\theta_2-\arg(a)}$.

Point (iii) is a natural consequence of point (ii) by selecting $a = e^{i\theta}$. In this case, the (real) Mellin transform of function $F(xe^{-i\theta})$ becomes $e^{s\theta}f^*(s)$ which is absolutely integrable by part (ii) of Theorem 11. Therefore, for all $z \in \mathcal{S}_{\theta_1,\theta_2}$ the function $z^{-s}f^*(s)$ is absolutely integrable with respect to $\Im(s)$. Finally, we conclude that the Mellin inverse identity holds in the complex case. ∎

The above extension enjoys all "nice" properties of Mellin transform of real x. In particular, it can be used to obtain an asymptotic expansion of $F(z)$ for $z \to 0$ and $z \to \infty$, as we know from the Mellin transform of a function of real variable. Indeed, this is proved formally below:

Theorem 13 *Let $f(x)$ be a function of real variable. Let $f^*(s)$ be its Mellin transform defined in the strip $\Re(s) \in (c, d)$. Let (θ_1, θ_2) be an interval containing 0. If for all $b \in (c, d)$ function $f^*(b+it)e^{\theta_1 t}$ and $f^*(b+it)e^{\theta_2 t}$ are absolutely integrable, then*
(i) *The analytic continuation $F(z)$ of $f(x)$ exists in cone $\mathcal{S}_{\theta_1,\theta_2}$ and $F(z) = O(z^{-b})$ for $|z| \to 0$ and $|z| \to \infty$ for all $b \in (c, d)$.*

(ii) *If the Mellin transform $f^*(s)$ has singularities that are regular isolated poles in the strip $\Re(s) \in (c', d')$ where $c' < c$ and $d' > d$, then the function $F(z)$ has the same asymptotic expansions as $f(x)$ up to $z^{-c'}$ at 0 and up to order $z^{-d'}$ at ∞.*

Proof: Defining $F(z) = \frac{1}{2i\pi} \int_{\Re(s)=b} z^{-s} f^*(s)ds$ one introduces an analytical function which matches $f(x)$ on the real line. By an elementary estimate of the integrand we have $F(z) = O(z^{-b})$ due to the assumption concerning the smallness of $f^*(b+it)$ as $t \to \pm\infty$. Part (ii) is a simple consequence of (i) and the fact that $f^*(s)$ is the complex Mellin transform of $F(z)$, as noticed above. ■

To illustrate a simple usage of such an extension, let us consider the functional equation studied in Section 6.1, that is:

$$f(z) = a(z) + f(pz) + f(qz)$$

where $a(z)$ is given. Denoting $f^*(s)$ the Mellin of $f(x)$ we arrive at

$$f^*(s) = a^*(s) + p^{-s} f^*(s) + q^{-s} f^*(s)$$

which leads to $f^*(s) = a^*(s)(1 - p^{-s} - q^{-s})^{-1}$ and a simply application of the residue technique allows to extract asymptotics of $f(z)$ for large z in a cone $\mathcal{S}_\theta$. In particular if $a^*(s) = \alpha(s)\Gamma(s)$ with $\alpha(s)$ of at most polynomial on $\pm i\infty$, then the asymptotic expansion remain valid in all cone strictly included in $\mathcal{S}_{-\pi/2,\pi/2}$ because $|\Gamma(it)|^2 = \pi(t\sinh(\pi t))^{-1} = O(e^{-\pi|t|})$ when t real tends to $\pm\infty$.

C References

References

[1] D. Aldous, *Probability Approximations via the Poisson Clumping Heuristic*, Springer Verlag, New York 1989.

[2] R. Arratia and S. Tavaré, Independent Processes Approximations for Random Combinatorial Structures, *Advances in Mathematics*, 104, 90-154, 1994.

[3] N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover, 1962.

[4] J. Capetanakis, Tree Algorithms for Packet Broadcast Channels. *IEEE Trans. Information Theory*, 25, 505-515, 1979.

[5] J. Capetanakis, Generalized TDMA: The Multiaccessing Tree Protocol, *IEEE Trans. Communications*, 27, 1476–1484, 1979.

[6] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York 1978.

[7] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Boston 1974.

[8] B. Davis, *Integral Transforms and Their Applications*, Springer Verlag, New York 1985.

[9] L. Devroye, The Expected Length of the Longest Probe Sequence When the Distribution Is Not Uniform, *Journal of Algorithms*, 6, 1-9, 1985.

