STABILITY CONDITIONS FOR SOME DISTRIBUTED SYSTEMS: BUFFERED RANDOM ACCESS SYSTEMS

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Wojciech Szpankowski*
Department of Computer Science
Purdue University
W. Lafayette, IN 47907
U.S.A.

Abstract

We consider the standard slotted ALOHA system with a finite number of buffered users. Stability analysis of such a system was initiated in 1979 by Tsybakov and Mikhailov. Since then several bounds on the stability region have been established, however, the exact stability region is known only for the symmetric system and two users ALOHA. This paper proves necessary and sufficient conditions for stability of the ALOHA system. We accomplish this by means of a novel technique based on three simple observations: applying mathematical induction to a smaller copy of the system, isolating single queue to which Loynes' stability criteria is adopted, and finally using stochastic dominance to verify the required stationarity assumptions in the Loynes' criterion. We also point out that our technique can be used to assess stability regions for other multidimensional systems. We illustrate it by deriving stability regions for buffered systems with conflict resolution algorithms (see also Georgiadis and Szpankowski (1992) for similar approach applied to stability of token passing rings).

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1. INTRODUCTION

A fundamental issue in the design of any distributed system is its *stability*, loosely defined as its ability to possess required properties in the presence of some disturbances. Hereafter, by stability we understand an ability of a system to keep a quantity of interest (e.g., queue length, waiting time, etc.) in a bounded region, or more precisely the existence of the limiting distribution for a quantity of interest.

In this paper, we investigate the buffered ALOHA system, propose a method of evaluating its stability, and show that this new approach can be extended to a larger class of distributed systems. Stability analysis of the buffered ALOHA system was initiated by Tsybakov and Mikhailov (1979) who obtained a simple bound for the stability region, and exact sufficient and necessary conditions for the ergodicity of the symmetric system (e.g., all input rates and probability of transmissions are the same). It was generalized in Szpankowski (1988) who derived some improved bounds for the stability region, and some new bounds for the instability region. Falin (1981) used the *Lyapunov test function* approach (cf. Tweedie (1976)) to obtain another bound for the stability region (cf. also Szpankowski (1988)). Recently, Rao and Ephremides (1989), using stochastic dominance method constructed the best up-to-date bound for the stability region for not-too-asymmetric buffered ALOHA system. Finally, for a simple arrival process Anantharam (1991) obtained the closure of the ergodicity region (i.e., a stability region for which *there exists* a vector of transmission probabilities that assures system stability).

The exact stability region for the ALOHA model is known *only* for $M = 2$ users system, and the symmetric model. The case $M = 2$ was solved by Tsybakov and Mikhailov (1979) who applied general stability criteria for two-dimensional homogeneous Markov chains derived by Malyshev in his seminal paper Malyshev (1972). These general stability criteria have been extended to higher dimensions by Mensikov (1974), and Malyshev and Mensikov (1981). For two-dimensional homogeneous Markov chains Vaninskii and Lazareva (1988), Fayolle (1989) and Rosenkrantz (1989) relaxed some of Malyshev's restrictions (e.g., boundness of the arrival process). We point out that so far the above criteria for dimension higher than two have been rarely applied in practice despite the fact that they are known for almost twenty years, and up-to-date only few real systems have been analyzed through this approach. However, contrary to token passing rings, stability region of ALOHA is *not* represented by a system of linear equations (with respect to input rates). Actually, it depends on the *distribution* of the arrival process, while in the case of token passing rings (also networks of queues, etc.) the stability region is a linear function of the arrival *rates*.

In this paper, we solve the stability problem of the ALOHA system originally posed by Tsybakov and Mikhailov (1979), that is, we prove necessary and sufficient condition for stability of the ALOHA system. Our approach is novel, and it was already outlined in Szpankowski (1990). This technique was recently rigorized in Georgiadis and Szpankowski (1992) where sufficient and necessary stability condition was established for token passing rings (cf. Georgiadis and Szpankowski (1993)).

Our technique is based on three simple observations. Informally speaking: (i) we use mathematical induction to reduce the dimensionality of the problem; (ii) we apply an old result of Loynes (1962) that allows to assess stability of an isolated general queue with stationary input process; (iii) finally, we use stochastic dominance to verify a technical stationarity requirement in Loynes' criteria and to apply induction arguments. It should be stressed, however, that within this general framework every multidimensional model requires subtle but significant modifications that are often far from obvious.

This paper is organized as follows. In the next section we present sufficient and necessary condition for stability of the buffered ALOHA system. In particular, we derived explicit formulas for the stability regions of the system with $M = 2$ and $M = 3$ users. The proof of our main result is delayed till Section 3 which discusses in length our approach. The proof of the sufficient part (cf. Section 3.1) resembles the idea already used in Georgiadis and Szpankowski (1992), but the necessary part (cf. Section 3.2) requires quite a new approach. From the discussion in Section 3 one should conclude that our method is not restricted to stability analysis of the ALOHA system, and other distributed systems can be approached in a similar manner. To illustrate this we present in Section 4 stability conditions for another distributed system, namely buffered multiqueue system with a conflict resolution algorithm (cf. Capetanakis (1979), Paterakis *et al.* (1987)).

2. MAIN RESULTS

This section presents sufficient and necessary stability conditions for the buffered ALOHA system. The proof of our main Theorem 1 is given in the next section.

2.1 Model Formulation

We start with a short description of the buffered ALOHA system (cf. Tsybakov and Mikhailov (1979), Szpankowski (1986)). The system consists of M distributed users, each having an infinite buffer for storing fixed-length packets. The channel is slotted, and a slot duration is equal to a packet transmission time. The arrival process is i.i.d. with respect to slots, and arrival processes are independent from a user to a user. A nonempty user transmits a packet with probability r_i in a slot, where $i \in \mathcal{M}$ and $\mathcal{M} = \{1, 2, \ldots, M\}$ is the set of users. If two or more users transmit simultaneously, then a collision occurs and the packets must be retransmitted in future. When exactly one packet is transmitted in a slot, then a successful transmission takes place, the packet is removed from its queue, and another packet, if the queue is nonempty, gets its chance to be served. For technical reasons we additionally introduce slot availability. We assume that a slot is available to all users with probability $1-b$, and blocked with probability b. In the standard ALOHA we have $b=0$.

Let N_j^t represent the queue length in the jth user at the beginning of the tth slot, where $t=0,1,\ldots$. Under the above assumptions, the M-dimensional process $\mathbf{N}^t=(N_1^t,N_2^t,\ldots,N_M^t)$ is a *Markov chain* (cf. Tsybakov and Mikhailov (1979), Szpankowski (1986)) since

$$N_j^{t+1}=(N_j^t-Y_j^t)^+ + X_j^t \tag{2.1}$$

where X_j^t represents the number of customers arriving during the tth slot to the jth user with $EX_j^t=\lambda_j<\infty$, and

$$Y_j^t=(1-B^t)R_j^t\left(1-\sum_{k\in\mathcal{M}-\{j\}}R_k^t\chi(N_k^t)\right)^+ . \tag{2.2}$$

In the above: $1-\Pr\{R_j^t=0\}=\Pr\{R_j^t=1\}=r_j$, and $\Pr\{B^t=0\}=1-\Pr\{B^t=1\}=1-b$, and for all $k\in I\!N_0$

$$\chi(k)=\begin{cases}0 & \text{if } k=0\\ 1 & \text{if } k\geq 1\end{cases}$$

where $I\!N_0$ is a set of nonnegative integers.

