THE INSULATION SEQUENCE OF A GRAPH

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ABSTRACT. In a graph G, a **k-insulated** set S is a subset of the vertices of G such that every vertex in S is adjacent to at most k vertices in S, and every vertex outside S is adjacent to at least $k+1$ vertices in S. The **insulation sequence** $i_0, i_1, i_2, \ldots$ of a graph G is defined by setting i_k equal to the maximum cardinality of a k-insulated set in G. We determine the insulation sequence for paths, cycles, fans, and wheels. We also study the effect of graph operations, such as the disjoint union, the join, the cross product, and graph composition, upon k-insulated sets. Finally, we completely characterize all possible orderings of the insulation sequence, and prove that the insulation sequence is increasing in trees.

Keywords: Insulation sequence; Maximal independent sets; k-insulated sets

1. Introduction

A. Jagota, G. Narasimhan and L. Soltes [1] define a **k-insulated set** of a graph $G(V, E)$ to be a set of vertices $S \subseteq G$ that satisfies two conditions: each vertex in S is adjacent to at most k other vertices in S, and each vertex not in S is adjacent to at least $k+1$ vertices in S. The **insulation sequence** $i_0, i_1, \ldots$ of a graph G is defined by setting i_k equal to the cardinality of a maximum k-insulated set in G. For example, the vertices of a 0-insulated set S_0 form an independent set, and each vertex in $G \backslash S_0$ is adjacent to at least one vertex in S_0. This means that a 0-insulated set is a maximal independent set, and i_0 is the independence number of G. Thus, the k-insulated set is a generalization of the maximal independent set. It is easy to show that for k greater than or equal to the maximum degree of a graph G, the only k-insulated set of G contains all the vertices of G. Note also that all vertices of degree at most k must be in all k-insulated sets. Jagota et al. prove the existence of a k-insulated set for every graph G and every positive integer k. They also provide algorithms to construct k-insulated sets.

As in [1], for a vertex v and a set S, let $d_S(v)$ denote the the number of vertices in S that are adjacent to vertex v. In this paper we will make extensive use of the *Algorithm B(k, S)* provided in [1], which, given a graph G, a positive integer k, and any set of vertices S in G, outputs a k-insulated set. The procedure is described below. (One should note that steps (2) and (3) below can be executed in either order.)

The Algorithm $B(k, S)$

(1) If S is a k-insulated set, the algorithm stops.
(2) If there exists a vertex $v \in S$ such that $d_S(v) > k$, then remove v from the current set S.
(3) If there exists a vertex $u \notin S$ such that $d_S(u) \leq k$, then put u into S.
(4) Return to step (1).

The algorithm runs until there are no more vertices that need to be removed from S or put into S, at which point a k-insulated set is obtained. Jagota et al.[1] use an energy function argument to show that the algorithm terminates.

We are going to use *Algorithm* $B(k, S)$ in order to prove that in any graph G, the maximum size of a 1-insulated set is less than or equal to the maximum size of a k-insulated set, for $k \geq 2$. Based on this result, we also prove that for almost any permutation of the terms of the insulation sequence, there is a graph G in which that ordering occurs. Further, we show that the insulation sequence is increasing in trees, and we provide a lower bound on the size of a k-insulated set.

2. The Insulation Sequence for some families of graphs

Let $i_k(G)$ denote the maximum size of a k-insulated set in a graph G. Let P_n be the path on n vertices. It is easy to see that

$$\begin{aligned} i_0(P_n) &= \left\lfloor \frac{n+1}{2} \right\rfloor, \\ i_1(P_n) &= \left\lfloor \frac{2(n+1)}{3} \right\rfloor, \\ i_k(P_n) &= n, \text{ for } k \geq 2. \end{aligned}$$

Similarly, let C_n be the cycle on n vertices. Then,

$$\begin{aligned} i_0(C_n) &= \left\lfloor \frac{n}{2} \right\rfloor, \\ i_1(C_n) &= \left\lfloor \frac{2n}{3} \right\rfloor, \\ i_k(C_n) &= n \text{ for } k \geq 2. \end{aligned}$$

Recall that a **fan** F_{n+1} consists of a vertex v and a path P_n, such that v is adjacent to every vertex of P_n, and a **wheel** W_{n+1} consists of a vertex u and a cycle C_n such that u is adjacent to every vertex of C_n. The insulation sequences for F_{n+1} and W_{n+1} are similar to those of P_n and C_n, respectively, because the vertex of degree n is only included in a maximum size k-insulated set for $k \geq n$. Thus, for fans we have

$$i_k(F_{n+1}) = \begin{cases} i_k(P_n) & \text{for } 0 \leq k < n, \\ n+1 & \text{for } k \geq n, \end{cases}$$

and for wheels we have

$$i_k(W_{n+1}) = \begin{cases} i_k(C_n) & \text{for } 0 \leq k < n, \\ n+1 & \text{for } k \geq n. \end{cases}$$

3. Graph operations and the insulation sequence

In this section we study the effect of the disjoint union, the join, the cross product, and the graph composition on the insulation sequences of arbitrary graphs. Recall that the **disjoint union** of graphs G and H, denoted $G \amalg H$, has the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Proposition 3.1. *Let* $G = G_1 \amalg G_2 \amalg \cdots \amalg G_n$. *Then* $i_k(G) = \sum_{j=1}^{j=n} i_k(G_j)$.

