The Incomplete Gamma Functions Since Tricomi

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ABSTRACT. The theory of the incomplete gamma functions, as part of the theory of confluent hypergeometric functions, has received its first systematic exposition by Tricomi in the early 1950s. His own contributions, as well as further advances made thereafter, are surveyed here with particular emphasis on asymptotic expansions, zeros, inequalities, computational methods, and applications.

1. The Incomplete Gamma Functions Up To 1950

Tricomi considered his work on the asymptotic behavior of Laguerre polynomials and their zeros among his "chief contributions to the theory of special functions" ([153, p. 56]). Nevertheless, the incomplete gamma function held a special fascination for him, as he was fond of calling it affectionately the Cinderella of special functions. I feel especially privileged to talk about this topic here, since the only time I met Tricomi in person was shortly before his death when he honored me by his presence in a colloquium lecture I gave in Turin. It was precisely the incomplete gamma functions and methods for computing them that I was talking about, a subject in which Tricomi still expressed a vivid interest.

The incomplete gamma functions arise from Euler's integral for the gamma function,

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt,$$

by decomposing it into an integral from 0 to x, and another from x to ∞,

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt, \quad \operatorname{Re} a > 0; \quad \Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt, \quad |\arg x| < \pi. \tag{1.1}$$

Historically, this decomposition was first studied in 1877 for $x = 1$ by Prym [118], apparently in an attempt to collect the poles at $a = 0, -1, -2, \ldots$ of the gamma function in the first (more manageable) integral, $\gamma(a, 1)$, leaving the second integral, $\Gamma(a, 1)$, an entire function. The functions (1.1), therefore, are sometimes referred to as Prym's functions. For general $x > 0$ (even for $x < 0$), however, the second integral in (1.1) already appears in Legendre's *Exercises* [85, pp. 399–343] and in some of his later works.

Noteworthy special cases of (1.1) are obtained when $a = 1 \pm n$ is an integer. Specifically, for $n \geq 0$,

$$\gamma(1 + n, x) = n![1 - e^{-x} e_n(x)], \tag{1.2}$$

$$\Gamma(1 + n, x) = n! e^{-x} e_n(x), \tag{1.3}$$

$$\Gamma(1 - n, x) = x^{1-n} E_n(x), \tag{1.4}$$

where $e_n(x) = 1 + x + x^2/2! + \cdots + x^n/n!$, $n = 0, 1, 2, \ldots$, are the partial sums of the exponential series, and

$$E_n(x) = \int_1^\infty e^{-xt} t^{-n} dt, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

the exponential integrals. The latter occur prominently in astrophysics and nuclear physics and include (for $n = 1$) such functions as the logarithmic, sine, and cosine integrals. The function $\gamma(a, x)$ has a pole when a is a negative integer or zero; see, however, (2.1) and (2.2). When $a = \frac{1}{2}$ one obtains the error functions

$$\operatorname{erf} x = \tfrac{1}{\sqrt{\pi}} \gamma(\tfrac{1}{2}, x^2), \quad \operatorname{erfc} x = 1 - \operatorname{erf} x = \tfrac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}, x^2) \tag{1.6}$$

and their close relatives such as the Fresnel integrals.

The older theory of the incomplete gamma function, including series expansions of various kinds, asymptotic expansions, differentiation and recurrence relations, continued fractions, etc., can be found in Nielsen [103, Kap. II,XV,XXI], and further material, especially integral representations, in Böhmer [15, Kap. V]. The basic theory, however, remained rather stable, until in the late 1940s, as a result of his involvement in the Bateman project, Tricomi fully recognized the importance of these functions and revitalized their theory by adding important contributions of his own (see §2) and by summarizing the knowledge as of 1950 in the second volume of the Bateman project [40, Ch. IX, pp. 133–151]. He gave a more detailed exposition, in the context of the theory of confluent hypergeometric functions, in his monograph [151, §§4.1–4.6].

One aspect of incomplete gamma functions, namely their real and complex zeros, does not receive an entirely adequate coverage in these works, in part, perhaps, because Tricomi's interest was in the x-zeros for fixed a, while work done in the early 1900s was exclusively concerned with a-zeros for fixed x. The earliest investigations dealt with the real negative zeros of $\gamma(a, x)$ for Prym's choice $x = 1$. Increasingly sharper localizations of these zeros were obtained in work of Haskins [59], Gronwall [56], and Walther [158]. Rasch [120] was the first to consider the case of arbitrary fixed $x > 0$, and Hille and Rasch [60] the case of $x < 0$. Complex zeros were already studied by Gronwall [56], who showed in the case $x = 1$ that there are exactly two conjugate complex pairs of them. They were subsequently computed to seven decimals by Franklin [43]. Nielsen [103] proved that all zeros of $\Gamma(a, x)$, for $x > 0$, lie in the half-plane $\operatorname{Re} a > x$. Rasch [120] gave an asymptotic formula for the number $M(x)$ of pairs of conjugate complex a-zeros of $\gamma(a, x)$ as $x \to \infty$. Hille and Rasch [60] already in 1929, and Mahler [96] in 1930, investigated the behavior of the zeros when x is a fixed complex number; they also identified zero-free regions in the complex a-plane.

Other texts on confluent hypergeometric functions are the one by Buchholz [17] published shortly before Tricomi's monograph, and one published later by Slater [125]. The former is based on Whittaker's definition [160, Ch. 16] of the confluent hypergeometric functions, a definition not favored by Tricomi; the latter also contains numerical tables. A detailed treatment of the probability integral and some of its generalizations, notably $\Phi(z, a) = \pi^{-1/2}\gamma(\frac{1}{2}a, z^2)$, can be found in a monograph by Hadži [58].

2. Tricomi's Contributions

2.1. *Normalization.* The integral $\gamma(a, x)$ has the inconvenience of not only having poles at the nonpositive integers $a = 0, -1, -2, \dots$, but also representing a multivalued function of the complex variable x, owing to the fractional power in the integrand. Both these inconveniences can be avoided by introducing, as Tricomi does in [146] and Böhmer before him in [15, pp. 124–125], the function

$$\gamma^*(a,x) = \frac{x^{-a}}{\Gamma(a)}\,\gamma(a,x), \tag{2.1}$$

which is an entire function in a as well as in x and real-valued for real a and real x (also for $x < 0$). In particular,

$$\gamma^*(-n,x) = x^n, \quad n = 0,1,2,\dots\ . \tag{2.2}$$

In terms of the function (2.1), both incomplete gamma functions in (1.1) can be represented as

$$\gamma(a,x) = \Gamma(a)x^a\gamma^*(a,x), \quad \Gamma(a,x) = \Gamma(a)[1 - x^a\gamma^*(a,x)], \tag{2.3}$$

where fractional powers of x, as always in this theory, are to be understood as having their principal values. Tricomi finds it useful to introduce yet another form of the incomplete gamma function, namely

$$\gamma_1(a,x) = \Gamma(a)x^a\gamma^*(a,-x), \tag{2.4}$$

for which, as he notes (cf. [151, p. 161]), one has

$$\gamma_1(a,x) = \int_0^x e^t t^{a-1} dt, \quad \mathrm{Re}\, a > 0. \tag{2.5}$$

This function allows the values of $\gamma(a,x)$ above and below the branch cut along the negative real axis to be expressed as

$$\gamma(a,-x \pm i0) = e^{\pm a\pi i}\gamma_1(a,x), \quad x > 0. \tag{2.6}$$

2.2. *Series expansions.* To the classical power series expansions Tricomi in [147] adds expansions in Bessel functions, which he obtains as special cases of similar expansions he derived for the confluent hypergeometric functions. Characteristically, Tricomi adopts a form of the Bessel functions which makes them entire functions of both the variable and the order, namely

$$J_\nu^*(x) = x^{-\nu/2} J_\nu(2\sqrt{x}). \tag{2.7}$$

(Tricomi's notation for them is $E_\nu(x)$.) In terms of these functions, he derives the expansion

$$\gamma^*(a,x) = e^{-x} \sum_{n=0}^{\infty} e_n(-1) x^n J^*_{a+n}(-x), \tag{2.8}$$

where $e_n(\,\cdot\,)$ is the $(n+1)$st partial sum of the exponential series (cf. §1). For real arguments a and x, one can write (2.8) as an expansion in (ordinary) Bessel functions $J_{a+n}(2\sqrt{|x|})$ if x is negative (see [147, 2d Eq. (39)], where, however, the factor x^n should read $x^{n/2}$), and as a similar expansion in modified Bessel functions $I_{a+n}(2\sqrt{x})$ if x is positive. Both converge rather well, when $0 \le a < 1$ (for other values of a the recurrence relation (6.4) can be used), but the former suffers increasingly from internal cancellations as $|x|$ becomes large.

For good measure, Tricomi obtains yet another expansion,

$$\gamma^*(a,x) = e^{-x/2} \sum_{n=0}^{\infty} c_n \left(\frac{x}{2}\right)^n J^*_{a+n}\left(\frac{a-1}{2}x\right), \tag{2.9}$$

where the coefficients c_n can be obtained recursively from[1]

$$\begin{aligned} c_0 &= 1, \quad c_1 = 0, \\ c_n &= c_{n-2} + L_n^{(1-a-n)}(1-a), \end{aligned} \tag{2.10}$$

and $L_n^{(\alpha)}$ are the Laguerre polynomials. The peculiar form $\lambda_n(y) = L_n^{(y-n)}(y)$ of the Laguerre polynomials appearing in (2.10) is studied by Tricomi in [148], where he derived the recursion

$$\begin{aligned} &\lambda_0(y) = 1, \quad \lambda_1(y) = 0, \\ &\lambda_{n+1}(y) = -\frac{1}{n+1}\left[n\lambda_n(y) + y\lambda_{n-1}(y)\right], \quad n = 1, 2, \ldots . \end{aligned} \tag{2.11}$$

(The same polynomials are also used by Temme [134] in uniform asymptotic expansions of Laplace transforms.) The series (2.9) seems to converge (for $0 \le a < 1$) somewhat faster than (2.8) and, for $x < 0$, also suffers less from

[1] There is a misprint in [147, line after Eq. (41)] in that A_1^* (our c_1) is erroneously defined to be 1 instead of 0.

internal cancellations. We used it (in IEEE Standard double precision) to produce the plots in §2.4 as well as the graphs in Fig. 6.1.