2.2 Stability Criteria

A multidimensional processes $\mathbf{N}^t=(N_1^t,\ldots,N_M^t)$ (not necessarily Markovian process) is stable if for $\mathbf{x}\in I\!N_0^M$ the following holds

$$\lim_{t\to\infty}\Pr\{\mathbf{N}^t<\mathbf{x}\}=F(\mathbf{x}) \qquad \text{and} \qquad \lim_{\mathbf{x}\to\infty}F(\mathbf{x})=1 \tag{2.3}$$

where $F(\mathbf{x})$ is the limiting distribution function, and by $\mathbf{x}\to\infty$ we understand that $x_j\to\infty$ for all $j\in\mathcal{M}=\{1,\ldots,M\}$. If a weaker condition holds, namely,

$$\lim_{\mathbf{x}\to\infty}\lim_{t\to\infty}\inf\ \Pr\{\mathbf{N}^t<\mathbf{x}\}=1 , \tag{2.4}$$

then the process is called *substable* by Loynes (1962), *tight* by Breiman (1968), and *bounded in probability* by Meyn and Tweedie (1992). Otherwise, the system is unstable (for more details see Loynes (1962), Borovkov (1976), Walrand (1988)).

We start with an informal overview of our approach. First of all, we construct a modified ALOHA system as follows. Let $\mathcal{P}=(\mathcal{S},\mathcal{U})$ be a partition of $\mathcal{M}$ such that users in $\mathcal{S}\neq\mathcal{M}$ work exactly in the same manner as in the original ALOHA model, while users in $\mathcal{U}$ *persistently* attempt to send (dummy) packets even if their buffers are empty (e.g., dummy packets). We call users in $\mathcal{U}$ *persistent* (or jamming or saturated), and users in $\mathcal{S}$ *nonpersistent.* Note that the system consisting of users in $\mathcal{S}$ forms a smaller copy of the original ALOHA system with a new probability of slot availability, namely $1-b=\prod_{j\in\mathcal{U}}(1-r_j)$. In passing, we observe that our sets $\mathcal{U}$ and $\mathcal{S}$ resemble *faces* and *induced Markov chains* of Malyshev and Mensikov (1981).

For a given partition $\mathcal{P}=(\mathcal{S},\mathcal{U})$ of $\mathcal{M}$ such that $\mathcal{S}\neq\mathcal{M}$, let $\overline{\mathbf{N}}^t_{\mathcal{P}}=(\overline{\mathbf{N}}^t_{\mathcal{S}},\overline{\mathbf{N}}^t_{\mathcal{U}})$ denote the queue lengths in the modified system where $\overline{\mathbf{N}}^t_{\mathcal{S}}$ (resp. $\overline{\mathbf{N}}^t_{\mathcal{U}}$) represents the queue lengths in $\mathcal{S}$ (resp. in $\mathcal{U}$). It can be proved (cf. Tsybakov and Mikhailov (1979) and Szpankowski (1986)) that $\overline{\mathbf{N}}^t_{\mathcal{P}}$ dominates the original queueing process $\mathbf{N}^t$, that is,

$$\mathbf{N}^t\leq_{st}\overline{\mathbf{N}}^t_{\mathcal{P}} \tag{2.5}$$

provided $\mathbf{N}^0=\overline{\mathbf{N}}^0_{\mathcal{P}}$.

Note that by our construction, the process $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is an $|\mathcal{S}|$-dimensional Markov chain that mimics the behavior of the ALOHA system. Since $|\mathcal{S}|<M$, we can apply mathematical induction to establish its stability. We further assume that the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is stationary and ergodic (this is explained in details in Section 3). We denote this stationary version as $\overline{\mathbf{N}}^0_{\mathcal{S}}$ to indicate that the process starts from the stationary distribution.

Let $Y^t_j(\mathcal{S})$ be the output process from the jth queue in the dominant system. We write $P^j_{succ}(\mathcal{S})=EY^t_j(\mathcal{S})$ to denote the probability of a successful transmission from the jth user in the dominant system $P^j_{succ}(\mathcal{S})=EY^t_j(\mathcal{S})$ provided $Y^t_j(\mathcal{S})$ is a stationary sequence. This is assumed to hold as a simple consequence of our assumption regarding stationarity of $\overline{\mathbf{N}}^t_{\mathcal{S}}$. Then, by (2.2) (cf. Tsybakov and Mikhailov (1979), Szpankowski (1986))

$$P^j_{succ}(\mathcal{S})=r_j\prod_{k\in\mathcal{U}-\{j\}}(1-r_k)\sum_{\mathbf{z}_{\mathcal{S}}\in\Theta_{\mathcal{S}}}\left(\Pr\{\chi(\overline{\mathbf{N}}^0_{\mathcal{S}})=\mathbf{z}_{\mathcal{S}}\}\prod_{k\in\mathcal{S}-\{j\}}(1-r_k)^{z_k}\right)\ , \tag{2.6}$$

where the summation is over the set

$$\Theta_{\mathcal{S}}=\{\mathbf{z}_{\mathcal{S}}:\mathbf{z}_{\mathcal{S}}=(z_1,\ldots,z_{|\mathcal{S}|}),\ z_j\in\{0,1\},\ j\in\mathcal{S}\}\ . \tag{2.7}$$

In the above, $\chi(\overline{\mathbf{N}}^0_{\mathcal{S}})=(\chi(\overline{N}^0_{i_1}),\ldots,\chi(\overline{N}_{i_{|\mathcal{S}|}}))$ where $i_j\in\mathcal{S}$ for all $j=1,\ldots,|\mathcal{S}|$, and by $k\in\mathcal{A}-\{j\}$ we mean that k belongs to $\mathcal{A}$ and it is *not* equal to j (even if $j\notin\mathcal{A}$). We set by convention $\Pr\{\chi(\overline{\mathbf{N}}^0_{\mathcal{S}})=1\}=1$ in the case $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is unstable.

Observe also that one can view (2.6) as the probability of success in an $\mathcal{S}$-copy of the ALOHA system with slot availability equal to $1-b_{\mathcal{U}}=\prod_{k\in\mathcal{U}}(1-r_k)$. Furthermore, if one considers a version of ALOHA with (external) slot availability equal to $1-b$, then the right-hand side of (2.6) must be multiplied by $1-b$.

To present our result in a compact form we introduce some more notation. Let $\mathcal{R}_{\mathcal{M}}$, $\mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}_{\mathcal{U}}$ be respectively stability regions for the whole system, and the systems consisting of nonpersistent users $\mathcal{S}$ and persistent ones $\mathcal{U}$ in the modified ALOHA system under the partition

$\mathcal{P} = (\mathcal{S}, \mathcal{U})$. We set $\mathcal{R}_{\mathcal{P}} = \mathcal{R}_{\mathcal{S}} \cap \mathcal{R}_{\mathcal{U}}$. Sometimes, we write more explicitly $\mathcal{R}_{\mathcal{M}}[b]$ and/or $\mathcal{R}_{\mathcal{S}}[b]$ for the stability region of a system with slot availability $1-b$. We use below $\mathcal{R}_{\mathcal{M}}$ ($\mathcal{R}_{\mathcal{S}}$) and $\mathcal{R}_{\mathcal{M}}[b]$ ($\mathcal{R}_{\mathcal{S}}[b]$) interchangeable to simplify notations as long as it does not lead to a confusion. In addition, we define $\mathcal{M}_k = \mathcal{M} - \{k\}$ so that the corresponding partition is $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$. Thus, $\mathcal{R}_{\mathcal{M}_k}[r_k]$ denotes the stability region of an $M-1$-dimensional ALOHA system with probability of blocking $b = r_k$. We also write $\boldsymbol{\lambda}_{\mathcal{S}} = \{\lambda_i\}_{i \in \mathcal{S}}$.

The next theorem is our main finding, and it provides sufficient and necessary condition for stability of the buffered ALOHA system. It is proved in Section 3. Below, we write iff* to mean: *if and only if* with possible exception of the boundary points of a stability region. (In fact, our technique usually does not establish stability of boundary points, however, in most cases only minor modifications are necessary to prove instability of boundary points; e.g., Georgiadis and Szpankowski (1992)).