Proof. It is easy to see that a maximum k-insulated set of G may be found by selecting a maximum k-insulated set S_j for each graph G_j, for $j = 1, 2, \ldots, n$, and taking the union of all the S_j's. □

The next graph operation we consider is the join. We define the **join** of two graphs G and H, denoted by $G + H$, to be the graph obtained from $G \amalg H$ by the addition of all edges uv, where $u \in G$ and $v \in H$. A graph $K_1 + G$ is called the **cone** of G. In particular, $K_1 + P_n$ is the fan F_{n+1}, and $K_1 + C_n$ is the wheel W_{n+1}.

Theorem 3.2. *Let* $G = G_1 + G_2$, *and let* $i_k(G_1) \geq i_k(G_2)$. *If* $i_k(G_1) \geq 2k$, *then* $i_k(G) = i_k(G_1)$.

Proof. Let S_k be a maximum k-insulated set in G_1. In G, every vertex in G_2 is adjacent to every vertex in S_k. The vertices outside a k-insulated set must be adjacent to at least $k+1$ vertices in S_k, so we have $|S_k| \geq k+1$ for k less than or equal to the maximum degree of G_1. Thus, S_k is a k-insulated set in G, and so $i_k(G) \geq i_k(G_1)$. Now, suppose there exist a maximum k-insulated set S' in G composed of the join of an induced subgraph G_1' of G_1 with q vertices, and an induced subgraph G_2' of G_2 with p vertices. Since every vertex in G_1' is adjacent to p vertices in G_2', $p \leq k$. For the same reason, $q \leq k$. Thus,

$$i_k(G) = |S'| = p + q \leq 2k \leq i_k(G_1).$$

□

The **cross product** of graphs G and H, denoted $G \times H$, has the vertex set $V(G) \times V(H)$ and vertex (u_1, v_1) is adjacent to vertex (u_2, v_2) if and only if u_1u_2 is an edge in G and $v_1 = v_2$, or v_1v_2 is an edge in H and $u_1 = u_2$. In particular, $P_n \times P_m$ is a grid. Let $\chi(G)$ denote the chromatic number of G.

Theorem 3.3. *Given a graph G on m vertices, let $n \geq 2\chi(G)$. Then $i_1(K_n \times G) = 2m$.*

Proof. We will consider the cross product $K_n \times G$ to be the graph G with each vertex replaced by a copy of K_n, and with edges between copies of K_n given by the definition of the cross product. In a complete graph K_n, every 1-insulated set has 2 vertices. Thus, the maximum size of a 1-insulated set in the disjoint union of m distinct copies of K_n is $2m$. Since i_1 cannot increase as edges are added to the graph, we obtain that $i_1(K_n \times G) \leq 2m$.

We will find a 1-insulated set S_1 of $K_n \times G$ that contains $2m$ vertices. Let $v_1, v_2, \ldots, v_m$ be the vertices of G. Choose a proper coloring $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ of G. Let Q_j be the copy of K_n corresponding to vertex v_j, and let $w_{j,1}, w_{j,2}, \ldots, w_{j,n}$ be the vertices of Q_j. For each copy Q_j of K_n, consider the pair of vertices $A_j = (w_{j,2c(v_j)-1}, w_{j,2c(v_j)})$. Note that these pairs of vertices are well defined since $n \geq 2\chi(G)$. Also, note that we chose these pairs of vertices such that there does not exist an edge between a vertex of A_i and a vertex of A_j in $K_n \times G$. Let S_1 be the union of all the pairs A_j, for $1 \leq j \leq m$. Note that $|S_1| = 2m$. We claim that S_1 is a 1-insulated set. Indeed, each vertex $v \notin S_1$ is in a copy Q_j of K_n, therefore v is adjacent to at least the 2 vertices of the pair A_j in S_1. Moreover, every vertex $u \in S_1$ is adjacent only to the other vertex of the pair A_j in which u lies. This completes the proof. □

Generalizing the result suggested by the above proof, we have the following theorem.