Although Tricomi refers to the polynomials $\lambda_n(y)$ as being nonorthogonal, Carlitz [19] showed that in fact $(-1)^{n+1}(n+2)x^{n+2}\lambda_{n+2}(x^{-2})$, $n = 0, 1, 2, \dots$, is a set of (monic) orthogonal polynomials relative to a measure that is discretely supported on the points $x_j = \pm j^{-1/2}$ with jumps $\frac{1}{2}j^{j-1}e^{-j}/j!$, $j = 1, 2, 3, \dots$. These polynomials occur also as "random walk polynomials" in the work of Karlin and McGregor [70, Appendix B] on birth and death processes. Their asymptotics and zero distribution are studied in [52] and [53].

Other series expansions which may be original with Tricomi are an expansion [146, Eq. (44)] of $\Gamma(a, x)$ in Laguerre polynomials $L_n^{(a)}(x)$, and an expansion (ibid., Eq. (45)) of $\gamma(a, \lambda x)$ in $\{\gamma(a + n, x)\}$.

2.3. *Asymptotics.* The asymptotic behavior of the incomplete gamma functions is elementary when only one of the two parameters a and x tends to infinity. More interesting (and also more difficult) is the behavior when $|a|$ and $|x|$ become large simultaneously. Here Tricomi shows in [146] (see also [151, §4.3]) that the matter depends on whether a and x are not near each other, or x is near $a - 1$, as $|a|$ and $|x|$ both tend to infinity. In the first case, he proves from the integral representation of $\Gamma(a + 1, x)$ that

$$\Gamma(a+1,x) = \frac{e^{-x}x^{a+1}}{x-a}\left[1 - \frac{a}{(x-a)^2} + \frac{2a}{(x-a)^3} + O\left(\frac{a^2}{(x-a)^4}\right)\right] \tag{2.12}$$

as the modulus of $\sqrt{a}/(x - a)$ tends to zero and its argument ultimately remains between $-3\pi/4$ and $3\pi/4$. He in fact has the complete asymptotic expansion in explicit (though complicated) form. In the second case there are two subcases depending on whether $\operatorname{Re} a$ is positive or negative. Equivalently, Tricomi considers the functions $\gamma(1 + a, x)$ and $\gamma_1(1 - a, x)$ separately, both under the assumption $\operatorname{Re} a > 0$. In the first subcase, again from the integral representation (1.1), he finds, when a and y are both real and y bounded, that

$$\gamma(a+1, a+\sqrt{2a}\,y) = \frac{1}{2}\Gamma(a+1)\left[1 + \operatorname{erf} y - \frac{2}{3}\sqrt{\frac{2}{a\pi}}(1+y^2)e^{-y^2} + O\left(\frac{1}{a}\right)\right], \quad a \to \infty. \tag{2.13}$$

(For a simplified derivation of (2.13), see also [152].) A similar result holds for complex a, y (with $\mathrm{Re}\, a > 0$), and again, Tricomi is able to write down the complete asymptotic expansion.

As a by-product of (2.13) and (1.3), one obtains a pretty asymptotic estimate for $e_n(x)$ near $x = n$, namely

$$e_n(n+\sqrt{2n}\,y) = \frac{1}{2}e^{n+\sqrt{2n}\,y}\left[\mathrm{erfc}\, y - \frac{2}{3}\sqrt{\frac{2}{n\pi}}(1+y^2)e^{-y^2} + O\left(\frac{1}{n}\right)\right], \quad n \to \infty. \tag{2.14}$$

In the second subcase, Tricomi finds, for real $a > 0$ and $y \in \mathbb{R}$ bounded, that

$$\begin{aligned}\gamma_1(1-a, a+\sqrt{2a}\,y) = \frac{1}{\Gamma(a)}\Bigg[&-\pi \cot a\pi + 2\sqrt{\pi}\int_0^y e^{t^2}dt \\ &+\frac{2}{3}\sqrt{\frac{2\pi}{a}}(1-y^2)e^{y^2} + O\left(\frac{1}{a}\right)\Bigg], \quad a \to \infty.\end{aligned} \tag{2.15}$$

2.4. *Zeros.* In [147] (see also [151, §4.4]) Tricomi studies the zeros of $\gamma^*(a,x)$, $a \in \mathbb{R}$, $x \in \mathbb{R}$, considered as a function of x for fixed a. Except possibly for $x = 0$, these zeros coincide with the zeros of $\gamma(a,x)$ or $\gamma_1(a,-x)$. Tricomi gives a complete description of these zeros and, more generally, a remarkable contour map of the function $\gamma^*(a,x)$, i.e., of the lines $\gamma^*(a,x) =$ const. In Fig. 2.1 we reproduce this map, and also provide the associated surface plot; they were generated by the MATLAB commands `contour` and `surf`, respectively. The function itself was computed with the help of the series expansion (2.9) for $0 \le a < 1$ and the recurrence relation (6.4) for other values of a.

From his asymptotic results in [146], Tricomi derives the following asymptotic approximations (as corrected by Kölbig [78]) for the real zeros: for the positive zeros $x_+(a)$ of $\gamma^*(a,x)$,

$$x_+(a) = \tau|a| - \frac{\tau}{1+\tau}\log\left[\frac{(1+\tau)\sqrt{(1+|a|)\pi/2}}{\sin a\pi}\right] + O\left(\left(\frac{\log|a|}{a}\right)^2\right),$$
$$a < 0, \quad \sin a\pi > 0, \quad a \to -\infty, \tag{2.16}$$

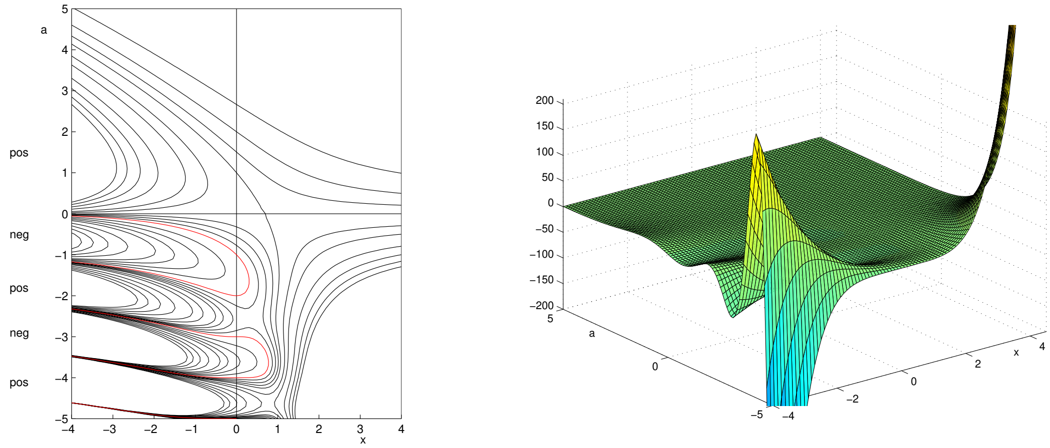

FIGURE 2.1. Contour map and surface plot of $\gamma^*(a, x)$. The lines in the contour map correspond to altitudes –6(1)–2(.5)0(.25)1(.5)2(1)6, the zero line being red.

where $\tau = .27846454\ldots$ is the unique positive root of the equation $1 + x + \log x = 0$; for the negative zeros,

$$x_-(a) = -(1 + |a|) - \sqrt{2(1 + |a|)}\, y_0 + O(|a|^{-1/2}), \quad a \to -\infty, \tag{2.17}$$

where $y_0 = y_0(a)$ is the unique root of $\int_0^y e^{t^2} dt = \frac{\sqrt{\pi}}{2} \cot(|a|\pi)$, provided $|a|$ is not too close to a positive integer.

2.5. *Inequalities and monotonicity.* Obviously,

$$g(a, x) := \frac{\gamma(a, x)}{\Gamma(a)}, \quad a > 0,\ x > 0, \tag{2.18}$$

is a probability distribution on $[0, \infty)$; thus, in particular, $0 \le g(a, x) \le 1$ and g is monotonically increasing in x. In [147] Tricomi proves that g is monotone also in a, namely decreasing. Interestingly enough, Tricomi uncovers similar monotonicity properties also in the regions $a < 0$, $x > 0$ and $a > 0$, $x < 0$. In the former region,

$$G(a, x) := -a e^x x^{-a} \Gamma(a, x), \tag{2.19}$$

and in the latter,

$$g_1(a, |x|) := ae^{-|x|}|x|^{-a}\gamma_1(a, |x|), \tag{2.20}$$

are both between 0 and 1 and are monotone in x as well as in a. More difficult (not surprisingly in view of Fig. 2.1) is the region $a < 0$, $x < 0$. Here Tricomi manages to prove that $|g^*(a, x)| \leq 1$, where

$$g^*(a, x) := \frac{e^x \gamma^*(a, x)}{\Gamma(|a| + 1)}. \tag{2.21}$$

Moreover, as a function of x, with a held fixed, g^* has one, or at most two, maxima or minima.