Theorem 1. *The stability region of the buffered ALOHA system with slot availability* $1-b$ *is*

$$\mathcal{R}_{\mathcal{M}}[b] = \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_k < P^k_{succ}(\mathcal{M}_k), \quad \textit{and} \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}[1-(1-b)(1-r_k)]\} \ . \tag{2.8a}$$

*In particular, the original ALOHA system is stable iff** $\boldsymbol{\lambda} \in \mathcal{R}_{\mathcal{M}}[0]$. ■

Remarks. (i) Actually, in Section 3 we prove that the stability region $\mathcal{R}_{\mathcal{M}}$ can also be represented as

$$\mathcal{R}_{\mathcal{M}} = \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_j < P^j_{succ}(\mathcal{M}_k), \ \text{ for all } \ j \in \mathcal{M}\} \ , \tag{2.8b}$$

where one has to interpret the above in the spirit of (2.8a). That is, for the partition $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$ the stability of (the smaller copy of the system) $\mathcal{M}_k$ is computed recursively under the assumption that $b = r_k$ (i.e., the kth queue is saturated). In fact, in Section 3 we prove even stronger results, namely, a local stability condition. More precisely, for a partition $\mathcal{P} = (\mathcal{S}, \mathcal{U})$ we prove that $\mathcal{R}_{\mathcal{U}} = \{\boldsymbol{\lambda}_{\mathcal{U}} : \ \lambda_j < P^j_{succ}(\mathcal{S}) \ \text{ for all } \ j \in \mathcal{U}\}$, and $\mathcal{R}_{\mathcal{P}} = \mathcal{R}_{\mathcal{U}} \cap \mathcal{R}_{\mathcal{S}}$, where $\mathcal{R}_{\mathcal{S}}$ (more specifically, $\mathcal{R}_{\mathcal{S}}[b_{\mathcal{U}}]$ with $b_{\mathcal{U}} = 1 - \prod_{j \in \mathcal{U}}(1-r_j)$) is given recursively by the induction.

(ii) Observe that $\mathcal{R}_{\mathcal{M}} = \bigcup_{\mathcal{P}_k} \mathcal{R}_{\mathcal{P}_k} = \bigcup_{\mathcal{P}} \mathcal{R}_{\mathcal{P}}$ where $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$, that is, only partitions $\mathcal{P}_k$ contributes to the final stability region. This follows from $(1-r_k)^{z_k} \geq (1-r_k)$ where $z_k \in \{0,1\}$, and the following monotonicity property of the probability of success $P^j_{succ}(\mathcal{S})$ (cf. Tsybakov and Mikhailov (1979), and Szpankowski (1986))

$$\mathcal{S}' \subset \mathcal{S} \quad \Longrightarrow \quad P^j_{succ}(\mathcal{S}') \leq P^j_{succ}(\mathcal{S}) \ .$$

(iii) From Theorem 1 one concludes that the system is unstable for $\boldsymbol{\lambda} \in \overline{\mathcal{R}}_{\mathcal{M}}$ where

$$\begin{aligned}
\overline{\mathcal{R}}_{\mathcal{M}} &= \bigcap_{k=1}^{M} \{\boldsymbol{\lambda} : \ \lambda_k > P^k_{succ}(\mathcal{M}_k) \quad \text{or} \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \notin \mathcal{R}_{\mathcal{M}_k}\} \\
&= \bigcap_{k=1}^{M} \{\boldsymbol{\lambda} : \ \lambda_k > P^k_{succ}(\mathcal{M}_k) \quad \text{and} \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}\} \cup \{\boldsymbol{\lambda}_{\mathcal{M}_k} \notin \mathcal{R}_{\mathcal{M}_k}\} \ .
\end{aligned}$$

It should be clear from our inductive arguments in Section 3 that to establish the above we need only to show that the *original* system is unstable for $\{\lambda_k > P^k_{succ}(\mathcal{M}_k) \ \text{ and } \ \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}\}$. This is proved in Section 3.2.

(iv) Our proof of Theorem 1 can be used to give a detailed characterization of the instability region. More precisely, we can partition the instability region $\overline{\mathcal{R}}_{\mathcal{M}}$ into subregions $\overline{\mathcal{R}}^{(k_1,\ldots,k_d)}_{\mathcal{M}}$ ($k_i \in \mathcal{M}$ for $i = 1, \ldots, d$) such that for $\lambda \in \overline{\mathcal{R}}^{(k_1,\ldots,k_d)}_{\mathcal{M}}$ only queues $N^t_{k_1}, \ldots, N^t_{k_d}$ are unstable while the remaining are stable. For example, of Section 3.2 we show that in the region

$$\overline{\mathcal{R}}^{(k)}_{\mathcal{M}} = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_k > P^k_{succ}(\mathcal{M}_k) \ \text{ and } \ \ \lambda_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}\}$$

only the kth queue is unstable while the others are stable.

(v) The stability region for the ALOHA system (and several others) can be simplified if one considers the so called *closure* of the stability region which is defined as the set of all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ such that *there exists* a vector of transmission probabilities $(r_1, \ldots, r_M)$ for which the ALOHA system is stable. Tsybakov and Mikhailov (1979) conjectured that the closure $\mathcal{C}$ of the stability region for the ALOHA can be characterized as follows

$$\mathcal{C} = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_j < r_j \prod_{k \neq j} (1 - r_k) \ , \ 0 \leq r_j \leq 1 \text{ for all } j \in \mathcal{M}\} \ .$$

Anantharam (1991) proved this fact for a simple input process. However, for general ALOHA it is still an open problem. We hope that our Theorem 1 will help to settle this conjecture.

2.3 Special Cases and Bounds

Provided Theorem 1 is proved, we shall show how to apply it to establish stability regions for $M = 2$ and $M = 3$, and the symmetric ALOHA system. We also discuss some bounds on $\mathcal{R}_{\mathcal{M}}$. Stability characterization for $M = 3$ is new. (We shall write below $\mathcal{R}$ instead of $\mathcal{R}_{\mathcal{M}}$ to simplify notations.)

We start with $M = 2$, and let $\mathcal{P}_1 = (\mathcal{M}_1, \{1\})$ and $\mathcal{P}_2 = (\mathcal{M}_2, \{2\})$ where $\mathcal{M}_1 = \{2\}$ and $\mathcal{M}_2 = \{1\}$. We write $\mathcal{R}_i$ to denote the stability region for the partition $\mathcal{P}_i$ for $i = 1, 2$. Below, only the construction of $\mathcal{R}_1$ is discussed in details. By (2.6) we have

$$\begin{aligned}
P^1_{succ}(\mathcal{M}_1) &= r_1(\Pr\{\overline{N}^0_2 = 0\} + (1 - r_2)\Pr\{\overline{N}^0_2 \geq 1\}) \\
P^2_{succ}(\mathcal{M}_1) &= r_2(1 - r_1) \ .
\end{aligned}$$