Theorem 3.4. *Given a graph G on m vertices, let $n \geq (k+1)\chi(G)$. Then $i_k(K_n \times G) = (k+1)m$.*

The proof of this result is similar to the proof of the previous theorem.

We will now obtain similar results for the composition of two graphs. The **composition** of graphs G and H, denoted by $G\{H\}$, has vertex set $V(G) \times V(H)$, and there exists an edge between vertices (u_1, v_1) and (u_2, v_2) if and only if one of the following conditions is met:

(1) we have u_1u_2 as an edge in G, or
(2) we have v_1v_2 as an edge in H and $u_1 = u_2$.

In Figure 1 we depict an example of the composition $C_4\{K_3\}$.

Theorem 3.5. *Let G and H be graphs, where H is not a cone graph. Then $i_1(G\{H\}) = i_0(G)i_1(H)$.*

Proof. We will use the term *adjacent copies* to mean that 2 copies of the graph H are joined in $G\{H\}$, i.e., there exists an edge in G between the vertices that correspond to the two respective copies of H. We prove that there cannot exist 2 adjacent copies of H that contain vertices of a 1-insulated set S_1. To this end, assume that there exist two adjacent copies of H, say H_1 and H_2, such that both contain vertices of S_1. Because every vertex of H_1 is adjacent to every vertex of H_2,

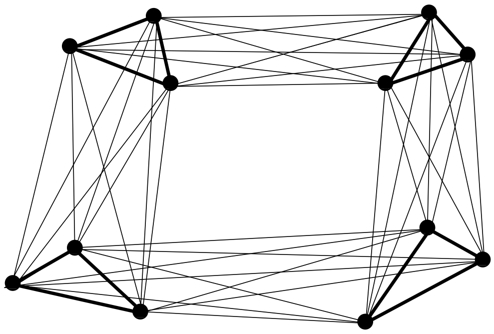

FIGURE 1. This graph represents the composition $C_4\{K_3\}$. Notice that each vertex of C_4 is replaced by a copy of K_3. Also, two copies of K_3 are joined whenever there is an edge between the corresponding vertices in C_4.

each of these two copies must contain exactly one vertex of an 1-insulated set. Let u be the vertex in $H_1 \cap S_1$ and let v be the vertex in $H_2 \cap S_1$. Since H is not a cone, there exists a vertex $w \in H_1 \backslash S_1$ which is adjacent to v and not adjacent to u. Therefore, there must exist another copy of H, say H_3, that contains a vertex $w_1 \in S_1$ which is adjacent to w. But as long as the edge ww_1 exists, all vertices of H_3 are adjacent to all vertices of H_1, and, in particular, there exists an edge uw_1 in the subgraph induced by S_1. This makes vertex $u \in S_1$ adjacent to two vertices in S_1, which contradicts the fact that S_1 is a 1-insulated set. Thus, there are no adjacent copies of H in $G\{H\}$ containing vertices of S_1. In this case, the only way to obtain a 1-insulated set in $G\{H\}$ is by considering a maximal independent set S_0 in G, and taking the vertices of a 1-insulated set in each copy of H that corresponds to a vertex in the independent set S_0. Thus, we have $i_1(G\{H\}) = i_0(G)i_1(H)$. $\square$

The proof of Theorem 3.5 suggests a way of constructing k-insulated sets in a composition-graph $G\{H\}$, which gives us the following result.

Theorem 3.6. *If G and H are graphs, then $i_k(G\{H\}) \geq i_0(G)i_k(H)$.*

One should note that the only change needed to adapt the proof of Theorem 3.5 for this theorem is to choose k-insulated sets in copies of H instead of choosing 1-insulated sets.

4. The Insulation Sequences for Arbitrary Graphs

In this section, we investigate the relative sizes of the i_k's. Recall that the insulation sequence is the sequence $i_0, i_1, i_2, \ldots$. In [1], Jagota et al. prove that the inequalities $i_0(G) \leq i_k(G) \leq (k+1)i_0(G)$ hold in any graph G. In our next result, we prove a further restriction on the relative sizes of the terms of the insulation sequence.

Theorem 4.1. *In any graph G, the inequality $i_1(G) \leq i_k(G)$ holds for all $k \geq 2$.*

Proof. The method used in this proof was suggested by David Moulton [2].

We will use *Algorithm* $B(k, S_1)$ as described in Section 1, where the initial set S_1 is a maximum 1-insulated set, and the final set S_k is a k-insulated set. We will prove that the size of the k-insulated set obtained by carrying out the *Algorithm* $B(k, S_1)$ is greater than or equal to the size of the initial 1-insulated set. Color the edges of the initial set S_1 blue, and color the remaining edges black. Let S be the set of vertices that the algorithm is currently operating on. For every set of vertices S, define $E(S) = |S| - \frac{B_S}{k}$, where B_S is the number of black edges that lie in S. During each iteration of the algorithm, one of two cases may occur:

Case 1.