2.6. *Applications.*

2.6.1. *Number theory.* It is known from a well-known theorem of Lagrange that each positive integer can be decomposed into a sum of (at most) four perfect squares, whereas only some integers are decomposable into a sum of two squares, and even fewer into a sum of two cubes, or two fourth powers, or, more generally, two kth powers, $k = 2, 3, 4, \ldots$. The problem of determining the distribution $N_k(x)$ of all positive integers $\leq x$ that are the sum of exactly two kth powers seems to have led Tricomi in 1938 to his first encounter of the incomplete gamma function. By probabilistic heuristics, and at times — as he says [151, p. 286], "acrobatic" — arguments, Tricomi in [143] indeed arrives at the approximation

$$N_k(x) \approx x - \frac{k}{k-2} A_k^{\frac{k}{k-2}} \Gamma\left(-\frac{k}{k-2}, A_k x^{\frac{2-k}{k}}\right), \quad k \geq 3, \tag{2.22}$$

where

$$A_k = \frac{[\Gamma(1/k)]^2}{2k^2\Gamma(2/k)}.$$

The nature of the approximation in (2.22), given its roundabout derivation, is of course unclear, but definitely is not asymptotic for $x \to \infty$, since a similarly derived approximation for $k = 2$ gives $N_2(x) \approx (1 - e^{-\pi/8})x$, whereas, by a result of Landau, $N_2(x) \sim bx/\sqrt{\log x}$ (cf. [86]), with a well-determined constant b approximately equal to .764 (cf. [151, p. 289]). Nevertheless, for x not too large, the formula (2.22) seems to give excellent results, as Tricomi demonstrates for $k = 3$ and $x \leq 2000$.

Precise asymptotic results have been obtained only more recently, for example in [61] for $k = 3$, and in [62] for $k\,(\text{odd}) \geq 5$.

2.6.2. *Random walks.* A problem of interest in physics, biology, and other areas of science, is the following. Given randomly n unit vectors in Euclidean space $\mathbb{R}^d$, what is the probability $P_n(r) = \Pr(\|s\| < r)$ that their sum s has length $< r$, where $0 \leq r \leq n$? The problem has been solved in 1906 for $d = 2$ by J.C. Kluyver (even for vectors of arbitrary fixed lengths), with full details, for $d = 3$, supplied later in 1919 by Lord Rayleigh. In the case of general d, the result is derived in Watson [159, p. 421], where $P_n(r)$ is given in the form of an integral involving Bessel functions (here written in terms of Tricomi's Bessel functions),

$$P_n(r) = [\Gamma(d/2)]^{n-1}(r^2/n)^{d/2}\int_0^\infty t^{(d/2)-1}J^*_{d/2}(r^2t/n)[J^*_{(d/2)-1}(t/n)]^n dt. \tag{2.23}$$

What is of particular interest in applications is the behavior of $P_n(r)$ as $n \to \infty$. Watson already studied this informally by applying the method of steepest descent and arriving at an asymptotic approximation involving ${}_1F_1\left(\frac{d}{2}; \frac{d}{2}+1; -\frac{r^2d}{2n}\right)$, hence the incomplete gamma function. In [149] Tricomi, by a more rigorous approach using power series and Laplace transform techniques, improves upon Watson's result, showing that

$$P_n(r) = x^{\frac{d}{2}}\left[\gamma^*\left(\frac{d}{2}, x\right) - \frac{1}{\Gamma(d/2)}\frac{e^{-x}}{n}\left(\frac{1}{2} - \frac{x}{d+2}\right)\right] + O(n^{-2}), \tag{2.24}$$

where

$$x = \frac{r^2d}{2n}.$$

The behavior of the leading term in (2.24) is illustrated in Fig. 2.2 for $n = 10$ and $d = 2, 3, 5, 10$.

From (2.24) it takes a quick calculation for Tricomi to determine the mean value $\overline{r}$ of r, namely

$$\overline{r} = \int_0^\infty r\frac{dP_n}{dr}dr = \frac{\Gamma((d+1)/2)}{\Gamma(d/2)}\sqrt{\frac{2n}{d}}\left[1 + \frac{1}{4(d+2)n} + O(n^{-2})\right]. \tag{2.25}$$

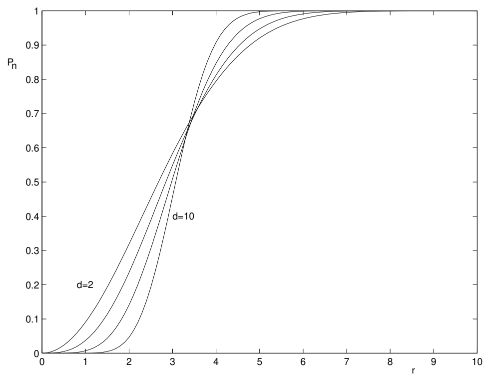

FIGURE 2.2. The probability $P_n(r)$ for $n = 10$ and $d = 2, 3, 5, 10$

2.6.3. *Laguerre's equation.* The Laguerre polynomial $L_n^{(\alpha)}(x)$, as is well known, is a solution of the linear second-order differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0, \tag{2.26}$$

a special case of the confluent hypergeometric equation. The second solution, $y_2(x)$, therefore, must be a confluent hypergeometric function, which Tricomi in [150], when α is not an integer, identifies explicitly in terms of the incomplete gamma function and products of Laguerre polynomials. Specifically,

$$y_2(x) = L_n^{(\alpha)}(x)\gamma_1(-\alpha, x) + e^x x^{-\alpha} \sum_{k=1}^{n} \frac{1}{k} L_{n-k}^{(\alpha+k)}(x) L_{k-1}^{(\alpha-k)}(-x). \tag{2.27}$$

(For γ_1, see (2.4).) There is an analogous formula involving $\Gamma(0, -x)$ when α is an integer.

2.7. *Miscellanea.* Without in any way wanting to disparage the results contained in this subsection, it seems fair to say that they lie at the fringes of the general theory of incomplete gamma functions and are therefore mentioned only in passing.

One concerns the gamma function itself, more precisely the ratio of two gamma functions, $\Gamma(z + a)/\Gamma(z + b)$, for which in [145] and [154] the complete asymptotic expansion in descending powers of z is derived, with an explicit characterization of the coefficients and the precise conditions of validity. Error bounds for this and similar expansions have later been obtained by Frenzen [44], [45].

In order to derive (2.15), Tricomi made use of the following (apparently new) integral representation of $\gamma^*(a,x)$ for real a and x,

$$\gamma^*(a,x) = \frac{e^{-x}}{\Gamma(a)\sin a\pi}\,\mathrm{Re}\left\{e^{-a\pi i}\int_0^\infty e^{-ixt}(1+it)^{a-1}dt\right\}. \tag{2.28}$$

Another curious integral representation is the one for the "norm" of the incomplete gamma function, $\Gamma(a,ix)\Gamma(a,-ix)$, which in [144] (or [40, §9.3, Eq. (6)]) is expressed in terms of the Laplace transform of a sum of two conjugate complex hypergeometric functions.

3. Asymptotics

3.1. *An improved approximation (2.13).* Formula (2.13), after division by $\Gamma(a+1)$, can be interpreted as an asymptotic approximation of the gamma distribution in terms of the normal distribution $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}dt$, the leading term in (2.13) indeed being $\Phi(\sqrt{2}y)$. A more accurate approximation has been derived by Pagurova [111] by statistical arguments; it involves derivatives of the normal distribution, hence Hermite polynomials $He_n(x)$,

$$\begin{aligned}\tfrac{\gamma(a,a+x\sqrt{a})}{\Gamma(a)} &= \Phi(x) - \tfrac{e^{-x^2/2}}{\sqrt{2\pi}}\Big\{\tfrac{1}{3\sqrt{a}}He_2(x) + \tfrac{1}{2a}[\tfrac{1}{2}He_3(x)+\tfrac{1}{9}He_5(x)] \\ &+\tfrac{1}{a\sqrt{a}}[\tfrac{1}{5}He_4(x)+\tfrac{1}{12}He_6(x)+\tfrac{1}{162}He_8(x)] + \tfrac{1}{6a^2}[He_5(x)+\tfrac{47}{80}He_7(x) \\ &+\tfrac{1}{12}He_9(x)+\tfrac{1}{324}He_{11}(x)] + \tfrac{1}{a^2\sqrt{a}}[\tfrac{1}{7}He_6(x)+\tfrac{19}{180}He_8(x)+\tfrac{31}{1440}He_{10}(x) \\ &+\tfrac{1}{648}He_{12}(x)+\tfrac{1}{29160}He_{14}(x)] + O(a^{-3})\Big\}.\end{aligned} \tag{3.1}$$

(The $a^{-1/2}$ term in (3.1), with $He_2(x) = x^2-1$, is consistent with the corresponding term in (2.13). This can be seen by applying the recurrence relation (6.4) to the left-hand side of (2.13), letting $x=\sqrt{2}y$, and applying elementary asymptotics to the additive term coming from the recurrence relation.) A similar, even more accurate, approximation (without the $a^{-1/2}$ term) is also derived, but it involves on the right-hand side a more complicated variable.

3.2. *Uniform asymptotics.* In deriving asymptotic results for large $|a|$, Tricomi found it necessary (cf. (2.12) and (2.13)) to distinguish cases according to the magnitude of $|x|$ relative to $|a|$. One of the major advances since

Tricomi's work in this area is the development of asymptotic expansions for large a that hold uniformly for, say, all $x \geq 0$. There is a price to be paid, however, for uniformity: For one, the expansion involves not only elementary but also transcendental functions, specifically the error function erfc in our case; for another, the calculation of the expansion coefficients is much more intricate.