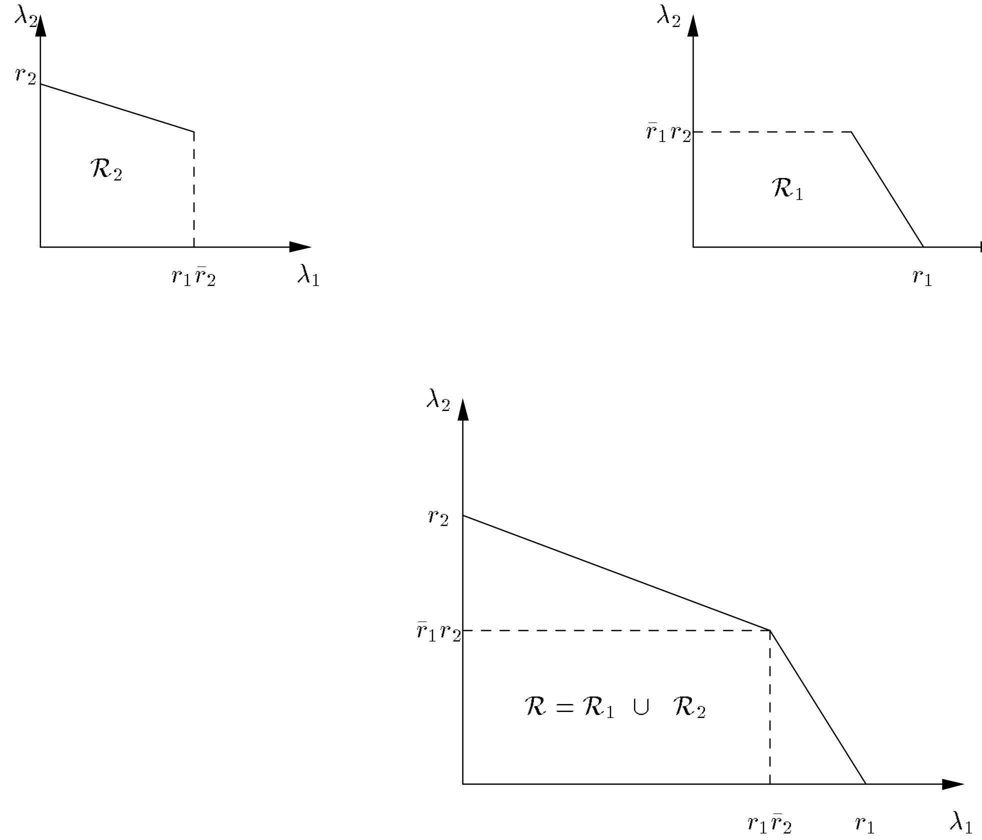

Figure 1: Stability region $\mathcal{R}$ for $M = 2$ as a union of stability regions $\mathcal{R}_1$ and $\mathcal{R}_2$.

But, for $\lambda_2 < P^2_{succ}(\mathcal{M}_1)$ the probability $\Pr\{\overline{N}^0_2 \geq 1\} = \lambda_2/(r_2(1-r_1))$, and hence one obtains the stability region $\mathcal{R}_1$ (see Corollary 2 below for the formula on $\mathcal{R}_1$ and Figure 1 for a graphical illustration). In a similar manner, considering $\mathcal{M}_2$ we obtain the stability region $\mathcal{R}_2$. The whole stability region $\mathcal{R}$ is the sum of $\mathcal{R}_1$ and $\mathcal{R}_2$ as illustrated in Figure 1.

Putting everything together, we summarize the stability result for $M = 2$ in the following corollary. Under stronger assumptions, this result was already known to Tsybakov and Mikhailov (1979) (cf. also Rao and Ephremides (1989)).

Corollary 2. *For $M = 2$, the buffered ALOHA system is stable for all $(\lambda_1, \lambda_2) \in \mathcal{R}$ such that*

$$\mathcal{R} = \{\lambda_1 < r_1(1-\lambda_2/\bar{r}_1) \text{ and } \lambda_2 < \bar{r}_1 r_2\} \cup \{\lambda_1 < r_1\bar{r}_2 \text{ and } \lambda_2 < r_2(1-\lambda_1/\bar{r}_2)\}\ , \qquad (2.9a)$$

and the system is unstable for $(\lambda_1, \lambda_2) \in \overline{\mathcal{R}}$ where

$$\overline{\mathcal{R}} = \{\lambda_1 > r_1(1-\lambda_2/\bar{r}_1) \text{ or } \lambda_2 > \bar{r}_1 r_2\} \cap \{\lambda_1 > r_1\bar{r}_2 \text{ or } \lambda_2 > r_2(1-\lambda_1/\bar{r}_2)\}\ , \qquad (2.9b)$$

where $\bar{r}_j = 1 - r_j$. ∎

Now, we consider the case $M = 3$ which is more intricate. We only investigate three partitions $\mathcal{P}_i = (\mathcal{M}_i, \{i\})$ where $\mathcal{M}_1 = \{2,3\}$, $\mathcal{M}_2 = \{1,3\}$ and $\mathcal{M}_3 = \{1,2\}$ (cf. Remark (ii)), and only the first partition will be discussed in details. As before, the stability region $\mathcal{R}$ is the sum of three regions $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ each corresponding to $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively.

Consider now $\mathcal{R}_1$. The dominant system can be viewed as a two-user ALOHA system with appropriate probability for the slot availability. Such a system was analyzed by Nain (1985). We first briefly summarize some of Nain's results adopted to our setting.

Let $F_1(x,y)$ denote the generating function of (N_2^t, N_3^t) with slot availability $1-b = 1-r_1$. Then, with a minor modification, it is proved in Nain (1985) (see also Szpankowski (1986)) that

$$\lambda_2 = \bar{r}_1 r_2 \bar{r}_3 (1 - F_1(0,1)) + \bar{r}_1 r_2 r_3 (F_1(1,0) - F_1(0,0)) \ , \qquad (2.10a)$$

$$\lambda_3 = \bar{r}_1 \bar{r}_2 r_3 (1 - F_1(1,0)) + \bar{r}_1 r_2 r_3 (F_1(0,1) - F_1(0,0)) \ , \qquad (2.10b)$$

where, as before, $\bar{r}_i = 1 - r_i$. Moreover, for $z_2, z_3 \in \{0,1\}$ we use the following simplified notation $P_1(z_2,z_3) = \Pr\{\chi(\overline{N}_2) = z_2, \chi(\overline{N}_3) = z_3\}$ with the first user being a persistent one. From Nain (1985) we have

$$P_1(0,0) = \left(1 - \frac{\lambda_2}{\bar{r}_3 \bar{r}_1} - \frac{\lambda_3}{r_3 \bar{r}_1}\right) \exp\left(\frac{\gamma(1)}{2\pi i}\right) \int_{|t|=1} \frac{\log g(t)}{t(t-\gamma(1))} dt \qquad (2.11a)$$

or

$$P_1(0,0) = \left(1 - \frac{\lambda_2}{r_2 \bar{r}_1} - \frac{\lambda_3}{\bar{r}_2 \bar{r}_1}\right) \exp\left(\frac{\gamma(1)}{2\pi i}\right) \int_{|t|=1} \frac{\log g_1(t)}{t(t-\gamma(1))} dt \qquad (2.11b)$$

depending whether $P_1(0,0)$ is computed from $F_1(0,y)$ or $F_1(x,0)$. We note that $\gamma(1)$ and $g(t)$ depend of the input rates λ_2 and λ_3 ! For example, $F_1(1,0)$ corresponding to (2.11a) becomes

$$\begin{aligned} F_1(1,0) \quad &= \quad \left(1 - \frac{\lambda_2}{\bar{r}_3 \bar{r}_1} - \frac{\lambda_3}{r_3 \bar{r}_1}\right) \left(\frac{1 - r_3 \bar{r}_1}{1 - r_3 \bar{r}_1 - r_2 \bar{r}_1}\right) \int_{|t|=1} \frac{\log g(t)}{((t-\gamma(1))^2} dt \\ &- \quad \frac{r_2 \bar{r}_1}{1 - r_2 \bar{r}_1 - r_3 \bar{r}_1} P_1(0,0) \end{aligned}$$

The region of validity of (2.11) is defined in Nain (1985). In (2.11), $\gamma(x)\,|_{x=1}$ is the inverse of a conformal mapping of a unit circle onto a curve L_x defined in Nain (1985) (p. 54 and Lemma 4.1). The functions $g(t)$ and $g_1(t)$ are defined in Nain (1985), too.