[10] G. Doetsch, *Handbuch der Laplace Transformation*, Verlag Birkhäuser, Basel 1950.

[11] G. Fayolle, P. Flajolet, M. Hofri and P. Jacquet, Analysis of Stack Algorithm for Random Multiple-Access Communication, *IEEE Trans. Information Theory*, 31, 244–254, 1985.

[12] G. Fayolle, P. Flajolet, and M. Hofri, On a Functional Equation Arising in the Analysis of a Protocol for a Multi-Access Broadcast Channel, *Adv. Appl. Prob.*, 18, 441-472, 1986.

[13] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York, 1971.

[14] J. Fill. H. Mahmoud, and W. Szpankowski, On the Distribution for the Duration of a Randomized Leader Election Algorithm, *Ann. Appl. Probab.*, 6, 1260–1283, 1996.

[15] P. Flajolet and G.N. Martin, Probabilistic Counting Algorithms for Data Base Applications, *J. Comp. Syst. Sci.*, 31, 182-209, 1985.

[16] P. Flajolet, X. Gourdon, P. Dumas, Mellin Transforms and Asymptotics: Harmonic Sums, *Theoretical Computer Science*, 144, 3-58, 1995.

[17] P. Flajolet, M. Régnier and D. Sotteau, Algebraic Methods for Trie Statistics, *Annals of Discrete Mathematics*, 25, 145-188, 1985.

[18] P. Flajolet and A. Odlyzko, Singularity Analysis of Generating Functions, *SIAM J. Disc. Methods*, 3, 216-240, 1990.

[19] P. Flajolet and B. Richmond, Generalized Digital Trees and Their Difference-Differential Equations, *Random Structures & Algorithms*, 3, 305-320, 1992.

[20] G.H. Gonnet The Expected Length of the Longest Probe Sequence in Hash Code Searching *J. of the ACM*, 28, 289-304, 1981.

[21] G. Gonnet and J. Munro, The Analysis of Linear Probing Sort by the Use of a New Mathematical Transform, *Journal of Algorithms*, 5, 451-470, 1984.

[22] M. Hofri, *Analysis of Algorithms*, Oxford University Press, New York, 1995.

[23] L. Holst, On Birthday, Collector's, Occupancy and other Classical Urn Problems, *International Statistical Review*, 54, 15-27, 1986.

[24] H-K. Hwang, Asymptotic Estimates of Elementary Probability Distributions, *Studies in Applied Mathematics*, to appear.

[25] P. Jacquet, *Contribution de l'Analyse d'Algorithmes a l'Evaluation de Protocoles de Communication*, Thèse Université de Paris Sud-Orsay, 1989.

[26] P. Jacquet and M. Régnier, Limiting Distributions for Trie Parameters, *Lecture Notes in Computer Science*, 214, 196-210, 1986.

[27] P. Jacquet and M. Régnier, Normal Limiting Distribution of the Size of Tries, *Proc. Performance'87*, 209-223, North Holland, Amsterdam, 1987

[28] P. Jacquet and W. Szpankowski, Ultimate Characterization of the Burst Response of an Interval Searching Algorithm: A Study of a Functional Equation, *SIAM J. Comp.*, 18, 777-791, 1989.

[29] P. Jacquet and W. Szpankowski, Analysis of Digital Tries with Markovian Dependency, *IEEE Transactions on Information Theory*, 37, pp. 1470-1475 (1991).

[30] P. Jacquet and W. Szpankowski, Autocorrelation on Words and Its Applications. Analysis of Suffix Trees by String-Ruler Approach, *J. Combin. Theory Ser. A*, 66, 237-269, 1994.

[31] P. Jacquet and W. Szpankowski, Asymptotic Behavior of the Lempel-Ziv Parsing Scheme and Digital Search Trees, *Theoretical Computer Science*, 144, 161-197, 1995.

[32] P. Jacquet and W. Szpankowski, Towards Analytical Information Theory: Entropy Computations, Purdue University, CSD-TR-96-085, 1996.

[33] S. Janson and W. Szpankowski, Analysis of an Asymmetric Leader Election Algorithm, Purdue University, CSD-TR-96-049, 1996.

[34] M. Kac, On the Deviations between Theoretical and Empirical Distributions, *Proc. Natl. Acad. Sci. USA*, 35, 252-257, 1949.

[35] P. Kirschenhofer, H. Prodinger and W. Szpankowski, Digital Search Trees Again Revisited: The Internal Path Length Perspective, *SIAM J. Computing*, 23, 598-616, 1994.