Uniform asymptotic expansions for the incomplete gamma functions were first derived by Temme [131], [132] (see also [140, §11.2.4]). His point of departure is the integral representation

$$\frac{\Gamma(a,z)}{\Gamma(a)} = \frac{e^{-a\phi(\lambda)}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{a\phi(t)} \frac{dt}{\lambda - t}, \quad 0 < c < \lambda, \tag{3.2}$$

where

$$\phi(t) = t - 1 - \ln t, \quad \lambda = \frac{z}{a}. \tag{3.3}$$

The integrand in (3.2) has a saddle point at $t = 1$. Changing the contour of integration into a path of steepest descent, and separating out the pole close to the saddle point (when $\lambda \approx 1$), Temme arrives at asymptotic representations of the type

$$\begin{aligned} \frac{\Gamma(a,z)}{\Gamma(a)} &= \tfrac{1}{2}\mathrm{erfc}(\eta\sqrt{a/2}) + R_a(\eta), \\ \frac{\gamma(a,z)}{\Gamma(a)} &= \tfrac{1}{2}\mathrm{erfc}(-\eta\sqrt{a/2}) - R_a(\eta), \\ R_a(\eta) &\sim \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \sum_{n=0}^{\infty} \frac{c_n(\eta)}{a^n}, \quad a \to \infty, \end{aligned} \tag{3.4}$$

where

$$\eta = (\lambda - 1)\sqrt{2\frac{\lambda - 1 - \ln\lambda}{(\lambda-1)^2}}\ . \tag{3.5}$$

(When $\lambda > 0$, then $\eta = \pm\sqrt{2(\lambda - 1 - \ln\lambda)}$ with the plus sign for $\lambda > 1$ and the minus sign for $\lambda < 1$.) The asymptotic expansion of $R_a(\eta)$ is valid for a going to infinity over positive values, and is uniform for all $\lambda \geq 0$, i.e., for all $z \geq 0$. Its validity, indeed, can be established for complex a and z as well; that is, (3.4) is valid as $a \to \infty$ uniformly in $|\arg a| \leq \pi - \varepsilon_1$ and $|\arg z/a| \leq 2\pi - \varepsilon_2$, where ε_1, ε_2 are positive numbers with $0 < \varepsilon_1 < \pi$, $0 < \varepsilon_2 < 2\pi$.

As to the coefficients $c_n(\eta)$ in (3.4), they are holomorphic functions of η and can be computed for small $|\eta|$ by a power series expansion, and for other values of $|\eta|$ by recurrence. For details, as well as for estimates of the remainder terms when the expansion in (3.4) is truncated at some finite $n = N - 1$, we must refer to the original paper [132]; also see [114]. An extensive set of Taylor coefficients for $c_n(\eta)$ is given in [30, Appendix F]. The growth of $c_n(\eta)$ as $n \to \infty$ is studied in [34].

For a rearranged version of the expansion (3.4), in the context of the Riemann zeta function, see also [113, Appendix A].

It is interesting to note the role played by the error function in (3.4). If $z = \lambda a$, with a and λ positive, then $\Gamma(a, \lambda a)$ as a function of λ exhibits a sharp decrease near the transition point $\lambda = 1$, the decrease being sharper the larger a. Elementary functions would have a hard time describing this kind of behavior, but the error function does a nice job of it; this is shown in Fig. 3.1.

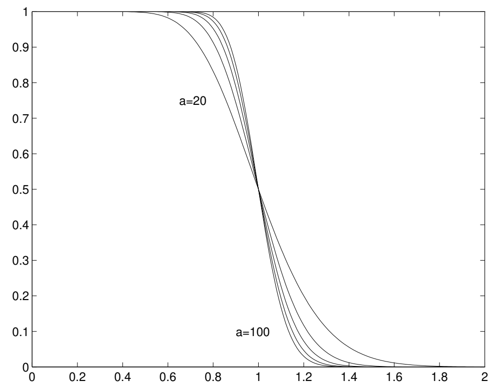

FIGURE 3.1. The leading term $\frac{1}{2}\mathrm{erfc}(\eta\sqrt{a/2})$, in (3.4) as a function of λ in $0 < \lambda \le 2$, for $a = 20(20)100$

At the transition point $\lambda = 1$ one has $\eta = 0$, and from the first of (3.4) one gets

$$\frac{\Gamma(a,a)}{\Gamma(a)} \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi a}} \sum_{n=0}^{\infty} \frac{c_n(0)}{a^n}, \quad a \to \infty.$$

A similar expansion in which the factor multiplying the series is replaced by the asymptotically equivalent factor $\Gamma(a+1)(e/a)^a$ is given in [89], together with an asymptotic representation of the coefficients for large indices and the first eleven coefficients expressed exactly in rational form.

The expansions (3.4) do not cover negative values of a, but there are similar uniform expansions for $\gamma(-a,-x)$ and $\Gamma(-a,-z)$, involving the same coefficients $c_n(\eta)$, that are valid in $|\arg a| \le \pi - \delta_1$, $|\arg \lambda| \le 2\pi - \delta_2$, with δ_1, δ_2 arbitrarily small positive constants (cf. [139]). Of special interest is the expansion for $\gamma^*(-a,-x)$, where a and x are positive and γ^* real but oscillatory [139, Eq. (3.11)].

An alternative derivation of (3.4) and, with similar methods, of Tricomi's expansions in §2.3, along with numerical comparisons, is given by Schell in [121].

Applying differential equations rather than integral representations, specifically the asymptotic theory of linear second-order differential equations with almost coalescent turning points, Dunster in [31] derives an alternative asymptotic approximation (not expansion) for $\Gamma(a,z)$ that also involves the complementary error function, but an auxiliary variable ζ rather more complicated than the η in Temme's expansion (3.4). The approximation holds, for example, when $a \to \infty$, uniformly for z in a domain containing the positive real axis, but there are other possible interpretations of its asymptotic character. For details, we refer to the original paper [31, Remarks on p. 1346].

3.3. *The generalized exponential integral.* If we take $n = p$ in (1.4) to be an arbitrary complex number, we are led to consider the generalized exponential integral

$$E_p(z) = z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt. \tag{3.6}$$

Even though closely related to the incomplete gamma function, it arises in this form in many applications and has attracted a considerable amount of interest in recent years.

3.3.1. *Asymptotic expansion for $p \to \infty$.* If p goes to ∞ over positive values $p > 1$, and x is an arbitrary nonnegative number, it was shown in [46] by elementary means, involving integration by parts, that

$$E_p(x) = \frac{e^{-x}}{x+p}\left[\sum_{k=0}^{n-1} H_k\left(\frac{x}{p}\right)p^{-k} + \Theta_n(x,p)p^{-n}\right], \tag{3.7}$$

where

$$\alpha_n \le \Theta_n(x,p) \le \beta_n\left(1+\frac{1}{x+p-1}\right). \tag{3.8}$$

Here,

$$H_k(u) = \frac{h_k(u)}{(1+u)^{2k}}\ , \quad k = 0, 1, 2, \dots\ ,$$

where $h_k(u)$ is a polynomial of degree $k-1$ (if $k > 0$) defined recursively by

$$h_0(u) = 1, \quad h_{k+1}(u) = (1 - 2ku)h_k(u) + u(1+u)h_k'(u), \quad k = 0, 1, 2, \dots\ ,$$

and α_n, β_n are lower and upper bounds, respectively, of $H_n(u)$ on the interval $u \geq 0$. The first eight polynomials $h_k(u)$ and respective constants α_k, β_k are given explicitly in [46]. For improvements, both in the error bounds and the approximations, see also [6, §3]. More recently, Dunster [32, Thm. 2.1] showed that the same[2] expansion (3.7) with a different bound on the error term (and different notations), holds uniformly for complex x in the domain $|\arg x| \leq \pi - \delta$, provided $p > (1 - \cos\delta)^{-1}$.

An alternative asymptotic approximation for $p\,(> 0) \to \infty$, valid for complex x in a domain containing the negative real axis, is given in [32, Thm. 3.1]. Similarly to the asymptotic approximation for the incomplete gamma function derived by the same author in [31], it also involves the complementary error function and a rather complicated auxiliary variable ζ. For an asymptotic expansion, including error bounds, see also [33, Thms. 5.1 and 5.2].

3.3.2. *Stokes's phenomenon and uniform exponential improvement.* For fixed p, as $z \to \infty$ in $|\arg z| \leq \frac{3}{2}\pi - \delta$, where δ is an arbitrarily small positive number, one has the classical asymptotic expansion

$$E_p(z) \sim \frac{e^{-z}}{z} \sum_{k=0}^{\infty} (-1)^k \frac{(p)_k}{z^k}, \tag{3.9}$$

where $(p)_k$ denotes the ascending factorial $p(p+1)\cdots(p+k-1)$. In the sector $\frac{1}{2}\pi + \delta \leq \arg z \leq \frac{7}{2}\pi - \delta$, which partly overlaps with the preceding sector, one has an asymptotic expansion just like (3.9) but with an additional term

$$\frac{2\pi i e^{-p\pi i}}{\Gamma(p)} z^{p-1}. \tag{3.10}$$

In the common sector $\frac{1}{2}\pi + \delta \leq \arg z \leq \frac{3}{2}\pi - \delta$ this term is exponentially small compared to the main term in (3.9). Nevertheless, as z crosses the

[2] There is a sign error in Eqs. (2.11) and (2.13) of [32], where the second term on the right should be subtracted instead of added.

line $\arg z = \pi$, there occurs a rapid, though smooth, change in the form of the asymptotic expansion. This is known as the Stokes phenomenon. It has been analyzed in a formal (but insightful) manner by Berry [11] and more rigorously by Olver [107], who writes the remainder term in (3.9), if truncated after the nth term, as follows,

$$E_p(z) = \frac{e^{-z}}{z}\sum_{k=0}^{n-1}(-1)^k\frac{(p)_k}{z^k} + \frac{2\pi i e^{-p\pi i}}{\Gamma(p)} z^{p-1} T_{n+p}(z), \tag{3.11}$$

where

$$T_q(z) = \frac{e^{q\pi i}\Gamma(q)}{2\pi i}\frac{E_q(z)}{z^{q-1}}. \tag{3.12}$$

