On EULER's attempt to compute logarithms by interpolation: A commentary to his letter of February 16, 1734 to DANIEL BERNOULLI

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Dedicated to Claude Brezinski on the occasion of his retirement

Abstract

In the letter to DANIEL BERNOULLI, EULER reports on his attempt to compute the common logarithm $\log x$ by interpolation at the successive powers of ten. He notes that for $x = 9$ the procedure, though converging fast, yields an incorrect answer. The interpolation procedure is analyzed mathematically, and the discrepancy explained on the basis of modern function theory. It turns out that Euler's procedure converges to a q-analogue $S_q(x)$ of the logarithm, where $q = 1/10$. In the case of the logarithm $\log_\omega x$ to base $\omega > 1$ (considered by Euler almost twenty years later), the limit of the analogous procedure (interpolating at the successive powers of ω) is $S_q(x)$ with $q = 1/\omega$. It is shown that by taking $\omega > 1$ sufficiently close to 1 and interpolating at sufficiently many points, the logarithm $\log x$ can indeed be approximated arbitrarily closely, although, if x, $1 < x < 10$, is relatively large, extremely high-precision arithmetic is required to overcome severe numerical cancellation. An alternative procedure for computing $\log x$ by interpolation at points in $[1, 10^\omega]$, $\omega > 0$, accumulating at the lower end point, is shown to converge to the desired limit, but also not without numerical complications.

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1. The handwritten original of the letter[1] in question is kept at the University Library of Basel under the signature Ms. L Ia 689 fol. 145–146v

[1] The letter is dated in the old style (Julian), since Euler wrote from Petersburg; the corresponding date in the new style (Gregorian) is February 27, 1734.