A vertex is taken out of S. Then

$$E(S - v) = |S| - 1 - \frac{B_S - x_1}{k},$$

where x_1 represents the number of black edges taken out. Note that $x_1 \geq k$, since a vertex v is taken out when $d_S(v) = k + 1$, and no two blue edges share a vertex. Therefore,

$$E(S - v) \geq E(S).$$

Case 2.

A vertex is added to S. Then

$$E(S \cup v) = |S| + 1 - \frac{B_S + x_2}{k},$$

where x_2 represents the number of black edges added to the set S. Note that $x_2 \leq k$, since a vertex v is added to S when $d_S(v) \leq k$. Therefore,

$$E(S \cup v) \geq E(S).$$

We conclude that the function $E(S)$ is non-decreasing; thus, when the algorithm produces a k-insulated set, we obtain $i_1 \leq |S_k| - \frac{B_{S_k}}{k}$ which implies $i_1 \leq i_k$. □

Corollary 4.2. *For graphs of maximum degree* 3, *the insulation sequence is monotonically increasing.*

Proof. It is obvious that for any graph with maximum degree D, we have $i_D > i_{D-1}$. We also know that $i_2 \geq i_1$ and $i_1 \geq i_0$ for every graph. Hence $i_0 \leq i_1 \leq i_2 < i_3$ for $D = 3$. □

By a similar argument as that in the proof of Theorem 4.1, we establish the following proposition.

Proposition 4.3. *Let H be an induced subgraph of a graph G. Then $i_1(H) \leq i_1(G)$.*

Proof. We will use *Algorithm* $B(k, S_1)$ to build a 1-insulated set in G from a maximum 1-insulated set S_1 in H. We call a vertex v *bad* if either $v \in S$ and $d_S(v) \geq 2$, or $v \notin S$ and $d_S(v) \leq 1$, where S is the vertex set at the current iteration of the algorithm (initially, $S = S_1$). Recall that steps (2) and (3) of the algorithm can be performed in either order. Thus, at each iteration of the algorithm, one can choose to remove a bad vertex of S (if such a vertex exists), instead of adding a vertex to S (if one needs to be added). We sequentially place in the set S the vertices $v \in G$ with $d_S(v) \leq 1$, and take out of the current S the vertices v with $d_S(v) = 2$ as soon as they occur in S. Once a vertex v is added to S, it can increase by 1 the degree of at most one vertex u in the subgraph induced by S. If u needs to be taken out from S, the number of vertices in the new S is at least the number of vertices in S before adding v. This proves that the size of the 1-insulated set in G output by the algorithm is at least the size of a maximum 1-insulated set in H. Therefore, $i_1(H) \leq i_1(G)$. □

Although the tendency is for the insulation sequence of a graph to increase, this is not true in general. In Figure 2, we see an example of a n-vertex graph M_k in which $i_k(M_k) > i_{k+1}(M_k)$. Take the complete bipartite graph $K_{k+1,k}$ and let $V = \{v_1, \ldots, v_{k+1}\}$ be the larger partite set. To each vertex v_i in V, join two other vertices v_{i1} and v_{i2}. Then we have

$$i_j(M_k) = \begin{cases} n - k & \text{for } 2 \leq j \leq k, \\ n - k - 1 & \text{for } j = k + 1, \\ n & \text{for } j \geq k + 2. \end{cases}$$

Thus, $i_k(M_k) > i_{k+1}(M_k)$.

The graphs M_k will be useful later in the paper when studying possible orderings of the insulation sequence. Our results thus far motivate the following question: Apart from the inequalities discussed above, are there any other constraints on the behavior of the insulation sequence? Before

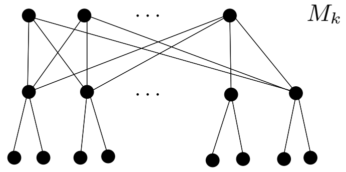

FIGURE 2. In the graph M_k, the insulation sequence is not monotonically increasing; for example, $i_k(M_k) > i_{k+1}(M_k)$.