The interest lies in the sector $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ (containing the "Stokes line" $\arg z = \pi$), where the factor T_{n+p} in (3.11) acts as a "Stokes multiplier" (cf. (3.10)). If n is chosen optimally, i.e., the series (3.9) is truncated just before its numerically smallest term, and if $z = \rho e^{i(\pi+\theta)}$, $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$, then $n = \rho - p + \alpha$, where α is complex and bounded in absolute value as $\rho \to \infty$. By a delicate analysis, Olver then finds an asymptotic representation of the Stokes multiplier in the form

$$T_{n+p}(z) \sim \tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}(\eta\sqrt{\rho/2}) + \frac{e^{-\frac{1}{2}\rho\eta^2}}{\sqrt{2\pi\rho}}\sum_{k=0}^{\infty} c_k(\theta,\alpha)\rho^{-k}, \quad \rho\to\infty, \tag{3.13}$$

which holds uniformly for $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and for $|\alpha|$ bounded. Here η is a complex-valued function of θ defined by

$$\tfrac{1}{2}\eta^2 = 1 + i\theta - e^{i\theta}.$$

(The branch with $\mathrm{Re}\,\eta > 0$ is taken for $\theta > 0$ and the one with $\mathrm{Re}\,\eta < 0$ for $\theta < 0$.) The (complex) error function in (3.13) plays a similar role as the error function in (3.4), the transition point now being at $\theta = 0$. Plots of the real and imaginary parts of the leading term in (3.13) are shown in Fig. 3.2 as functions of λ, where $\theta = \lambda\pi/2$.

An alternative discussion of Stokes's phenomenon is given more recently by Dunster in [32, §§4 and 5].

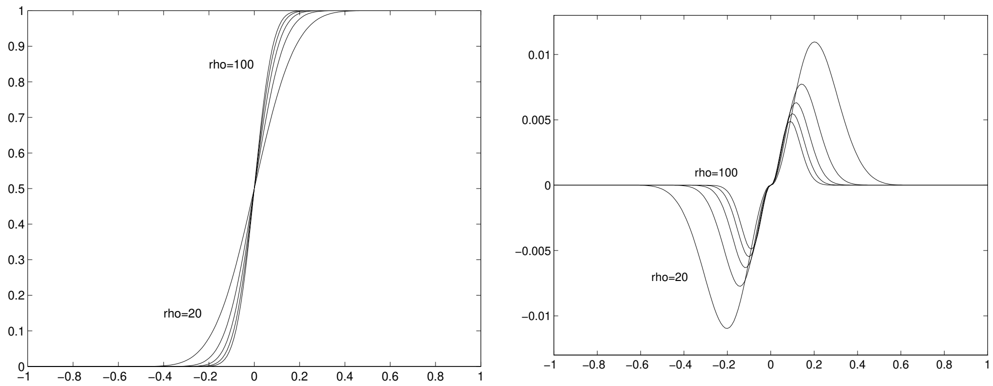

FIGURE 3.2. The real and imaginary parts of the leading term in (3.13) as functions of λ, $\theta = \lambda\pi/2$, $-1 \le \lambda \le 1$, for $\rho = 20(20)100$

In [108] it is shown that choosing n optimally as described, and expanding the remainder term in (3.11) in descending powers of ρ, provides in the domain $|\arg z| \le \pi - \delta$ a "uniformly exponentially improved" approximation in the sense that truncating the remainder expansion at any fixed term yields, in combination with the (optimally) truncated expansion (3.9), an approximation of $E_p(z)$ whose relative error is exponentially small, uniformly in the domain indicated. In the same work the asymptotic nature of the expansion (3.13) is further analyzed, its validity extended to the sector $-2\pi + \delta \le \theta \le 2\pi - \delta$, and new, more convenient, formulae are given for the coefficients. (Note, however, that the notation in [108] differs somewhat from the notation in [107].)

Much of what is discussed in this §3.3, and more, is nicely summarized by Olver in [110]. See also §7.2 for repeated re-expansion of remainders.

4. Inverse Functions and Zeros

4.1. *Inverse functions.* For any $a > 0$, the functions

$$P(a,x) = \frac{\gamma(a,x)}{\Gamma(a)}, \quad Q(a,x) = \frac{\Gamma(a,x)}{\Gamma(a)}, \tag{4.1}$$

satisfying $P(a,x) + Q(a,x) = 1$, are cumulative probability functions on the interval $x \ge 0$. For example, $a = \frac{\nu}{2}$, $x = \frac{1}{2}\chi^2$, where $\nu \ge 1$ is an integer, yields the chi-square probability functions with ν degrees of freedom. For this reason, their inverse functions are important in statistical applications, where, given any p with $0 < p < 1$, or $q = 1 - p$, one is interested in determining x such that

$$P(a,x) = p, \quad \text{or (equivalently)}, \quad Q(a,x) = q. \tag{4.2}$$

In principle, this amounts to solving a nonlinear equation, for which many iterative methods are available such as Newton's method. In practice, however, these would require good initial approximations as well as repeated evaluations of the incomplete gamma function, both of which can render the inversion costly. It is desirable, therefore, to be able to solve the equations (4.2) more directly and economically.

The case $a = \frac{1}{2}$ of the error function (cf. (1.6)) is particularly simple, as the inverse error function is a function of a single real variable on $[0, 1]$ and hence accessible to approximation-theoretic methods. Thus, Strecok in [129] uses Chebyshev expansions in appropriate variables and ranges to obtain accuracies in the region $[0, 1-10^{-300}]$ of at least 18 significant decimal digits, whereas Blair et al. [13] use rational approximations to obtain even higher accuracies of up to 23 digits on a larger domain, $[0, 1 - 10^{-10000}]$. They also provide an asymptotic series approximation accurate to at least 25 digits for the remaining interval $[1 - 10^{-10000}, 1]$.

For general a, Temme [136] employs his uniform asymptotic expansion (3.4) to do the inversion. Thus, the second equation in (4.2), for example, in combination with the first in (3.4), takes the form

$$\tfrac{1}{2}\mathrm{erfc}(\eta\sqrt{a/2}) + R_a(\eta) = q. \tag{4.3}$$

This is first solved for η, whereupon (3.5) is used to solve for λ, which by (3.3) finally yields $x = a\lambda$. To solve (4.3) for η, one can take as initial approximation η_0 the solution of

$$\tfrac{1}{2}\mathrm{erfc}(\eta_0\sqrt{a/2}) = q, \tag{4.4}$$

which is computable in terms of the inverse error function previously discussed. Then for η one seeks an asymptotic expansion of the form

$$\eta \sim \eta_0 + \frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{a^2} + \frac{\varepsilon_3}{a^3} + \cdots, \quad a \to \infty. \tag{4.5}$$

The determination of the coefficients ε_i is laborious. It is shown in [136] that they are analytic functions of η_0 for $|\eta_0| < 2\sqrt{\pi}$ and therefore can be expanded in powers of η_0. For the first four coefficients ε_i, the power series are given in exact rational form up to 17, 11, 9, and 7 terms, respectively. For larger values of η_0, the same four coefficients are expressed algebraically in terms of η_0, $\varepsilon_1 = \eta_0^{-1} \ln f(\eta_0)$ and $f(\eta_0) = \frac{\eta_0}{\lambda_0 - 1}$, where λ_0 is the solution of (3.5) for $\eta = \eta_0$. The procedure is particularly effective for large values of a, but yields typically 3 to 4 correct decimal digits already for $a = 1$ or $a = 2$.

For the chi-square distribution an alternative asymptotic inversion is described in [41], which is valid for small q and a fixed.

4.2. *Real zeros.* The unique positive zero of $\mathrm{Ei}(x) = \frac{1}{2}[E_1(-x + i0) + E_1(-x - i0)]$ is given to 30 decimal places in [29, p. 300]. Asymptotic approximations to the positive zeros of the sine and cosine integrals can be found in [37] and [127], respectively.

Tricomi's interest, as noted in §1, was in the zeros of $\gamma(a, x)$ considered as a function of x for fixed $a < 0$. Little (to the author's knowledge) has been done on this problem beyond Tricomi's work. Curiously, though, the negative zero $x_-(a)$ of $\gamma(a, x)$ for $-1 < a < 0$ has received some scrutiny in connection with a probability density (encountered by Mandelbrot) whose Fourier transform is $[\Gamma(1+a)\gamma^*(a, -is)]^{-1}$. Lew [87] indeed shows that $x_-(a)$ decreases monotonically in $[-1, 0]$ (which can also be read off from the contour map of Fig. 2.1) and satifies the inequalities

$$1 - \frac{1}{|a|} < x_-(a) < \ln |a|, \qquad -1 < a < 0. \tag{4.6}$$

4.3. *Complex zeros.* The study of complex zeros becomes interesting already for some of the special cases (1.2)–(1.6) of the incomplete gamma function. Thus, e.g., the complex zeros of $e_n(nz)$ (cf. (1.3)) and their asymptotics as $n \to \infty$ have received a great deal of attention; see, e.g., Varga's monograph [156, Ch. 4]. Asymptotic approximations to the zeros of the complex error function $w(z) = e^{-z^2}\mathrm{erfc}(-iz)$ and to those of $\mathrm{erf}(z)$ are given in [42], including tables[3] to 11 significant digits of the first 100 of them. For the complex zeros of the Fresnel integrals, see [80].