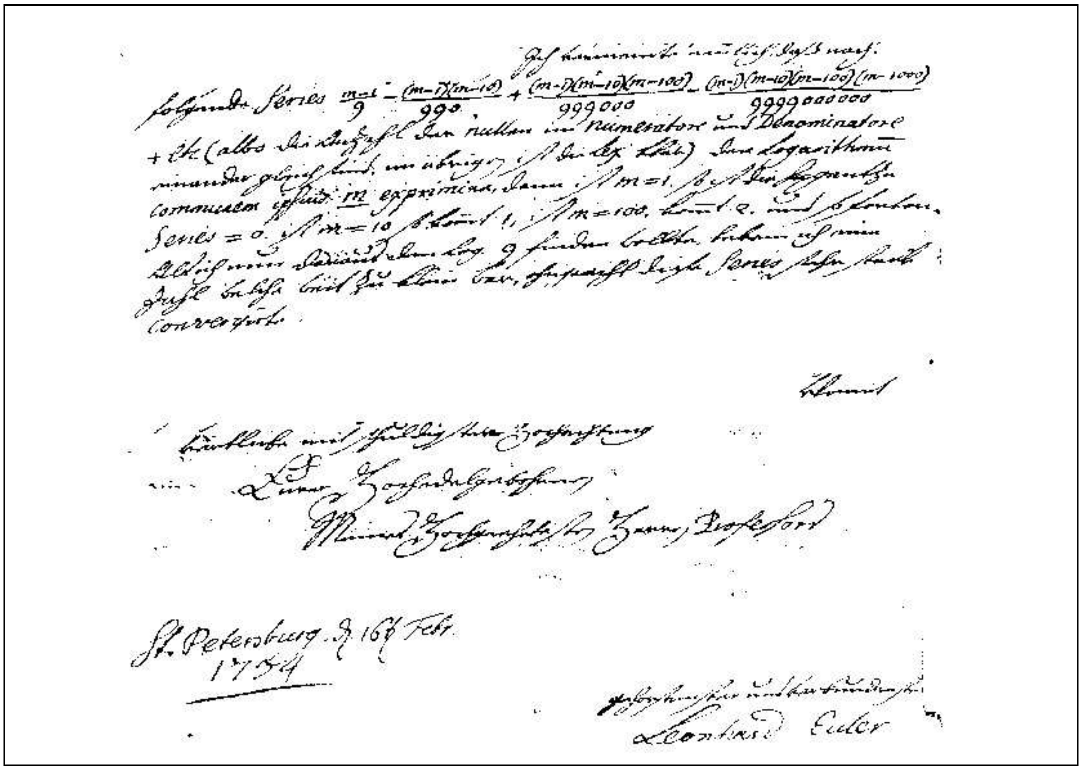

Figure 1: Excerpt from EULER's letter to D. BERNOULLI

and has been published by G. ENESTRÖM in [2]. Fig. 1 shows the passage relevant to us, including the rather formal closing phrases "Womit / verbleibe mit schuldigster Hochachtung / Eurer Hochedelgebohrnen / Meines Hochgeehrtesten Herren Professors / gehorsamster und verbundenster / Leonhard Euler". [*Author's translation*: Herewith I remain in most obliged respect your Honorable's and my most highly esteemed Professor's most obedient and indebted Leonhard Euler.]

The mathematical passage reads as follows: "Ich vermeinte neulich, daß nachfolgende *Series*

$$\frac{m-1}{9} - \frac{(m-1)(m-10)}{990} + \frac{(m-1)(m-10)(m-100)}{999\,000}$$
$$- \frac{(m-1)(m-10)(m-100)(m-1000)}{9\,999\,000\,000} + \text{etc.}$$

(alwo die Anzahl der nullen im *Numeratore* und *Denominatore* einander gleich sind, im übrigen ist die *Lex* klar) den *Logarithmum communem ipsius* m exprimiere, dann ist $m = 1$, so ist die gantze *Series* $= 0$, ist $m = 10$

so kommt 1, ist $m = 100$, kommt 2, und so fortan. Als ich nun daraus den *Log*[*arithmum*] 9 finden wollte, bekam ich eine Zahl welche weit zu klein war, ohngeacht diese *Series* sehr stark convergirte". [*Author's translation*: I recently thought that the following series

$$\frac{m-1}{9} - \frac{(m-1)(m-10)}{990} + \frac{(m-1)(m-10)(m-100)}{999\,000}$$
$$-\frac{(m-1)(m-10)(m-100)(m-1000)}{9\,999\,000\,000} + \text{etc.}$$

(where the number of zeros in the numerator and in the denominator is the same—the law, after all, is clear) would represent the common logarithm of m, for, when $m = 1$, the whole series is $= 0$, if $m = 10$, it becomes 1, if $m = 100$ it is 2, and so on. Now when I wanted to find from it the logarithm of 9, I obtained a number which is much too small, even though the series converged very strongly.]

2. EULER's intention, in modern terminology, is to compute the common logarithm by interpolating a certain number (ideally infinitely many) of known values of the logarithm. Fearless (even reckless) as so often was the case, EULER takes the simplest values, $\log 10^k = k$, $k = 0, 1, 2, 3, \ldots$, and for $\log x$, $x > 0$, writes down the infinite series

$$S(x) = \sum_{k=1}^{\infty} a_k (x-1)(x-10)\cdots(x-10^{k-1}) \tag{1}$$

whose nth partial sum is NEWTON's interpolation polynomial of degree n, hence

$$a_k = [x_0, x_1, \ldots, x_k]f$$

the divided difference of order k for the function $f(x) = \log x$ and abscissae $x_r = 10^r$, $r = 0, 1, \ldots, k$. This may have been the way in which EULER determined the first four coefficients

$$a_1 = \frac{1}{9},\ a_2 = -\frac{1}{990},\ a_3 = \frac{1}{999\,000},\ a_4 = -\frac{1}{9\,999\,000\,000}\,.$$

The "law", which he alludes to, apparently is

$$a_n = \frac{(-1)^{n-1}}{10^{n(n-1)/2}(10^n - 1)}\,, \quad n = 1, 2, 3, \ldots\,. \tag{2}$$