[36] P. Kirschenhofer, H. Prodinger and W. Szpankowski, Analysis of a Splitting Process Arising in Probabilistic Counting and Other Related Algorithms, *Random Structures & Algorithms*, 9, 379-402, 1996.

[37] D. Knuth, *The Art of Computer Programming. Sorting and Searching,* Vol. 3., Addison-Wesley 1973.

[38] G. Louchard and W. Szpankowski, Average Profile and Limiting Distribution for a Phrase Size in the Lempel-Ziv Parsing Algorithm, *IEEE Information Theory*, 41, 478-488, 1995.

[39] G. Louchard and W. Szpankowski, On the Average Redundancy Rate of the Lempel-Ziv Code, *IEEE Information Theory*, 43, 2-8, 1997.

[40] G. Louchard, W. Szpankowski, and J. Tang, Average Profile for the Generalized Digital Search Trees and the Generalized Lempel-Ziv Algorithm, *SIAM J. Computing*, to appear.

[41] H. Mahmoud, *Evolution of Random Search Trees*, John Wiley & Sons, New York 1992.

[42] P. Mathys, P. Flajolet, Q-ary Collision Resolution Algorithms in Random-Access Systems with Free and Blocked Channel Access, *IEEE Trans. Information Theory*, 31, 217–243, 1985.

[43] A. Odlyzko, Asymptotic Enumeration, in *Handbook of Combinatorics*, Vol. II, (Eds. R. Graham, M. Götschel and L. Lovász), Elsevier Science, 1063-1229, 1995.

[44] P. V., Poblete, Approximating Functions by Their Poisson Transform, *Information Processing Letters*, 23, 127–130, 1986.

[45] P. Poblete, A. Viola and J.I. Munro, The Diagonal Poisson Transform and its Application to the Analysis of a Hashing Scheme, *Random Structures & Algorithms*, 10, 221–256, 1997.

[46] H. Prodinger, How to Select a Loser. *Discrete Math.*, 120, 149–159, 1993.

[47] B. Rais, P. Jacquet and W. Szpankowski, A Limiting Distribution for the Depth in PATRICIA Tries, *SIAM J. Discrete Mathematics*, 6, 197-213, 1993.

[48] M. Regnier and P. Jacquet, New Results on the Size of Tries, *IEEE Trans. Information Theory*, 35, 203-205 (1989).

[49] R. Remmert, *Theory of Complex Functions*, Springer Verlag, New York 1991.

[50] S. Ross, *Stochastic Processes*, John Wiley & Sons, New York 1983.

[51] U. Schmid, The Average CRI-length of a Tree Collision Resolution Algorithm in Presence of Multiplicity-Dependent Capture Effects, *Proc. ICALP 92*, Vienna, 223-234, 1992.

[52] R. Sedgewick and P. Flajolet, *Analysis of Algorithms*, Addison Wesley, 1995.

[53] R. Stanley, Enumerative Combinatorics, Wadsworth, Monterey, 1986.

[54] W. Szpankowski, On a Recurrence Equation Arising in the Analysis of Conflict Resolution Algorithms, *Stochastic Models*, 3, 89-114, 1987.

[55] W. Szpankowski, Patricia Trie Again Revisited, *J. of the ACM*, 37, 691-711, 1990.

[56] B. Tsybakov, and V. Mikhailiov, Random Multiple Packet Access: Part-and-Try algorithm. *Prob. Information Transmission*, 16, 305–317, 1980.

[57] J. Vitter and P. Flajolet, Analysis of Algorithms and Data Structures, in *Handbook of Theoretical Computer Science* (ed. J. van Leeuwen). vol. A: Algorithms and Complexity, 431-524, North Holland, Amsterdam 1990.

[58] H. Wilf, *generatingfunctionology*, Academic Press, Boston 1990.

[59] A.D. Wyner and J. Ziv, Some Asymptotic Properties of the Entropy of a Stationary Ergodic Data Source with Applications to Data Compression, *IEEE Trans. Information Theory*, 35, 1250-1258 (1989).

[60] J. Ziv and A. Lempel, A Universal Algorithm for Sequential Data Compression, *IEEE Trans. Information Theory*, 23, 337-343, 1977.

[61] J. Ziv and A. Lempel, Compression of Individual Sequences via Variable-rate Coding, *IEEE Trans. Information Theory*, 24, 530-536, 1978.