For an asymptotic analysis of the complex zeros of $\Gamma(a, z)$, Temme in [138] uses the same method as described previously in §4.1, except that in

[3] The heading of Table 2 in [42] is incorrect; it should be "Zeros of $w(z)$" or "Zeros of $\mathrm{Erfc}(-iz)$".

(4.4) he puts $q = 0$ and takes for $\eta_0\sqrt{a/2}$ one of the complex zeros of the complementary error function. In particular, curves in the complex λ-plane are identified which are approached by the λ-zeros of $\Gamma(a, \lambda a)$ as $a \to \infty$ over positive values. A branch of this curve is the Szegő curve known from [156], which has been studied in connection with integer values of a.

In the work of Kölbig [76], [77], [78] the focus is on the complex zeros of $\gamma(a, x)$ considered as a function of a for fixed real x. From the contour map in Fig. 2.1 it is evident that the line $x =$ const, for x suitably chosen, has two intersections with the zero level curve of γ^* in each of the intervals $-2m < a < 1 - 2m$, $m = 1, 2, 3, \ldots$ (visible in Fig. 2.1 explicitly for $m = 1$ and $m = 2$). These intersections move toward each other as x is increased and eventually coalesce. If the point of coalescence is denoted by (a_m^*, x_m^*), the double zero of γ^* (or γ) at this point will split into a pair of conjugate complex zeros upon further increase of x beyond x_m^*. Thus, for each $m = 1, 2, 3, \ldots$ there is a pair of conjugate complex trajectories in the complex a-plane emanating from a_m^* along which γ vanishes. Using Tricomi's result (2.16), Kölbig in [78] gives the approximations $a_m^* \sim 1 - 2m - .623021$ and $x_m^* \sim .556929m - .108906 \ln m - .299840$ and in [76] he provides graphs and tables of the first five (eight in [77]) trajectories $a = a_m(x)$, $x \geq x_m^*$, in the upper half-plane. In [78] the concern is with the trajectories $a = a_m(x)/x$, i.e., the zero curves of $\gamma(xa, x)$ in the complex a-plane, and plots of the first eight of them are shown. As $m \to \infty$, according to a result of Mahler [96], they approach a limiting curve, which is also shown.

5. Inequalities and Monotonicity

5.1. *Inequalities.* An early inequality of some generality for the incomplete gamma function is the author's inequality [47]

$$\frac{1}{2a}[(x+2)^a - x^a] < e^x\Gamma(a, x) \leq \frac{c_a}{a}[(x + c_a^{-1})^a - x^a], \quad 0 \leq x < \infty,\ 0 < a < 1, \tag{5.1}$$

where

$$c_a = [\Gamma(1 + a)]^{\frac{1}{1-a}}.$$

For $a = \frac{1}{2}$, the second inequality reduces to one of Pollak [117], the first to one of Komatu [79], for "Mills' ratio" $e^{x^2}\int_x^\infty e^{-t^2}dt$. For sharper bounds regarding this ratio, see also [16], and for related inequalities, [82]. As $a \uparrow 1$, both bounds tend to 1, which is the value of $e^x\Gamma(1,x)$, since $\Gamma(1,x) = e^{-x}$. As $a \downarrow 0$, one obtains an inequality for the exponential integral,

$$\tfrac{1}{2}\ln\left(1+\tfrac{2}{x}\right) \le e^x E_1(x) \le \ln\left(1+\tfrac{1}{x}\right), \quad 0 < x < \infty, \tag{5.2}$$

which sharpens an inequality due to E. Hopf [63, p. 26]. Another special case of (5.1) obtains by setting $x = 0$ and using $\Gamma(1+a) \le 1$ on $[0,1]$,

$$2^{a-1} \le \Gamma(1+a) \le 1, \quad 0 \le a \le 1.$$

This has been sharpened and generalized in [47] to[4] (ψ is the logarithmic derivative of the gamma function)

$$x^{1-a} \le \frac{\Gamma(x+1)}{\Gamma(x+a)} \le \exp[(1-a)\psi(x+1)], \quad 0 \le a \le 1,\ x > 0, \tag{5.3}$$

which in turn has been the subject of numerous improvements and extensions; see, e.g., [39], [155], [124], [64], [84], [74], [91], [81], [82], [142], [98, §§2,3], [116, §3], [112], [51]. Further inequalities for the gamma function can be found in [101, §3.6], [72], [20], [71], [94], [73], [126], [123], [69], [119], [36], [83], [93], [38, §3], [54], [55].

An alternative inequality for the incomplete gamma function was recently obtained by Alzer [4], who proved

$$(1-e^{-s_a x})^a < \frac{\gamma(a,x)}{\Gamma(a)} < (1-e^{-r_a x})^a, \quad 0 \le x < \infty,\ a > 0,\ a \ne 1, \tag{5.4}$$

where

$$r_a = \begin{cases} [\Gamma(1+a)]^{-1/a} & \text{if } 0 < a < 1, \\ 1 & \text{if } a > 1, \end{cases} \qquad s_a = \begin{cases} 1 & \text{if } 0 < a < 1, \\ [\Gamma(1+a)]^{-1/a} & \text{if } a > 1. \end{cases}$$

For $a = \frac{1}{2}$, this reduces to inequalities of Chu [28] for the error function $\operatorname{erf} x$. As $a \to 1$, both bounds tend to $1 - e^{-x}$, which is the value of $\gamma(1,x)$. Rewriting (5.4) in terms of $\frac{\Gamma(a,x)}{\Gamma(a)} = 1 - \frac{\gamma(a,x)}{\Gamma(a)}$, and letting $a \downarrow 0$, yields a new inequality for the exponential integral,

$$-\ln(1-e^{-cx}) \le E_1(x) \le -\ln(1-e^{-x}), \quad 0 < x < \infty, \tag{5.5}$$

[4] Actually, (5.3) was proved in [47] only for x an integer $n = 1, 2, 3, \ldots$, but the proof given is valid for arbitrary $x > 0$ (cf. *Math. Reviews 21*, Review 2067).

where c is expressible in terms of Euler's constant γ as $c = e^\gamma = 1.7810724\dots$. In the domain $0 < a < 1$ the inequalities for $\Gamma(a, x)$ derivable from (5.4) are sharper than those in (5.1) if x is small, but weaker if x is large. The right inequality in (5.5) is always weaker than the corresponding inequality in (5.2), whereas the left inequality is sharper for small x and weaker for large x.

Upper bounds for $\Gamma(a, x)$ in the domain $x \geq 0$, $a \geq 1$ are also derived in [14]. The rather special, but pretty, sequence of inequalities,

$$\frac{\Gamma(n, n)}{\Gamma(n)} < \frac{1}{2} < \frac{\Gamma(n, n-1)}{\Gamma(n)}, \quad n = 1, 2, 3, \dots\ , \tag{5.6}$$

is proved in [157] and attributed to G. Lochs.

5.2. *Monotonicity.* Monotonicity, convexity, and higher monotonicity results abound for the gamma function, but seem to be scarce for incomplete gamma functions. Absolute monotonicity, i.e., positivity of all derivatives, has been shown in [35, §4b] for the sum of squares of "Hermite functions", which are expressible in terms of confluent hypergeometric functions. Lorch [90] has monotonicity results for ratios of Whittaker functions. Convexity and logarithmic convexity of $\Gamma(a+1, a)/\Gamma(a+1)$ on $[0, \infty)$ are shown by Temme [133]. The fact that this function decreases monotonically from 1 to $\frac{1}{2}$ has been shown previously by Van De Lune [95]. According to the well-known Bohr-Mollerup-Artin theorem, logarithmic convexity, on the other hand, lies at the heart of the gamma function, as, together with the difference equation and normalization, it characterizes the gamma function uniquely (cf. Artin [9]).

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders in I and $(-1)^n f^{(n)}(x) \geq 0$ on I for $n = 0, 1, 2, \dots$. It is called strictly completely monotonic if strict inequality holds for each n. Remarkably, many functions involving the gamma and/or the psi function are completely monotonic. Bustoz and Ismail [18], for example, prove this for the functions

$$\left(1 - \frac{1}{2x}\right)^{-1/2} \frac{\Gamma^2(x+\frac{1}{2})}{\Gamma(x)\Gamma(x+1)} \quad \text{and} \quad \left(1 + \frac{1}{2x}\right)^{-1/2} \frac{\Gamma(x)\Gamma(x+1)}{\Gamma^2(x+\frac{1}{2})} \tag{5.7}$$

on the interval $(\frac{1}{2}, \infty)$ and $(0, \infty)$, respectively. Likewise, they show that

$$\frac{\Gamma(x+s)}{\Gamma(x+1)} \exp[(1-s)\psi(x+\tfrac{1}{2}(s+1))] \quad \text{and} \quad \frac{\Gamma(x+1)(x+\frac{1}{2}s)^{s-1}}{\Gamma(x+s)}, \quad 0 \leq s \leq 1, \tag{5.8}$$

are completely monotonic on $(0,\infty)$, and strictly so if $0 < s < 1$. Furthermore,

