Cauchy-Schwarz inequality for dual norm

Yudong Cao[1]

During the lecture we are concerned with proving that $\mathbf{x}^T\mathbf{y} \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|_*$ for some operator norm $\|\cdot\|$ and its dual $\|\cdot\|_*$. This can be shown with the following argument: Let $\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and then we have

$$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|(\mathbf{u}^T\mathbf{y}). \tag{1}$$

The definition of dual norm is such that

$$\|\mathbf{y}\|_* = \sup_{\|\mathbf{u}\|\leq 1} \mathbf{u}^T\mathbf{y}. \tag{2}$$

Since $\|\mathbf{u}\| = \|\frac{\mathbf{x}}{\|\mathbf{x}\|}\| = 1$, we have $\mathbf{u}^T\mathbf{y} \leq \|\mathbf{y}\|_*$ by Equation (2). Then Equation (1) becomes $\mathbf{x}^T\mathbf{y} \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|_*$.

[1]cao23@purdue.edu