We assert, somewhat more generally, that for arbitrary integer-valued $r \geq 0$,

$$[x_r, x_{r+1}, \ldots, x_{r+n}]f = \frac{(-1)^{n-1}}{10^{rn+n(n-1)/2}(10^n - 1)}. \tag{3}$$

One proves (3) by mathematical induction on n. For $n = 1$, the assertion is evidently true. The validity of (3) for some n and arbitrary $r \geq 0$, and a well-known property of divided differences (see, e. g., [4, (2.64)]), then imply

$$\begin{aligned}
&[x_r, x_{r+1}, \ldots, x_{r+n}, x_{r+n+1}]f \\
&\quad = \frac{[x_{r+1}, x_{r+2}, \ldots, x_{r+n+1}]f - [x_r, x_{r+1}, \ldots, x_{r+n}]f}{x_{r+n+1} - x_r} \\
&\quad = \frac{(-1)^{n-1}}{10^{rn+n(n-1)/2}(10^n - 1)} \, \frac{1 - 10^n}{10^n(10^{r+n+1} - 10^r)} \\
&\quad = \frac{(-1)^n}{10^{rn+n(n-1)/2} \, 10^{n+r}(10^{n+1} - 1)} \\
&\quad = \frac{(-1)^n}{10^{r(n+1)+n(n+1)/2} \, (10^{n+1} - 1)},
\end{aligned}$$

which is precisely (3) with n replaced by $n+1$. This proves (3), and therefore also (2).

3. It suffices, of course, to assume $1 \leq x < 10$, since every other positive number x' can be written in the form $x' = x \cdot 10^p$ with some integer $p \neq 0$, and $\log x' = p + \log x$. The series (1) then converges uniformly on $[1, 10]$ and, as Euler remarks, very fast. The nth term $t_n(x)$ of (1), when a_n is given by (2), in fact computes to

$$t_n(x) = -\frac{\prod_{k=0}^{n-1}(1 - x/10^k)}{10^n - 1} = -\frac{q^n}{1 - q^n} \, (x; q)_n, \quad q = \frac{1}{10}, \tag{4}$$

where

$$(x; q)_n = \prod_{k=0}^{n-1}(1 - xq^k) \tag{5}$$

is the q-shifted factorial (cf. [1, §10.2]). There holds, for $1 \leq x < 10$ and $n \geq 2$,

$$|t_n(x)| < \frac{9}{10^n - 1}\left(1 - \frac{1}{10^{n-1}}\right) < \frac{9}{10^n}, \tag{6}$$

so that the nth partial sum of the series

$$S(x) = \sum_{k=1}^{\infty} t_k(x) \tag{7}$$

approximates its limit up to an error less than $9 \cdot 10^{-(n+1)}(1 + 10^{-1} + 10^{-2} + \cdots) = 10^{-n}$. For EULER's special value $x = 9$ one so obtains

$$S(9) = 0.897778586588\ldots,$$

a value which is significantly smaller than $\log 9 = 0.954242509439\ldots$; the relative error is 5.92%.

One can speculate what prompted EULER to communicate his computation for the special value $x = 9$. Very likely, he also tried other (integer-valued) $x < 9$, but had to observe that the results are then even worse. As a matter of fact, the relative deviation of the limit value from the true value of the logarithm increases monotonically as x decreases over the natural numbers from 9 to 2, and at $x = 2$ is about 10 times as large as at $x = 9$, and at $x = 0$ even 100%.