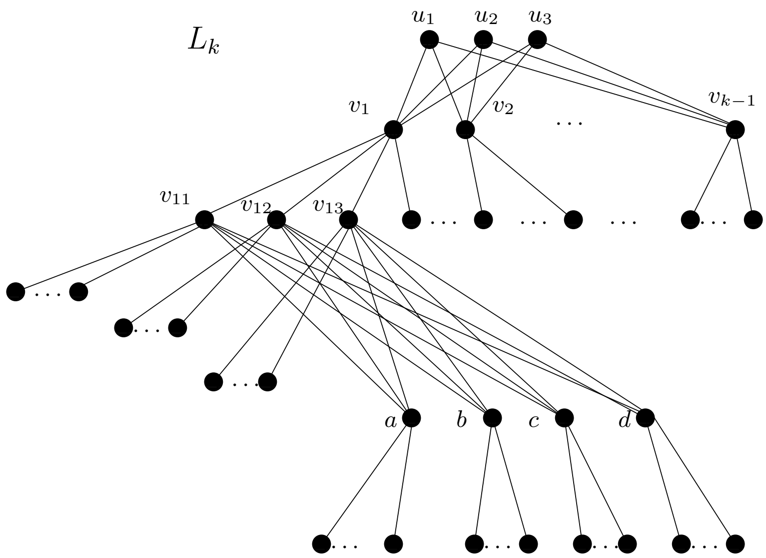

FIGURE 3. In the graph L_k, we have $i_k(L_k) > i_{k+1}(L_k) > i_{k+2}(L_k)$.

answering this question, we construct families of graphs for which the insulation sequence is strictly increasing, and we also construct graphs for which the sequence is not monotone.

First, we construct a n-vertex graph with a strictly increasing insulation sequence. Begin with the path P_m with the consecutive vertices $v_1, v_2, \dots, v_m$ and join $i-2$ other vertices to each vertex v_i, for $i \geq 3$. In this way we obtain

$$i_k = n - m + 1 + k, \text{ for } 1 \leq k \leq m-1.$$

Now consider the n-vertex graph L_k ($k \geq 7$) depicted in Figure 3. Start with $K_{3,k-1}$ and let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, \dots, v_{k-1}\}$ be the two parts of the complete bipartite graph. Join every vertex v_i in V to the $k-3$ distinct other vertices $v_{i1}, v_{i2}, \dots, v_{i(k-3)}$. Join $(k-7)$ other distinct vertices to each of v_{11}, v_{12}, and v_{13}, and connect four other vertices a, b, c, d to each of v_{11}, v_{12} and v_{13} (the set $\{v_{11}, v_{12}, v_{13}, a, b, c, d\}$ is a $K_{3,4}$ graph). Let W be the set of the $3(k-7)$ vertices of degree 1 that have just been added. Then join each of a, b, c, and d to $(k-3)$ other distinct vertices. Call this last set of vertices of degree one W_1.

Proposition 4.4. *In the graph L_k, the following inequality holds:*

$$i_{k-1} < i_{k-2} < i_{k-3}.$$

Proof. The vertices in U are of degree $(k-1)$, so they must be counted in i_{k-1}. Also, the vertices $v_{i1}, v_{i2}, \dots, v_{i(k-3)}$, for $i = 1, 2, \dots, k-1$, are of degree at most $k-2$, so they must be counted in both i_{k-1} and i_{k-2}. Moreover, the vertices in W and W_1 are of degree 1, so they have to be counted in i_{k-1}, i_{k-2}, and i_{k-3}. The other vertices of L_k cannot be in the maximum $(k-1)$-insulated set. Thus,

$$i_{k-1} = n - (k-1) - 4 = n - k - 3.$$

Now all of the following are counted in i_{k-2}: the vertices of V, the vertices in the set $\{v_{i1}, v_{i2}, \ldots, v_{i(k-3)}\}$ for $i = 1, 2, \ldots, k-1$, and the vertices in W. So, $i_{k-2} = n - 7$. Also, for i_{k-3}, we count all vertices of L_k except those in U and v_{11}, v_{12}, and v_{13}. Thus, $i_{k-3} = n - 6$, and we have

$$i_{k-1} < i_{k-2} < i_{k-3}.$$

□

As we have noticed in the above example, it is nontrivial to find connected graphs for which the insulation sequence is decreasing in some specific places. However, the disjoint union is a very useful tool in constructing graphs in which the insulation sequence has a given ordering of its terms. We will show how to obtain a graph G in which $i_2 > i_3 > i_4$ by use of the disjoint union. We start with the graph M_2 introduced before, in which we have $i_2(M_2) - i_3(M_2) = 1$ and $i_4(M_2) - i_3(M_2) = 3$. In the graph M_3, it is true that $i_2(M_3) = i_3(M_3) = i_4(M_3) + 1$, so by taking the disjoint union of 4 copies of M_3 with the graph M_2, we obtain the following:

$$\begin{aligned} &i_2(M_2 \amalg 4M_3) = i_2(M_2) + 4i_2(M_3) \\ &> i_3(M_2 \amalg 4M_3) = i_3(M_2) + 4i_3(M_3) \\ &> i_4(M_2 \amalg 4M_3) = i_4(M_2) + 4i_4(M_3). \end{aligned}$$

We will use a similar method to show that, apart from $i_0 \leq i_1 \leq i_k$ and $i_{D-1} < i_D$, there are no other constraints on the insulation sequence for arbitrary graphs. Namely, we prove the following.