Now we are ready to present the stability region for $M = 3$ ALOHA system. By (2.6) for $\mathcal{P}_1 = (\mathcal{M}_1, \{1\})$ we have

$$P^1_{succ}(\mathcal{M}_1) = r_1 \left(P_1(0,0) + P_1(1,0)(1-r_2) + P_1(0,1)(1-r_3) + P_1(1,1)(1-r_2)(1-r_3)\right) \ ,$$

$$P^2_{succ}(\mathcal{M}_1) = r_2(1-r_1)(1-(1-F_1(1,0))r_3) \ ,$$

$$P^3_{succ}(\mathcal{M}_1) = r_3(1-r_1)(1-(1-F_1(0,1))r_2) \ ,$$

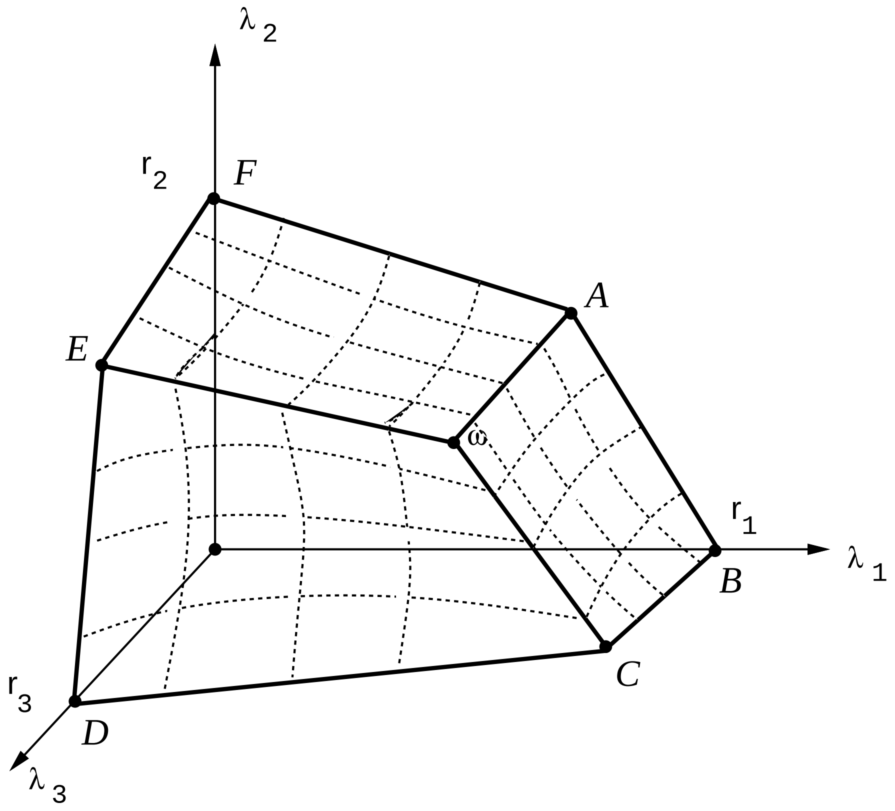

Figure 2: Stability region for $M = 3$ users in the slotted ALOHA system.

where the probabilities $P_1(z_2, z_3)$ must be computed according to (2.10)-(2.12). For example, using (2.10) we can show that

$$P^1_{succ}(\mathcal{M}_1) = r_1 \left(1 - \frac{\lambda_2 \bar{r}_2/\bar{r}_1 + \lambda_3 \bar{r}_3/\bar{r}_1 + r_2 r_3 (P_1(0,0) - 1)}{1 - r_2 - r_3}\right) . \tag{2.12}$$

In a similar manner, we can express $P^2_{succ}(\mathcal{M}_1)$ and $P^3_{succ}(\mathcal{M}_1)$ in terms of $P_1(0,0)$.

In summary, we obtain the following corollary.

Corollary 3. *The buffered ALOHA system is stable iff* $(\lambda_1, \lambda_2, \lambda_3) \in R = \bigcup_{i=1}^3 \mathcal{R}_i$ *where*

$$\begin{aligned}
\mathcal{R}_1 &= \{\lambda_1 < P^1_{succ}(\mathcal{M}_1),\ \lambda_2 < r_2\bar{r}_1(1-(1-F_1(1,0))r_3),\ \lambda_3 < r_3\bar{r}_1(1-(1-F_1(0,1))r_2)\} \\
\mathcal{R}_2 &= \{\lambda_1 < r_1\bar{r}_2(1-(1-F_2(1,0))r_3),\ \lambda_2 < P^2_{succ}(\mathcal{M}_2),\ \lambda_3 < r_3\bar{r}_2(1-(1-F_2(0,1))r_1)\} \\
\mathcal{R}_3 &= \{\lambda_2 < r_1\bar{r}_3(1-(1-F_3(1,0))r_2),\ \lambda_2 < r_2\bar{r}_3(1-(1-F_3(0,1))r_1),\ \lambda_3 < P^3_{succ}(\mathcal{M}_3)\}
\end{aligned}$$

where the appropriate probabilities above are computed from the Nain's (1985) model as discussed in (2.10)-(2.12) with some obvious modifications. ■

Stability region $\mathcal{R}$ for $M = 3$ is shown in our Figure 2. Note that the following points belong to the boundary of the stability region: $\omega = (\lambda_1, \lambda_2, \lambda_3) = (r_1\bar{r}_2\bar{r}_3, \bar{r}_1 r_2 \bar{r}_3, \bar{r}_1\bar{r}_2 r_3)$,

$A = (r_1\bar{r}_2, \bar{r}_1 r_2, 0)$, and $B = (r_1, 0, 0)$, $C = (r_1\bar{r}_3, 0, \bar{r}_1 r_3)$, $D = (0, 0, r_3)$, $E = (0, r_2\bar{r}_3, \bar{r}_2 r_3)$ and $F = (0, r_2, 0)$. In passing, we stress the fact that the probability of success $P^j_{succ}(\mathcal{M}_k)$ *does* depend explicitly on the probability $P_1(0,0)$, which is a *nonlinear* function of the input rates. This implies that the boundaries of the stability region for the ALOHA system for $M \geq 3$ are *not* linear functions of $(\lambda_1, \lambda_2, \lambda_3)$.

A generalization of the above to $M > 3$ is even more challenging, since we need to estimate the probability of empty/nonempty buffers in three and higher dimensional ALOHA systems. Nevertheless, for $M > 3$ some bounds are easy to obtain from Theorem 1. For example, the bounds derived by Tsybakov and Mikhailov (1979) (cf. also Szpankowski (1988)) directly follow from (2.6) and (2.8). Indeed, since $\prod_{k\in\mathcal{M}_j}(1-r_k)^{z_k} \geq \prod_{k\in\mathcal{M}_j}(1-r_k)$, one immediately proves from (2.6) that $\lambda_j < r_j \prod_{k\in\mathcal{M}_j}(1-r_k)$ for $j \in \mathcal{M}$ is sufficient for stability of the ALOHA system. On the other hand, since $\prod_{k\in\mathcal{M}_j}(1-r_k)^{z_k} \leq 1$ we prove that $\lambda \geq r_j$ for some $j \in \mathcal{M}$, is sufficient for instability of the ALOHA system.