4. From today's perspective it is not surprising that the series (7) does not converge to the expected value. The nth term of the series, after all, is a polynomial of degree n, thus an analytic function of the complex variable x, and the series itself converges uniformly on each disk $|x| \le R$. In fact,

$$\left| \prod_{k=1}^{n-1} (1 - x/10^k) \right| \le \prod_{k=1}^{n-1} (1 + R/10^k),$$

and the product on the right converges absolutely when $n \to \infty$. Therefore, $C \cdot \sum_{n=1}^{\infty} 1/(10^n - 1)$, where $C = (R + 1) \prod_{k=1}^{\infty} (1 + R/10^k)$, is a convergent majorant of the series. By Weierstraß's double-series theorem, $S(x)$ thus represents a function which is analytic in every domain $|x| \le R$, hence an entire function. Consider now $d(z) = S(z) - \log z$ in the domain $\mathcal{D} = \{z \in \mathbb{C} : |\arg z| < \pi\}$, where log denotes the principal branch of the logarithm. If we had $d(z) = 0$ at infinitely many points which have a point of accumulation in $\mathcal{D} \backslash \{\infty\}$ (for example, in an arbitrarily small interval of the real line), it would follow that $d(z) \equiv 0$ for all $z \in \mathcal{D}$. This evidently is impossible since $d(x) \to \infty$ when $x \downarrow 0$. Consequently, $S(x)$ cannot be identically equal to $\log x$ on any interval, however small.

Interestingly, however, the function $S(x)$ may be interpreted as a q-analogue of the logarithm, where $q = 1/10$; cf. §5.

5. The motivation for EULER's bold choice $x_r = 10^r$ of the abscissae of interpolation is of course clear: not a single logarithm needs to be computed in order to generate the interpolation data. Almost equally simple would be the choice $x_r = \omega^r$, $\omega > 0$, which requires only one single logarithm, $\log \omega$. It is natural, then, to consider interpolation to the logarithm to base ω, that is, to $\log_\omega x = \log x / \log \omega$. What is the interpolation series[2] in this case and how does it behave?

To analyze the function $S(x;\omega)$ represented by the interpolation series, it is useful to introduce a q-analogue of the logarithm as defined by E. Koelink and W. Van Assche (see [6], where in §6 other definitions of the q-logarithm, used in the physics literature, are also discussed),

$$S_q(x) = -\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\,(x;q)_n, \tag{8}$$

where $(x;q)_n$ is the q-shifted factorial (5). One verifies, at least formally, that

$$\lim_{q\to 1}(1-q)S_q(x) = -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = \ln x, \quad 0 < x < 2, \tag{9}$$

and

$$S_q(q^{-n}) = n, \quad n = 0, 1, 2, \ldots, \tag{10}$$

which motivates the name "q-analogue of the logarithm". On the other hand,

[2] EULER returns to this series almost twenty years later in his memoir [3] (where a is written in place of ω). He derives very elegantly the logarithmic nature (10) of $S(x;\omega)$, emphasizing repeatedly that it holds only for positive integer values of n, and he computes (in §10) $S_q(q^{-n})$ also for negative n, explicitly for $n \geq -5$. He missed, however, the close connection of $\log \omega \cdot S(x;\omega)$ to $\log x$ when $\omega \downarrow 1$ (cf. (13) and (16) below), which in view of the strange numerical behavior of $\log \omega \cdot S(x;\omega)$ as $\omega \downarrow 1$ (cf. §6) is easy to understand. Instead, he used the series $S(x;\omega)$ as a springboard to derive all sorts of identities for it, among others two special cases (in §§17 and 26) of what today is known as the "q-binomial theorem". He also finds the expansion of $S(x;\omega)$ in powers of x and from known infinite products deduces new infinite series. At the end of the memoir EULER calculates Lambert's series $-S(0;\omega) = \sum_{n=1}^{\infty} 1/(\omega^n - 1)$ for $\omega = 10$ to 30 decimal places, but not before developing a convergence acceleration scheme for the more general series $\sum_{n=1}^{\infty} 1/(\omega^n - z)$.

