

**The Decay Rate of the Solution  
to a Tridiagonal Linear System  
With A Very Special Right  
Hand Side**

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# THE DECAY RATE OF THE SOLUTION TO A TRIDIAGONAL LINEAR SYSTEM WITH A VERY SPECIAL RIGHT HAND SIDE

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**1. Introduction.** This is a short note which deals with a detail in the analysis of the truncated SPIKE algorithm [2], [3] for systems which are strictly diagonally dominant by rows. It contains the proof of Theorem 3.9 [1].

**2. The main result.** There is only one result namely the following theorem  
THEOREM 2.1. *Let  $\{(a_i, b_i, c_i)\}_{i=1}^n$  be a finite sequence, such that  $a_i \neq 0$ , and*

$$\max_{i=1, \dots, n} \frac{|b_i| + |c_i|}{|a_i|} = \epsilon < 1.$$

If the vector  $x$  given by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & & & \\ c_2 & \ddots & \ddots & & \\ & \ddots & & b_{n-1} & \\ & & & c_n & a_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_n \end{bmatrix},$$

exhibits the smallest possible decay rate, i.e. if

$$|x_1| = \epsilon^n \tag{2.1}$$

then

$$c_i = 0, \quad \text{and} \quad |b_i| = \epsilon |a_i|, \tag{2.2}$$

for  $i = 1, 2, \dots, n$ .

*Proof.* We prove the theorem using the Thomas algorithm [4], which is designed to solve tridiagonal systems of the form

$$c_i x_{i-1} + a_i x_i + b_i x_{i+1} = f_i, \quad i = 1, 2, \dots, n,$$

where  $x_0$ , and  $x_{n+1}$  are given in advance. If the system is strictly diagonally dominant by rows with degree  $d = \epsilon^{-1} > 1$  then the solution can be computed as follows

$$x_i = p_i x_{i+1} + q_i, \quad i = 1, \dots, n,$$

where the coefficient  $p_i$  and  $q_i$  are given by

$$p_0 = 0, \quad p_i = \frac{-b_i}{a_i + c_i p_{i-1}}, \quad i = 1, 2, \dots, n,$$

and

$$q_0 = x_0, \quad q_i = \frac{f_i - c_i q_{i-1}}{a_i + c_i p_{i-1}}, \quad i = 1, 2, \dots, n.$$

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We claim that  $|p_i| \leq \epsilon$ , for  $i = 0, 1, 2, \dots, n$ . If  $b_i = 0$  then  $p_i = 0$ , and there is nothing to show. Assuming  $|p_{i-1}| \leq \epsilon < 1$ , and  $b_i \neq 0$ , we have

$$\begin{aligned} |p_i| &= \frac{|b_i|}{|a_i + c_i p_{i-1}|} \leq \frac{|b_i|}{|a_i| - |c_i| \epsilon} \\ &\leq \frac{|b_i|}{\epsilon^{-1}(|b_i| + |c_i|) - |c_i| \epsilon} = \frac{|b_i|}{\epsilon^{-1}|b_i| + (\epsilon^{-1} - \epsilon)|c_i|} \leq \epsilon, \end{aligned} \quad (2.3)$$

because  $\epsilon \leq 1$ , implies  $(\epsilon^{-1} - \epsilon)|c_i| \geq 0$ .

In our case  $x_0 = x_{n+1} = 0$ , and  $f_i = 0$  for  $i = 1, 2, \dots, n-1$ , while  $f_n = b_n$ . It follows that

$$q_i = 0, \quad i = 0, 1, 2, \dots, n-1,$$

while

$$q_n = \frac{b_n}{a_n + c_n p_{n-1}}, \quad \text{and} \quad |q_n| \leq \epsilon.$$

It follows that

$$x_n = q_n, \quad x_i = \left( \prod_{j=i}^{n-1} p_j \right) q_n,$$

which implies that

$$|x_i| \leq \epsilon^{n-i+1}.$$

Now suppose  $|x_1|$  assumes the largest possible value, namely

$$|x_1| = \epsilon^n$$

then we must have

$$|p_i| = \epsilon, \quad i = 1, 2, \dots, n-1, \quad \text{and} \quad |q_n| = \epsilon.$$

Now, we claim that this can only happen if  $c_i = 0$ , for  $i = 1, 2, \dots, n$ . From (2.3) we see that we actually have

$$\frac{|b_i|}{|a_i| - |c_i| \epsilon} = \epsilon,$$

for  $i = 1, 2, \dots, n-1$ , as well as  $i = n$ . It follows, that

$$\epsilon^2 |c_i| = \epsilon |a_i| - |b_i|.$$

However,  $\epsilon |a_i| \geq |b_i| + |c_i|$ , leaving us with

$$\epsilon^2 |c_i| = \epsilon |a_i| - |b_i| \geq |c_i|,$$

from which we deduce  $|c_i| = 0$ , because  $\epsilon < 1$ .  $\square$

In short, if a tridiagonal matrix which is strictly diagonally dominant by rows, exhibits the slowest possible decay rate, then it is actually bidiagonal, and the ratio  $|b_i|/|a_i|$  is fixed. In our experience the spikes always decay much faster than the worst case.

## REFERENCES

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