The above simple bounds can be used to establish sufficient and necessary conditions for stability of the symmetric ALOHA system. We prove the following result which was already known to Tsybakov and Mikhailov (1979)

Corollary 4. *Let $r_j = r$ and $\lambda_j = \lambda$ for all $j \in \mathcal{M}$. Then, such a symmetric ALOHA system is stable if and only if the following holds*

$$\lambda < r(1-r)^{M-1} \ . \tag{2.13}$$

Proof. This directly follows from Theorem 1 since in the symmetric case we really deal with a one dimensional problem. The stability region in this case is the intersection of the line $\lambda_1 = \lambda_2 = \cdots = \lambda_M = \lambda$ with the region $\mathcal{R}_\mathcal{M}$ as defined in (2.8) for $r_1 = r_2 = \cdots = r_M = r$. Even simpler proof can be obtained, by noting that (2.13) is a direct consequence of the above upper bound $\lambda_j < r_j \prod_{k\in\mathcal{M}_j}(1-r_k)$ for $j \in \mathcal{M}$. Setting in this inequality the symmetric model assumptions, we obtain (2.13). From Tsybakov and Mikhailov (1979), and Szpankowski (1986) we also know that (2.13) is necessary for the stability. ∎

To obtain more sophisticated bounds for the stability region $\mathcal{R}_\mathcal{M}$, one needs a tighter estimate for the probability $\Pr\{\chi(\overline{\mathbf{N}}^t_\mathcal{S}) = \mathbf{z}_\mathcal{S}\}$ in (2.6) (cf. Szpankowski (1988), and Rao and Ephremides (1989)). For example, in Rao and Ephremides (1989) all probabilities $\Pr\{\chi(\overline{\mathbf{N}}^t_\mathcal{S}) = \mathbf{z}_\mathcal{S}\}$ in (2.6) are skillfully bounded by a one dimensional probability $\Pr\{\overline{N}^t_k \geq 1\}$ for $k \in \mathcal{S}$.

3. ANALYSIS: AN ALTERNATIVE APPROACH

In this section, we present our approach to compute stability conditions for multiqueue systems. We concentrate on proving Theorem 1, however, generality of the approach should be apparent from our discussion. We first prove sufficient conditions (cf. Section 3.1), and then necessary conditions (cf. Section 3.2) for stability. In the proof below, we shall use two general results, namely the so called *isolation lemma* (cf. Lemma 5), and Loynes' scheme Loynes (1962) (cf. Borovkov (1976)) adopted to our situation (cf. Lemma 6).

We start with the isolation lemma which states that for substability of a multidimensional process $\mathbf{N}^t$ (not necessary Markovian) one requires substability of its components. More precisely, we prove the following.

Lemma 5. (i) *If for all $j \in \mathcal{M}$ the one dimensional processes N_j^t are substable, then the M-dimensional process $\mathbf{N}^t = (N_1^t, N_2^t, \ldots, N_M^t)$ is substable.*

(ii) *If for some j, say j^*, N_{j*}^t is unstable, then $\mathbf{N}^t$ is also unstable.*

(iii) *If $\mathbf{N}^t$ is a Markov chain defined on a countable state space, then the stability of N_j^t for all $j \in \mathcal{M}$ implies stability of the multidimensional Markov chain $\mathbf{N}^t$.*

Proof. We first prove part (i). By definition (2.4), for all $j \in \mathcal{M}$ we have $\lim_{x_j,t\to\infty} \sup \Pr\{N_j^t > x_j\} = 0$. But,

$$\begin{aligned} 1 \geq \lim_{x_j\to\infty} \liminf_{n\to\infty} \Pr\{N_j^n \leq x_j, \text{ for } \quad j & = 1, 2, \ldots, M\} \\ & \geq 1 - \sum_{j=1}^{M} \lim_{x_j\to\infty} \limsup_{n\to\infty} \Pr\{N_j^n > x_j\} = 1 \ . \end{aligned}$$

Thus, $\lim_{x\to\infty} \lim_{t\to\infty} \inf \Pr\{\mathbf{N}^t < \mathbf{x}\} = 1$, as needed. If $\mathbf{N}^t$ is a Markov chain defined on a countable state space, then substability implies stability since such a Markov chain always converges to a random variable, which might be dishonest. This proves (iii). For results concerning Markov chains defined on more general spaces the reader is referred to Meyn and Tweedie (1992).

For part (ii) we notice that instability of N_{j*}^t implies $\lim_{x_{j*}\to\infty} \lim_{t\to\infty} \inf \Pr\{N_{j*}^t < x_{j*}\} < 1$. Hence,

$$\lim_{x_{j*}\to\infty} \lim_{t\to\infty} \inf \Pr\{\mathbf{N}^t < \mathbf{x}\} \leq \lim_{x_{j*}\to\infty} \lim_{t\to\infty} \inf \Pr\{N_{j*}^t < x_{j*}\} < 1$$

which proves Lemma 5. ∎

Our second result is a simple consequence of the Loynes' scheme (cf. Loynes (1962) and Borovkov (1976)). It provides stability criteria for a general single queue described by the following stochastic equation

$$N_j^{t+1} = (N_j^t - Y_j^t)^+ + X_j^t \tag{3.1}$$

with stationary input sequences.

Lemma 6. [Loynes 1962), Borovkov (1976)]. *Let the pair* (X_j^t, Y_j^t) *be a strictly stationary and ergodic process. We denote by* $\lambda_j = EX_j$ *and* EY_j *the corresponding mean values of* X_j^t *and* Y_j^t *respectively. Then the following holds*

(i) *if* $\lambda_j < EY_j$*, then the queue length* N_j^t *is stable in the sense of definition (2.3),*

(ii) *if* $\lambda_j > EY_j$*, then the queue is unstable,* **and** $\lim_{t\to\infty} N_j^t = \infty$ *(a.s.).*

Proof. The proof is a direct consequence of well known results. Indeed, set $W_j^{t+1} = N_j^{t+1} - X_j^t$ and $U_j^t = X_j^{t-1} - Y_j^t$. Then,

$$W_j^{t+1} = (W_j^t + U_j^t)^+ \ . \tag{3.2}$$

and, by Loynes (1962) and Borovkov (1976) N_j^t is stable for $EU_j^t < 0$ and unstable for $EU_j^t > 0$. To prove the second part of (ii), we note that $W_j^{t+1} \geq W_j^0 + \sum_{k=0}^t U_j^t$, and by Birkhoff's *Individual Ergodic Theorem* (Breiman (1968)) $\sum_{k=0}^t U_j^t \to \infty$ (a.s.) provided $EU_j^t > 0$. ■

3.1 Sufficient Conditions

We use mathematical induction to establish sufficient condition for stability of the Markov chain $\mathbf{N}^t$ describing the ALOHA system. This part of the proof resembles the idea already used by Georgiadis and Szpankowski (1992).

For $M = 1$ the proof is easy. It suffices to note that the average drift becomes $E\{N^{t+1} - N^t | N^t \geq 1\} = \lambda - EY^t$. Hence, the proof follows from the Lyapunov function method (cf. Tweedie (1976), Szpankowski (1990)).

Now, we apply induction. We assume that for any $\mathcal{S}$ such that $|\mathcal{S}| < M$ the set $\mathcal{R}_\mathcal{S}$ (more specifically, $\mathcal{R}_\mathcal{S}[b_\mathcal{U}]$ where $b_\mathcal{U} = 1 - \prod_{j\in\mathcal{U}}(1 - r_j)$) is the stability region of the $\mathcal{S}$-copy of the ALOHA system under the partition $\mathcal{P} = (\mathcal{S}, \mathcal{U})$. We shall prove that the stability region $\mathcal{R}_\mathcal{M}$ of the original M-users ALOHA system is given by formula (2.8a) in Theorem 1.