Theorem 4.5. *Given $D \geq 4$, for any permutation π of $\{2, 3, \ldots, (D-1)\}$, there exists a graph G of maximum degree D such that its insulation sequence satisfies:*

$$i_{\pi(2)} > i_{\pi(3)} > \cdots > i_{\pi(D-1)}.$$

Furthermore, G can be constructed such that the inequalities become equalities in any positions.

We must first prove a technical lemma.

Lemma 4.6. *Given a graph G of maximim degree D and two integers p and n such that $p \geq 1$ and $2 \leq n \leq D-2$, there exists a graph H of maximum degree D and a constant $c \geq 0$ such that $i_j(H) = i_j(G) + c$ for $2 \leq j \leq n$, and $i_{n+1}(H) = p + c$.*

Proof. We will construct H by using G, the families of graphs M_n presented earlier in the paper, and the families of star-graphs $R_n = K_{1,n}$, where n is the number of vertices of degree 1 adjacent to the common vertex. Recall that

$$i_j(M_k) = \begin{cases} i_k(M_k) & \text{for } 2 \leq j \leq k, \\ i_k(M_k) - 1 & \text{for } j = k+1, \\ i_k(M_k) + k & \text{for } j \geq k+2, \end{cases} \tag{1}$$

and also note that the maximum degree of M_k is $k+2$. Furthermore,

$$i_j(R_k) = \begin{cases} i_k(R_k) & \text{for } 2 \leq j \leq k-1, \\ i_k(R_k) + 1 & \text{for } j \geq k, \end{cases} \tag{2}$$

and the maximum degree of R_k is k.

We distinguish three cases:

Case 1. $i_{n+1}(G) = p$, and we are done.

Case 2. $i_{n+1}(G) > p$. Then letting $m = i_{n+1}(G) - p$, and taking the disjoint union of G with m copies of M_n and using Proposition 3.1 and Equation 1 we obtain the desired H. Indeed,

$$i_j(H) = i_j(G) + mi_j(M_n), \text{ for } 2 \leq j \leq n.$$

So, $c = mi_2(M_n) = \cdots = mi_n(M_n)$, and

$$i_{n+1}(H) = i_{n+1}(G) + mi_{n+1}(M_n) = i_{n+1}(G) + m(i_n(M_n) - 1) = p + c.$$

Case 3. $i_{n+1} < p$. Then, let $m = p - i_{n+1}(G)$, and let $H = G \amalg mR_n$. We obtain

$$i_j(H) = i_j(G) + mi_j(R_n), \text{ for } 2 \leq j \leq n.$$

So, $c = mi_2(R_n) = \cdots = mi_n(R_n)$ and

$$i_{n+1}(H) = i_{n+1}(G) + mi_{n+1}(R_n) = i_{n+1}(G) + m(i_n(R_n) + 1) = p + c.$$

Observe that in all the cases the maximum degree of the graph H is D. □

Proof of Theorem 4.5. We will prove the theorem for the strict inequalities and show how the inequalities can become equalities. Inducting on n, we will construct a graph G whose insulation sequence satisfies the required ordering. Our induction hypothesis will be that for any permutation σ of $i_2, \ldots, i_n$, with $n \leq D - 2$, there exists a graph H with the maximum degree D, and $i_{\sigma(2)} > i_{\sigma(3)} > \cdots > i_{\sigma(n)}$. For the base case, $n = 2$, we have $i_2(R_2 \amalg R_D) < i_3(R_2 \amalg R_D)$ and $i_2(M_3 \amalg R_D) > i_3(M_3 \amalg R_D)$. Let q be such that $\pi(q) = n + 1$. By the induction hypothesis, there exists a graph N with maximum degree D such that

$$i_{\pi(2)} > \cdots > i_{\pi(q-1)} > i_{\pi(q+1)} > \cdots > i_{\pi(n)}.$$

Then we only need to apply the lemma with p such that $i_{\pi(q-1)} < p < i_{\pi(q+1)}$. Note that if $|i_{\pi(q-1)} - i_{\pi(q+1)}| = 1$, we can take the disjoint union of 2 copies of N and the insulation sequence of the graph obtained satisfies the same ordering as the insulation sequence of N. The graph that satisfies the conditions of the theorem is obtained by taking the disjoint union of N (or $2N$ if this is the case) with a certain number of copies of either M_n or R_n as described in the lemma.