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\exp[(s-1)\psi(x+\sqrt{s})] \text{ and } \frac{\Gamma(x+s)[x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}]^{1-s}}{\Gamma(x+1)}, \quad 0<s<1, \tag{5.9}$$

are strictly decreasing on $(0,\infty)$, which, together with (5.8), generalizes inequalities of Kershaw [74]. Other examples are given in [66] and [2]. Far-reaching results are proved by Alzer in [3]. Thus, for example, $x[\ln x - \psi(x)]$ is shown to be strictly completely monotonic on $(0,\infty)$, which extends a monotonicity and convexity result of Anderson et al. [8, §3]. The convexity on $(0,\infty)$ of $x\psi(x)$, proved by the author [48], is generalized to

$$0 < (-1)^n x^{n-1}[x\psi(x)]^{(n)} < (n-2)!, \quad x>0, \ n\geq 2 \tag{5.10}$$

([3, Thm. 4]). All remainders in the asymptotic expansion of $\ln\Gamma(x)$ for $x\to\infty$, are completely monotonic. More precisely, if

$$R_n(x) = \ln\Gamma(x) - (x-\tfrac{1}{2})\ln x + x - \tfrac{1}{2}\ln(2\pi) - \sum_{k=1}^{n}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \qquad n=0,1,2,\dots\,, \tag{5.11}$$

where B_{2k} are the Bernoulli numbers, then $(-1)^n R_n(x)$ is completely monotonic on $(0,\infty)$ ([3, Thm. 8]). This was proved earlier by Muldoon [102] for $n=0$, whereas convexity and concavity for general n were shown by Merkle [98]. Another remarkable result is the complete monotonicity on $(0,\infty)$ of

$$\prod_{\nu=1}^{n}\frac{\Gamma(x+a_\nu)}{\Gamma(x+b_\nu)} \tag{5.12}$$

([3, Thm. 10]), provided

$$0\leq a_1\leq a_2\leq\cdots\leq a_n, \quad 0\leq b_1\leq b_2\leq\cdots\leq b_n,$$
$$\sum_{\nu=1}^{\mu} a_\nu \leq \sum_{\nu=1}^{\mu} b_\nu \quad \text{for} \quad \mu=1,2,\dots,n.$$

This generalizes a result of Bustoz and Ismail [18, Thm. 6] for $n=2$ and monotonicity results of Stolarsky [128, §8] and Maligranda et al. [97, Thm. 2].

A monotonicity result of Kershaw and Laforgia [75], according to which on $(0,\infty)$ the function $[\Gamma(1+\frac{1}{x})]^x$ decreases, and $x[\Gamma(1+\frac{1}{x})]^x$ increases, extends an earlier inequality of Minc and Sathre [100]. See also [116, §5] for additional monotonicity results of this kind. Logarithmic convexity on $\mathbb{R}_+$ of $\Gamma(2x)/x\Gamma^2(x)$ and logarithmic concavity of $\Gamma(2x)/\Gamma^2(x)$ are proved by Merkle [99].

The q-analogue of the gamma function also enjoys inequalities and higher monotonicity properties, many of which extend those in this §5.2. For a good account of this, the reader is referred to Ismail and Muldoon [65].

6. Numerical Methods

6.1. *General procedures.* As with other special functions, numerical methods for computing incomplete gamma functions rely on a variety of standard tools. Thus, asymptotics is used by Takenaga [130] to evaluate $\Gamma(a,x)$ for large a. In a series of papers, Chiccoli et al. [23], [24], [25], [26], [27] use asymptotic approximations, Taylor and other series expansions (including Tricomi's series (2.8)), and recurrence relations, to evaluate the generalized exponential integral $E_p(x)$ for arbitrary positive p and x. A combination of forward and backward recurrence is the principal tool in the work of Amos [5], [7] to compute the exponential integrals for integer values $p=n$ of p and positive, resp. complex x. Difficulties near the negative real axis are overcome by an analytic continuation scheme. Allasia and Besenghi [1] propose quadrature methods, in particular the composite trapezoidal rule, to evaluate $\Gamma(a,x)$ for $a<-1$, $x>0$ and provide detailed error analyses. The use of (unstable) forward recurrence to compute the "molecular integrals" $\{\gamma(n+1,x)\}$ is analyzed in [88]. A fairly comprehensive procedure for evaluating incomplete gamma functions in the domain $-\infty<a<\infty$, $x\geq 0$ is described in [49]. If there is a weakness in this procedure, it is the fact of becoming computer-intensive when a and x are both very large and almost equal. This, however, has been corrected by DiDonato and Morris [30], who use, among other things, Temme's uniform asymptotic expansion (cf. §3.2) to compute $\gamma(a,x)/\Gamma(a)$ and $\Gamma(a,x)/\Gamma(a)$ for $a>0$, $x\geq 0$, and by Temme himself [135], who uses (3.4) in the critical region, with the asymptotic series (in a) for $R_a(\eta)$ replaced by a more manageable Taylor series (in η). DiDonato and Morris also describe an inversion procedure which uses a third-order iterative method along with an elaborate scheme of computing a good initial value. Expansion in Chebyshev polynomials is used by Barakat

[10] to compute $\gamma(a, z)$ for real a and purely imaginary z. Techniques based on continued fractions are employed by Jones and Thron [68] and Jacobsen et al. [67], and still other, especially asymptotic, techniques by Temme [137], to compute $\gamma(a, z)$ and $\Gamma(a, z)$ for complex a and z. There is an extensive literature dealing with the computation of special univariate cases of the incomplete gamma function, such as the exponential integral $E_1(x)$ and the error function and their close relatives, both for real and complex arguments. For this, as well as for relevant software, including software for incomplete gamma functions, we refer to the comprehensive documentation in [92]. Here we concentrate on real parameters and the stable use of recurrence relations.

6.2. *Recurrence relations.* The recurrence relations satisfied by incomplete gamma functions are linear, inhomogeneous, first-order difference equations of the form

$$y_n = a_n y_{n-1} + b_n, \quad n = 1, 2, 3, \ldots, \quad a_n \neq 0, \tag{6.1}$$

where the coefficients a_n, b_n depend on x and/or a. Given y_0, the recurrence (6.1) defines uniquely the sequence $\{y_n\}_{n=0}^{\infty}$. It is important, however, to know how robust the recurrence is to small perturbations such as rounding errors. An informative answer to this is provided by the "amplification factors" $\omega_{s\to t}$, which determine the effect of a small relative error ε at $n = s$ ("s" for "starting") upon the value at $n = t$ ("t" for "terminal"), assuming exact arithmetic. Thus, if the desired solution of (6.1) is $\{f_n\}$, and if $y_s = f_s(1+\varepsilon)$, then $y_t = f_t(1 + \omega_{s\to t} \cdot \varepsilon)$ in exact arithmetic. Here s may be less than t, which is the case in forward recursion, or $s > t$, in which case (6.1) is applied in reverse order (computing y_{n-1} in terms of y_n). An easy calculation (cf. [50]) will show that

$$\omega_{s\to t} = \frac{\rho_t}{\rho_s}, \tag{6.2}$$

where

$$\rho_n = \frac{f_0 h_n}{f_n}, \quad h_n = a_n a_{n-1} \cdots a_0 \ \ (a_0 = 1) \tag{6.3}$$

(assuming $f_0 \neq 0$). Here, h_n is a solution of the homogeneous equation (6.1) (with all $b_n = 0$). Knowledge of the quantities $\{\rho_n\}$ is thus sufficient to determine all amplification factors in (6.2). Note that $\rho_n = \omega_{0\to n}$.

A first example is $\gamma^*(a,x)$, which satisfies the recurrence relation

$$\gamma^*(a+1,x) = \frac{1}{x}\left[\gamma^*(a,x) - \frac{e^{-x}}{\Gamma(a+1)}\right]. \tag{6.4}$$

Once we know $\gamma^*(a,x)$ for $0 \le a < 1$, repeated application of this relation allows us to obtain $\gamma^*(a,x)$ for any $a \ge 1$, and also for any $a < 0$ if we apply (6.4) in the reverse order. Consider first the case of positive parameters,

$$\gamma_n^* = \gamma^*(a+n,x), \quad n = 0,1,2,\ldots, \quad 0 \le a < 1. \tag{6.5}$$

Then (6.4) yields

$$\gamma_n^* = \frac{1}{x}\left[\gamma_{n-1}^* - \frac{e^{-x}}{\Gamma(a+n)}\right], \quad n = 1,2,\ldots\ ; \quad \gamma_0^* = \gamma^*(a,x), \tag{6.6}$$

a relation of the form (6.1). Since here $h_n = x^{-n}$, we get

$$\rho_n = \frac{\gamma^*(a,x)}{\gamma^*(a+n,x)x^n}, \quad n = 0,1,2,\ldots\ . \tag{6.7}$$

The behavior of the corresponding amplification factors $|\omega_{0\to n}| = |\rho_n|$ is similar for all values of a in $[0,1)$; Fig. 6.1 shows them (on a logarithmic scale) for $a = \frac{1}{2}$ and for selected negative values of x on the left, and positive values of x on the right. (The case $x = 0$ is uninteresting, since $\gamma^*(a,0) = 1/\Gamma(a+1)$.) In either case, $|\rho_n| \to \infty$ as $n \to \infty$, but in the first case there

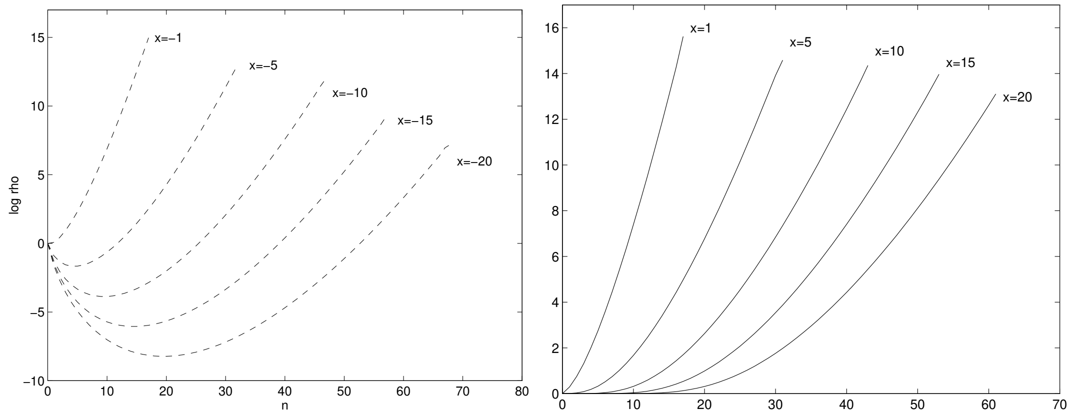

FIGURE 6.1. Amplification factors $|\omega_{0\to n}|$ for the incomplete gamma function $\{\gamma^*(\frac{1}{2}+n,x)\}$

is a significant downward dip to a minimum at about $n = |x|$ before $|\rho_n|$ grows monotonically to ∞, whereas in the second case $|\rho_n|$ increases monotonically from the start. This has important computational implications. The fact that $|\rho_n| \to \infty$ by (6.2) indeed implies that $|\omega_{s\to t}|$ for t fixed becomes arbitrarily small as $s \to \infty$. In effect, this means that backward recurrence in (6.6) from some large $n = \nu$ down to any fixed n produces arbitrarily accurate results if ν is chosen large enough, regardless of the choice of starting value. The latter, therefore, may conveniently be taken to be zero. This procedure is particularly effective for positive x, because of the monotonicity of $|\rho_n|$. When $x < 0$, backward recurrence should not proceed below $n \approx |x|$, since otherwise, by the nature of the dashed curves in Fig. 6.1, one would run into a regime of significant error amplification. The values of γ_n^* for n smaller than $|x|$ therefore must be generated by forward recurrence.