We consider the modified system under the partition $\mathcal{P} = (\mathcal{S}, \mathcal{U})$ that stochastically dominates the original system, as stated in (2.5). Hence, we concentrate on proving stability condition for the dominant system represented by $\overline{\mathbf{N}}_\mathcal{P}^t = (\overline{\mathbf{N}}_\mathcal{S}^t, \overline{\mathbf{N}}_\mathcal{U}^t)$. We first consider users in $\mathcal{S}$, and establish stability condition for $\overline{\mathbf{N}}_\mathcal{S}^t$. Note that the set of users restricted to $\mathcal{S}$ is a *smaller* copy of the ALOHA model, with the probability of slot availability equal to $1 - b_\mathcal{U} = \prod_{k\in\mathcal{U}-\{j\}}(1 - r_k)$. Hence, the probability of success $P_{succ}^j(\mathcal{S})$ is given by (2.6). Since $|\mathcal{S}| < M$, by mathematical induction arguments $\overline{\mathbf{N}}_\mathcal{S}^t$ is stable for $\boldsymbol{\lambda}_\mathcal{S} \in \mathcal{R}_\mathcal{S}[b_\mathcal{U}]$ where $\boldsymbol{\lambda}_\mathcal{S} = (\lambda_{i_1}, \ldots, \lambda_{i_{|\mathcal{S}|}})$ with $i_j \in \mathcal{S}$ for $1 \leq i_j \leq |\mathcal{S}|$.

We now derive stability condition for a persistent queue $j \in \mathcal{U}$. The idea is to apply Lemma 6 to an isolated persistent queue $j \in \mathcal{U}$. For this we need to establish stationarity and ergodicity of the output process $Y_j^t(\mathcal{S})$.

We proceed as follows. Let $\boldsymbol{\lambda}_{\mathcal{S}} \in \mathcal{R}_{\mathcal{S}}[b_{\mathcal{U}}]$, and consider a persistent user, say $j \in \mathcal{U}$. As proved above, the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{S}}$ is stable in $\mathcal{R}_{\mathcal{S}}[b_{\mathcal{U}}]$. More precisely, this Markov chain is ergodic, hence there exists a unique honest stationary distribution π for this process. But, $(\overline{\mathbf{N}}^t_{\mathcal{S}}, Y_j^{t+1}(\mathcal{S}))$ is another $|\mathcal{S}|+1$-dimensional Markov chain. It is easy to see that this chain is irreducible and aperiodic, too. Moreover, $Y_j^t(\mathcal{S})$ is bounded from the above by one, hence $(\overline{\mathbf{N}}^t_{\mathcal{S}}, Y_j^{t+1}(\mathcal{S}))$ is substable. By Lemma 5(iii), this implies that the $|\mathcal{S}|+1$-dimensional Markov chain is stable, and also ergodic. Let its stationary distribution be $\bar{\pi}$.

Now we construct a stationary version of the process $(\overline{\mathbf{N}}^t_{\mathcal{S}}, Y_j^{t+1}(\mathcal{S}))$ by starting it with the initial distribution $\bar{\pi}$. This will imply that the process $Y_j^t(\mathcal{S})$ for $j \in \mathcal{U}$ is stationary and ergodic for such an initial distribution. Then, by Lemma 6 the jth persistent queue is stable if $\lambda_j < EY_j^t(\mathcal{S}) = P^j_{succ}(\mathcal{S})$ where $P^j_{succ}(\mathcal{S})$ is given by (2.6). This is true for every persistent queue $j \in \mathcal{U}$, and hence by Lemma 5 the process $\overline{\mathbf{N}}^t_{\mathcal{U}}$ is stable for $\boldsymbol{\lambda}_{\mathcal{U}} \in \mathcal{R}_{\mathcal{U}}$ where

$$\mathcal{R}_{\mathcal{U}} = \{\boldsymbol{\lambda}_{\mathcal{U}} : \ \lambda_j < P^j_{succ}(\mathcal{S}) \ \text{ for all } \ j \in \mathcal{U}\} \ , \tag{3.3}$$

and the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{P}} = (\overline{\mathbf{N}}^t_{\mathcal{S}}, \overline{\mathbf{N}}^t_{\mathcal{U}})$ is stable for $\boldsymbol{\lambda} \in \mathcal{R}_{\mathcal{P}} = \mathcal{R}_{\mathcal{S}}[b_{\mathcal{U}}] \cap \mathcal{R}_{\mathcal{U}}$. This is particularly true for $\mathcal{S} = \mathcal{M}_k$.

So far, we have established stability condition for a *given* partition $\mathcal{P} = (\mathcal{S}, \mathcal{U})$ of $\mathcal{M}$. Clearly, the dominant system represented by $\overline{\mathbf{N}}^t$ is stable if *there exists* a partition $\mathcal{P}$ such that $\overline{\mathbf{N}}^t_{\mathcal{P}}$ is stable. Therefore, the stability region for $\overline{\mathbf{N}}^t$ becomes $\mathcal{R}_{\mathcal{M}} = \bigcup_{\mathcal{P}} \mathcal{R}_{\mathcal{P}}$ where the sum is over all partitions of $\mathcal{M}$ such that $\mathcal{S} \neq \mathcal{M}$. However, from Remark (ii) we know that $\mathcal{R}_{\mathcal{M}} = \bigcup_{\mathcal{P}_k} \mathcal{R}_{\mathcal{P}_k} = \bigcup_{\mathcal{P}} \mathcal{R}_{\mathcal{P}}$, hence sufficient part of Theorem 1 is proved.

3.2 Necessary Conditions

In this section we prove necessary conditions for stability of the ALOHA system. We establish the following result, which in view of Remark (iii) directly implies the necessary part of Theorem 1.

Theorem 7. *If* $\boldsymbol{\lambda} \in \overline{\mathcal{R}}^{(k)}_{\mathcal{M}}$ *where*

$$\overline{\mathcal{R}}^{(k)}_{\mathcal{M}} = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_k > P^k_{succ}(\mathcal{M}_k) \quad \textit{and} \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}\} \tag{3.4}$$

then the kth queue $N_k^t \to \infty$ *with a positive probability, hence the original Markov chain* $\mathbf{N}^t$ *is unstable.* ∎

We carry out the proof of Theorem 7 in two steps. The main idea behind the proof is to show that:

(i) if (3.4) holds, then the kth queue under the partition $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$ is unstable *in the dominant system* $\overline{\mathbf{N}}^t_{\mathcal{P}_k}$,

(ii) the last assertion can be extended to the original system .

Step 1. *Instability of the dominant system*

Let us consider the dominant system under the partition $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$. The arrival process to a persistent queue (e.g., the kth one) does not effect the stability condition for nonpersistent queues represented by the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{M}_k}$. Hence, for $\boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}$ the Markov chain $\overline{\mathbf{N}}^t_{\mathcal{M}_k}$ is *stable*, as proved in Section 3.1. Using the same construction as above, we can assure stationarity and ergodicity of $\overline{\mathbf{N}}^t_{\mathcal{M}_k}$ as well as the output process $Y^t_k(\mathcal{M}_k)$. Therefore, the stationarity and ergodicity assumptions of Loynes' scheme hold, and we can apply Lemma 6 to the kth queue. In particular, part (ii) implies that if $\lambda_k > EY^t_k(\mathcal{M}_k) = P^k_{succ}(\mathcal{M}_k)$, then $\overline{N}^t_k \to \infty$ (a.s.).

Step 2. *Instability of the kth queue in the original system*

We now prove that for $\boldsymbol{\lambda} \in \overline{\mathcal{R}}^{(k)}_{\mathcal{M}}$ the kth queue is also unstable in the *original* system. We shall use a coupling argument to show that with positive probability the dominant system and the original system are *indistinguishable* for $\boldsymbol{\lambda} \in \overline{\mathcal{R}}^{(k)}_{\mathcal{M}}$.