To obtain equality instead of strict inequality between, say, $i_{\pi(a)}$ and $i_{\pi(a+1)}$, we take the graph H_1 such that

$$i_{\pi(2)}(H_1) > \cdots > i_{\pi(a)}(H_1) > i_{\pi(a+1)}(H_1) > \cdots > i_{\pi(D-1)}(H_1),$$

and the graph H_2 such that

$$i_{\pi(2)}(H_2) > \cdots > i_{\pi(a+1)}(H_2) > i_{\pi(a)}(H_2) > \cdots > i_{\pi(D-1)}(H_2).$$

Finally, to show that we can replace the inequality with equality at any position, scale H_1 by a factor $\alpha = i_{\pi(a+1)}(H_2) - i_{\pi(a)}(H_2)$, and H_2 by a factor $\beta = i_{\pi(a)}(H_1) - i_{\pi(a+1)}(H_1)$. Then, in the disjoint union $G = \alpha H_1 \amalg \beta H_2$, we will have

$$i_{\pi(2)}(G) > \cdots > i_{\pi(a)}(G) = i_{\pi(a+1)}(G) > \cdots > i_{\pi(D-1)}(H_1),$$

as desired. □

In particular, Theorem 4.5 proves the existence of insulation sequences with arbitrarily many consecutive decreasing terms.

5. The Insulation sequence for trees

In this section, we provide results about the insulation sequence of trees.

Theorem 5.1. *Let T be a tree, let k be any nonnegative integer, and let S_k and S_{k+1} be k- and $(k+1)$-insulated sets of T, respectively. Then $|S_k| \leq |S_{k+1}|$ for any $k \geq 0$.*

Proof. For $k \geq 1$, every k-insulated set contains all the leaves of the tree, and therefore $S_k \cap S_{k+1} \neq \emptyset$. Suppose there exists a vertex $v \in S_k \backslash S_{k+1}$. Then it must be adjacent to at least $k+2$ vertices in S_{k+1}. However, v is also adjacent to at most k vertices in S_k, and so it is adjacent to at least 2 vertices in $S_{k+1} \backslash S_k$. Let $|S_k \backslash S_{k+1}| = p$ and $|S_{k+1} \backslash S_k| = q$. Let $\epsilon(S)$ denote the number of edges induced by any vertex set S. Then

$$p + q > \epsilon((S_k \backslash S_{k+1}) \cup (S_{k+1} \backslash S_k)) \geq 2p \tag{3}$$

because the number of vertices in a forest is greater than the number of edges of the forest. The above inequality implies $q > p$, and so $|S_{k+1}| > |S_k|$. If there does not exist a vertex $v \in S_k \backslash S_{k+1}$, i.e. $p = 0$, then $S_k \subseteq S_{k+1}$, and therefore $|S_k| \leq |S_{k+1}|$. For $k = 0$ we might have either $S_0 \cap S_1 = \emptyset$, or $S_0 \cap S_1 \neq \emptyset$, but the argument above works for this case as well, and so $|S_0| \leq |S_1|$. □

As an immediate corollary we have the following.

Corollary 5.2. *In any tree, the insulation sequence is nondecreasing.*

A connected **unicyclic** graph is a graph obtained by adding only one edge to a tree. A similar result as above can be obtained for unicyclic graphs.

Corollary 5.3. *In any connected unicyclic graph, the insulation sequence is nondecreasing.*

Proof. The proof follows easily from the proof of Theorem 5.1. Notice that in an unicyclic graph the number of the vertices is equal to the number of the edges. Thus, we can have equality in the inequality (3), and still obtain $q \geq p$. This again implies $|S_k| \leq |S_{k+1}|$. □

Next we provide a lower bound on the size of a k-insulated set in a tree.

Proposition 5.4. *Let S_k be a k-insulated set in a tree T on n vertices. Then $|S_k| \geq \frac{kn+1}{k+1}$. In particular, $i_k \geq \frac{kn+1}{k+1}$.*

Proof. A *neighbor-component* of vertex $a \in T \backslash S_k$ is a connected component of S_k containing a vertex adjacent to a. Let $c_1, c_2, \ldots, c_t$ be the connected components of S_k, let $V = T \backslash S_k$, and label the vertices in V by $v_1, v_2, \ldots, v_{n-|S_k|}$. We construct a graph T' as follows: T' has vertices $U = \{u_1, u_2, \ldots, u_{n-|S_k|}\}$ and $W = \{w_1, w_2, \ldots, w_t\}$; there exists an edge between u_i and u_j if and only if there exists an edge between v_i and v_j in T; and there exists an edge between u_i and w_j if and only if c_j is a neighbor-component of v_i in T (see Figure 4).