The case of negative parameters can be expected to be more complicated since we are getting into regions containing zero curves (cf. Fig. 2.1 and §2.4). Here the recursion for $\gamma_n^* = \gamma^*(a-n,x)$, $0 \le a < 1$, is

$$\gamma_n^* = x\gamma_{n-1}^* + \frac{e^{-x}}{\Gamma(a-n+1)}, \quad n = 1,2,\dots\ ; \quad \gamma_0^* = \gamma^*(a,x), \tag{6.8}$$

where the second term on the right is to be replaced by zero if $a = 0$. Limited exploration suggests that the amplification factors $|\rho_n| = |\omega_{0\to n}|$, when $x > 0$, are of the order of magnitude 1 for a while before decreasing rapidly, while for $x < 0$ they also eventually decrease at a similar speed, but may assume relatively large values (especially if $|x|$ is large) before they do so. Nevertheless, the recurrence (6.8), overall, is reasonably stable in forward direction.

What has been said about $\gamma^*(a,x)$ holds also for $\gamma(a,x)$, since the quantities ρ_n in (6.3) are invariant with respect to any scaling transformation $y_n \mapsto c_n y_n$, $c_n \ne 0$.

A second example is the complementary incomplete gamma function $\Gamma(a,x)$, for which the recurrence relation reads

$$\Gamma(a+1,x) = a\Gamma(a,x) + x^a e^{-x}. \tag{6.9}$$

Its use for generating $\Gamma_n^+ = \Gamma(a+n,x)$ and $\Gamma_n^- = \Gamma(a-n,x)$, $n = 0,1,2,\dots$, $0 \le a < 1$, can be discussed along lines similar to the preceding, except that x is restricted to positive values. One finds that the respective amplification factors $|\rho_n^+|$ and $|\rho_n^-|$ behave much like those in Fig. 6.1, but upside-down. That is, $|\rho_n^+| = \rho_n^+$ decreases monotonically whereas $|\rho_n^-|$ initially increases to a maximum near $n = |x|$ before decreasing to zero, the maximum being larger the larger $|x|$. The monotonicity of $\rho_n^+ = \frac{\Gamma(a,x)}{\Gamma(a)} / \frac{\Gamma(a+n,x)}{\Gamma(a+n)}$ follows from the monotonicity of $\Gamma(a,x)/\Gamma(a)$ as a function of a, proved by Tricomi (cf. §2.5). This means that the recurrence for Γ_n^+ is perfectly stable in forward direction, whereas the one for Γ_n^- should be started at a value of n near $|x|$, with backward [forward] recurrence being applied for the smaller [larger] values of n. The starting value can be computed by a continued fraction, for example. Note that Γ_n^- is related to the generalized exponential integral by $\Gamma_n^- = x^{n-a}E_{1-a+n}(x)$ (cf. Eq. (3.6)).

7. Applications

Many of the special cases of the incomplete gamma function are widely used in the applied sciences. Thus, the exponential integrals $E_p(x)$ for $p > 0$ play a significant role in transport theory and fluid flow, and for negative integer values of p furnish basic auxiliary functions in molecular physics. The error functions are frequently encountered in heat conduction, and the Fresnel integrals in Fresnel diffraction, problems. The complex error function $e^{-z^2}\mathrm{erfc}(-iz)$ is important in plasma wave problems, where it is known as "plasma dispersion function", in astrophysics and Lorentz/Doppler line broadening, where the real and imaginary parts go under the name of "Voigt functions", and in the design of particle accelerators. Finally, the incomplete gamma function ratios and their special cases are used extensively in statistical applications. Rather than reviewing these "external" applications (a nearly impossible task), we limit ourselves to a few recent "internal" applications that we happen to be familiar with, i.e., applications within the theory of special functions.

7.1. *Expansions in incomplete gamma functions.*

7.1.1. *The Riemann zeta function on the critical line.* Efforts to verify the Riemann Hypothesis, according to which all zeros of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ in $\mathrm{Re}\, s > 0$ lie on the critical line $\mathrm{Re}\, s = \frac{1}{2}$, require high-precision calculation of the zeta function for $s = \frac{1}{2} + it$ and t very large. Presently, the most efficient methods are based on the Riemann-Siegel formula and some of its recent improvements; see, e.g., Odlyzko [104] and Berry [12]. A promising alternative method has been developed by Paris [113] and Paris and Cang [115], who use an expansion of the zeta function in incomplete gamma functions in combination with (essentially) Temme's uniform asymptotic expansion (cf. §3.2). Once a reliable estimate of the truncation error becomes available, the expansion could provide a useful tool for the rigorous verification of the Riemann Hypothesis.

For an expansion in incomplete gamma functions of more general Dirichlet series, see also [57, p. 106].

7.1.2. *A generalization of the exponential integral.* The following generalization of the exponential integral,

$$\Gamma(a,x;b)=\int_x^\infty t^{a-1}e^{-t-b/t}dt,\quad x>0,\ a>0,\ b\ge 0, \tag{7.1}$$

has been studied in [21]. By expanding $e^{-b/t}$ into a Taylor series in t^{-1} one obtains an expansion in incomplete gamma functions [21], [22],

$$\Gamma(a,x;b)=\sum_{n=0}^\infty \frac{(-b)^n}{n!}\Gamma(a-n,x). \tag{7.2}$$

It is just the Maclaurin expansion of $\Gamma(\,\cdot\,,\,\cdot\,;b)$, an entire function of b. When $a>0$ and $b\ge 0$ are restricted to bounded intervals, then (7.2) can also be viewed as an asymptotic expansion for $x\to\infty$. The incomplete gamma functions $\Gamma(a-n,x)$ required in (7.2) can be generated recursively as discussed in §6.2. There is also an asymptotic expansion of $\Gamma(a,x;b)$ for large a analogous to (3.4), involving the complementary error function [22, Eq. (5.2)].

7.1.3. *Fermi-Dirac integrals.* The Fermi-Dirac integral

$$F_{p-1}(x)=\frac{1}{\Gamma(p)}\int_0^\infty \frac{t^{p-1}}{1+e^{t-x}}dt,\quad p>0,\ x\in\mathbb{R}, \tag{7.3}$$

for negative values of x is easily evaluated by the series

$$F_{p-1}(x)=\sum_{n=1}^\infty \frac{(-1)^{n-1}e^{xn}}{n^p},\quad x<0.$$

More difficult is the case of (large) positive x and p. Writing $p=ax$ and assuming $N-1<a<N$ for some integer $N\ge 1$, Temme and Olde Daalhuis [141], improving on previous work of Schell [122], obtain the representation

$$F_{p-1}=\sum_{n=1}^{N-1}\frac{(-1)^{n-1}e^{xn}}{n^p}+G_{p-1}(x)+H_{p-1}(x), \tag{7.4}$$

where the terms in the sum on the right (if $N \geq 3$) decrease monotonically. By contour integration in the complex plane, the function $G_{p-1}(x)$ is expressible as the term with $n = N$ in the summation of (7.4) multiplied by an incomplete gamma function ratio,

$$G_{p-1}(x) = \frac{(-1)^{N-1} e^{xN}}{N^p} \cdot \frac{\Gamma(p, Nx)}{\Gamma(p)}. \tag{7.5}$$

Here Temme's uniform asymptotic expansion for $p \to \infty$ (cf. §3.2) is applicable, also when $a = N$. The last term, $H_{p-1}(x)$, in (7.4) can be formally expanded in descending powers of x. Both terms G_{p-1}, H_{p-1} are negligible when N is large.

7.2. *Hyperasymptotics.* A process of successive re-expansion of remainder terms in asymptotic expansions, called hyperasymptotics, is developed in [109] for solutions of the confluent hypergeometric equation, and in [105], [106] for solutions of more general linear homogeneous second-order differential equations having an irregular singularity of rank one at infinity. By truncating the classical Poincaré expansion after a judiciously selected number of terms, one re-expands the corresponding remainder term to obtain a "first-level" expansion, the Poincaré expansion being at level zero. This first-level expansion is a series in generalized exponential integrals (cf. §3.3). If that series in turn is judiciously truncated, its remainder term is re-expanded to obtain a "second-level" expansion; it proceeds in functions called "hyperterminants", which are repeated infinite integrals of the generalized exponential integral. The process can be repeated indefinitely. An important feature of this sequence of re-expansions is that at each step the error is reduced by an exponentially small factor, of which the "exponential improvement" of the first-level expansion, mentioned at the end of §3.3.2, is just one example.

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