Except for formalism – that is left for the interested reader – we proceed as follows. Consider the dominant system $\overline{\mathbf{N}}^t_{\mathcal{P}_k}$ with $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$. From Step 1, we know that for $\boldsymbol{\lambda} \in \overline{\mathcal{R}}^{(k)}_{\mathcal{M}}$ the kth queue length in the dominant system tends to infinity (a.s.). Therefore, there exist $B, T > 0$ such that $\overline{N}^t_k \geq B$ for $t \geq T$ (a.s.). We now shift the input process $U^t_k = X^{t-1}_k - Y^t_k$ (cf. (3.2)) by T, that is, we consider U^{t+T}_k which is distributed as U^t_k due to stationarity of the pair (X^t_k, Y^t_k). Thus $\overline{N}^0_k \geq B$, and clearly on a set (of sample paths) of positive probability the queue $\overline{N}^t_k$ never returns to zero for all $t \geq 0$. Let now $\mathbf{N}^0 = \overline{\mathbf{N}}^0_{\mathcal{P}_k}$ with $N^0_k = \overline{N}^0_k \geq B$, so on this set of positive probability the dominant system and the original system are *identical*, that is, $\mathbf{N}^t = \overline{\mathbf{N}}^t_{\mathcal{P}_k}$ for all $t \geq 0$. In conclusion, we observe that $N^t_k \to \infty$ with positive probability.

4. FURTHER RESULTS AND CONCLUSION

The technique proposed Section 3 is clearly not limited to the ALOHA system. It is, however, restricted to *queueing systems* since the real engine of our proof is the Loynes' scheme which allows for the treatment of non-Markovian subsystems (of persistent queues). In Georgiadis and Szpankowski (1992) we used this technique to prove the stability of token passing rings. In fact, our technique can be successfully used to derive stability regions for a large class of queueing networks such as token passing rings with contractive and monotonic policies (cf. Georgiadis and Szpankowski (1993)), general network of queues, coupled processors systems (cf. Szpankowski (1990)), rings with spatial reuse and quotas, and so forth. We shall return to some of these problems in forthcoming papers.

To generalize our scheme, we repeat main ingredients of our proof of Theorem 1. As the first step, we partition the set of users $\mathcal{M}$ into nonpersistent users $\mathcal{S}$ and persistent ones $\mathcal{U}$. It is assumed that users in $\mathcal{S}$ form a smaller copy of the original system, while persistent users in $\mathcal{U}$ attempt sending dummy packets even when empty. We denote by $\overline{\mathbf{N}}^t_{\mathcal{P}}$ the queueing process representing this modified system. For our technique to work, we need the following three properties:

(P1) *Monotonicity Condition*

The queue lengths in a system increase whenever dummy messages are sent by a persistent user, that is $\mathbf{N}^t \leq_{st} \overline{\mathbf{N}}^t_{\mathcal{P}}$ for all $t \geq 0$.

(P2) *Stationary Version of $\mathcal{S}$*

The set of users $\mathcal{S}$ is a smaller copy of the original systems. Then, we request that the departure process from $\mathcal{S}$ that effects a persistent queue $k \in \mathcal{U}$ is stationary and ergodic (so Loynes' scheme can be used for establishing stability of a persistent queue).

(P3) *Identical Behaviors When Non-Empty*

The dominant system and the original system are identical as long as their queues do not empty out.

If the above three properties are satisfied, then we can carry out our analysis, and establish sufficient and necessary condition for stability of a system. We illustrate this below, on another multiqueue systems (we leave, however, details of the proof to an interested reader who should follow our footsteps from Section 3). In passing, we note that the above properties can be weakened. For example, our analysis is still valid if in (P1) we assume only that $\mathbf{N}^t \leq_{st} \overline{\mathbf{N}}^t_{\mathcal{P}} + X$ for some random variable X that is bounded (a.s.). In property (P2) asymptotic stationarity suffices to carry out the analysis. Finally, in (P3) both, the dominant and the original systems, should behave identically only for sufficiently large queues.

As an example, we now investigate a buffered multiaccess system with a conflict resolution algorithm. The system works as the ALOHA except that it adopts another multiaccess protocol, namely the so called *blocked conflict resolution algorithm* (blocked CRA) of Capetanakis (1979), and Tsybakov and Mikhailov (1978).

More specifically, the system works as follows. If there is a collision, then all users *not* involved in it are blocked, and are not allowed to transmit until the current collision is resolved. The collision is solved by a divide-and-conquer algorithm, that is, all users involved in a collision flip a coin and only those who flipped "tails" are allowed to transmit in the next slot. This process is continued until all users in the current collision are successfully transmitted. The quality of such a systems depends on the length L^t of the conflict resolution session (CRS), where t is a nonnegative integer representing the beginning of the tth CRS. More details can be found in Capetanakis (1979), Tsybakov and Mikhailov (1978).

It is easy to verify that properties (P1) and (P3) hold for the above system. To establish (P2) (more precisely, the "smaller copy property"), we consider a slightly generalized system. Namely, we assume that in addition to M regular users there are K persistent users that always send dummy messages. Clearly, now (P2) holds in such a system, and its stability region we denote as $\mathcal{R}_{\mathcal{M}}[K]$. Thus, reproducing our previous analysis, we can establish the following result.

Theorem 8. *The buffered multiaccess system with a blocked conflict resolution algorithm is stable iff* $\boldsymbol{\lambda} \in \mathcal{R}_{\mathcal{M}}[0]$ *with*

$$\begin{aligned}\mathcal{R}_{\mathcal{M}}[0] &= \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_j EL(\mathcal{M}_k) < 1 \quad \textit{for all} \ \ j \in \mathcal{M}\} \\ &= \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} : \ \lambda_k EL(\mathcal{M}_k) < 1 \quad \textit{and} \quad \boldsymbol{\lambda}_{\mathcal{M}_k} \in \mathcal{R}_{\mathcal{M}_k}[1]\}\end{aligned}$$

where $EL(\mathcal{M}_k)$ *denotes the average session length under the partition* $\mathcal{P}_k = (\mathcal{M}_k, \{k\})$ *in the stationary regime.* ■

The following result is a direct consequence of Theorem 8 and elementary properties of the protocol (cf. Capetanakis (1979), Tsybakov and Mikhailov (1978))

Corollary 9. *The buffered CRA system with* $M = 2$ *users is stable if and only if* $(\lambda_1, \lambda_2) \in \mathcal{R}$ *with*

$$\begin{aligned}\mathcal{R} &= \{\lambda_1 < 1/\ell_2 = 0.2 \ \ \textit{and} \quad \lambda_2 < 1 - 4\lambda_1\} \cup \{\lambda_1 < 1 - 4\lambda_2 \quad \textit{and} \quad \lambda_2 < 1/\ell_2 = 0.2\} \\ &= \{\lambda : \ \lambda_2 + 4\lambda_1 < 1 \quad \textit{or} \quad \lambda_1 + 4\lambda_2 < 1\}\end{aligned}$$

where the Markov chain $\mathbf{N}^t$ *is imbeded at the beginning of CRS.* ■

Furthermore, observing that $EL_m < \alpha m + \beta$, where $\alpha = 2.8867$ and $\beta = 1.2336$ (cf. Capetanakis (979) and Szpankowski (1987)), we obtain the following bound for the stability region.

Corollary 10. *The buffered CRA system is stable if* $\boldsymbol{\lambda} \in \mathcal{R}_1$*, where*

$$\mathcal{R}_1 = \bigcup_{k=1}^{M} \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) : \ \lambda_j < \frac{1 - \alpha \sum_{i \neq k} \lambda_i}{\alpha + \beta} \quad \textit{for all} \quad j \in \mathcal{M}\}$$

where the constants α *and* β *are given above.* ■

Finally, it should be mentioned that our technique can be extended to multiply channels (i.e., multiserver case) when several packets are sent simultaneously. This even works for infinite number of servers (cf. Georgiadis and Szpankowski (1992)). We point out that for such multidimensional models the Malyshev and Mensikov (1981) criteria do not work in such a system since homogeneity property is not preserved.

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