We claim that T' is a tree. By construction T' is connected. Now, suppose there exists a cycle C in T'. The vertices of U form a tree, and the vertices of W are not adjacent to each other. Then C must contain a vertex in W, say, w_j. Let $u_m, u_n \in U$, be adjacent to w_j and contained in C. Then there exists a path between u_m and u_n in U, and therefore, there exists a path between v_m and v_n in T. Also, v_m and v_n have a common neighbor component c_j in S_k. If we denote by x_m and x_n the vertices from c_j that are adjacent to v_m and v_n in T, respectively, then there exists a path P between x_m and x_n in c_j. Thus, there exist 2 paths between v_m and v_n in T, namely one in V and one containing v_m, x_m, P, x_n, and v_n in order. This contradicts the fact that T is a tree, and therefore T' is a tree.

Note that a vertex in V cannot be adjacent to two distinct vertices in the same component of S_k (otherwise there would exist cycles in T). Thus, the number of edges in T' is equal to $\epsilon(T) - \epsilon(S_k)$,

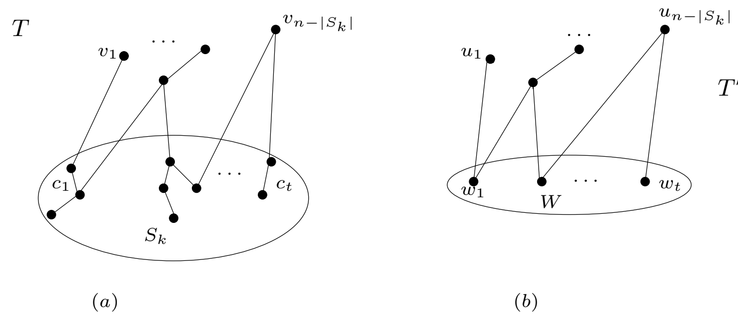

FIGURE 4. In (a) is shown the tree T, a k-insulated set S_k, and the connected components of S_k, namely $c_1, c_2 \dots c_t$. In (b), each component of S_k is replaced by a single vertex.

where $\epsilon(S)$ denotes the number of edges in the subgraph induced by any vertex set S. As S_k is a k-insulated set in T, we have $\epsilon(T) - \epsilon(S_k) \geq (k+1)(n - |S_k|)$. Since T' is a tree, the number of vertices in T' is one more than the number of edges in T'. Therefore,

$$\epsilon(T) - \epsilon(S_k) = t + n - |S_k| - 1 \geq (k+1)(n - |S_k|).$$

We finally obtain $k(n - |S_k|) + 1 \leq t \leq |S_k|$, and so $|S_k| \geq \frac{kn+1}{k+1}$.

□

6. Open Questions

In this section we present some open questions regarding k-insulated sets and insulation sequences.

Question 6.1

We proved in Theorem 4.5 that the insulation sequence may have almost any ordering among its terms. It would be interesting to find necessary and sufficient conditions for a graph G to have $i_k(G) > i_{k+1}(G)$ in the ordering of the insulation sequence.

Question 6.2

Let $m_k(G)$ denote the minimum size of a k-insulated set in a graph G. It is interesting to observe that, although the inequality $i_0 \leq i_1$ holds in any graph, it is not always true that $m_0 \leq m_1$. Figure 5 depicts an example of a graph in which the size of the minimum 0-insulated set is greater than the size of the minimum 1-insulated set. Notice that in this case the minimum size of a 1-insulated set is 4, while the minimum size of a 0-insulated set is 5. For a given permutation of the sizes of $m_k(G)$, for $k \geq 0$, is there a graph in which this ordering occurs?

Question 6.3

What orderings of the insulation sequence are possible in d-regular graphs, for $d \geq 4$?

Question 6.4

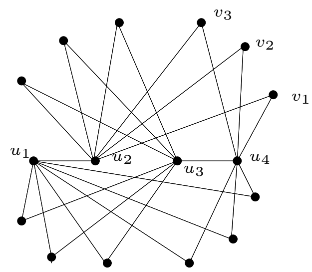

FIGURE 5. In this graph the minimum size of a 0-insulated set, formed by the vertices u_1, u_3, v_1, v_2, v_3, is greater than the minimum size of a 1-insulated set, formed by the vertices u_1, u_2, u_3, u_4.

Corollary 5.2 states that the insulation sequence is nondecreasing in trees. For a given positive integer k, what is the minimum number of edges that need to be added to a tree on n vertices, in order to obtain $i_k > i_{k+1}$?

Acknowledgements

This research was done in the REU of Prof. Joseph Gallian at the University of Minnesota, Duluth with financial support from Bard College. The author thanks Stephen Wang, Mike Develin, Philip Matchett, Geir Helleloid, Denis Chebikin, David Moulton and Joseph Gallian for useful suggestions.

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