in analogy to (4) one obtains

$$S(x;\omega)=\sum_{n=1}^{\infty}t_n(x;\omega),\quad t_n(x;\omega)=-\frac{q^n}{1-q^n}\,(x;q)_n,\quad q=\frac{1}{\omega}\,, \tag{11}$$

so that

$$S(x;\omega)=S_{1/\omega}(x). \tag{12}$$

Note that (10) with $q=1/\omega$ are precisely the interpolation conditions which produced the interpolation series $S(x;\omega)$ in the first place. It is evident from (11) and (5) that when $\omega<1$, hence $q>1$, the terms $t_n(x;\omega)$ converge to 1 if $x=0$, or to infinity in absolute value if $x\neq 0$, so that the series in (11) diverges. This definitely rules out the temptation of choosing $x_r=10^{-r}$.

Assume, therefore, that $\omega>1$. By an argument analogous to the one in §4 the series $S(x;\omega)$, and hence also $S_{1/\omega}(x)$, is seen to be an entire function (now depending on the parameter ω). It is true that larger values of ω yield faster convergence of the series in (11), but (9) and (12) suggest that better approximations to the logarithm can be expected for values of $\omega>1$ closer to 1. We now show indeed that $\log x$ can be approximated by Euler's interpolation process as accurately as we wish by taking $\omega>1$ sufficiently close to 1 and taking sufficiently many terms in the series of (11). We prove this for $0<x<2$, and provide numerical evidence for it when $x\geq 2$.

Since $\log x=\log\omega\cdot\log_\omega x$, the nth-degree interpolation approximation to the common logarithm $\log x$ is

$$s_n=\log\omega\cdot S_n(x;\omega), \tag{13}$$

where $S_n(x;\omega)$ is the nth partial sum of $S(x;\omega)$. Now

$$\frac{\ln\frac{1}{q}}{\ln 10}\,S_q(x)=\frac{\ln\frac{1}{q}}{(1-q)\ln 10}\cdot(1-q)S_q(x),$$

so that as $q\to 1$, since $\ln q^{-1}/(1-q)\to 1$, it follows from (9) that

$$\lim_{q\to 1}\frac{\ln\frac{1}{q}}{\ln 10}\,S_q(x)=\frac{\ln x}{\ln 10}=\log x,\quad 0<x<2.$$

Therefore, since $q=1/\omega$ and using (12), $\lim_{\omega\downarrow 1}\log\omega\cdot S(x;\omega)=\log x$, so that, given any $\varepsilon>0$, we can choose $\omega>1$ sufficiently close to 1 to have

$$|\log\omega\cdot S(x;\omega)-\log x|\leq\frac{\varepsilon}{2}\,. \tag{14}$$

On the other hand, n can be taken large enough so that

$$|\log \omega \cdot S_n(x;\omega) - \log \omega \cdot S(x;\omega)| \le \frac{\varepsilon}{2}\,. \tag{15}$$

Combining (14) and (15) yields

$$|s_n - \log x| = |\log \omega \cdot S_n(x;\omega) - \log \omega \cdot S(x;\omega) + \log \omega \cdot S(x;\omega) - \log x|$$
$$\le |\log \omega \cdot S_n(x;\omega) - \log \omega \cdot S(x;\omega)| + |\log \omega \cdot S(x;\omega) - \log x| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \tag{16}$$

as was to be shown.

6. We have seen that the interpolation procedure converges for $0 < x < 2$ (to the correct value $\log x$) as $\omega \downarrow 1$ and $n \to \infty$. Interestingly, the same seems to persist also for $x \ge 2$, but not without considerable numerical obstacles. Before discussing this, we note a simple scheme to evaluate $t_n(x;\omega)$ and thus, by summation, $S(x;\omega)$. Letting $u_n = (\omega^n - 1)t_n(x;\omega)$, one obtains from (11) the recursive procedure

$$\begin{aligned} t_1(x;\omega) &= \frac{x-1}{\omega-1}\,, \\ \left.\begin{aligned} u_n &= (1 - x/\omega^{n-1})u_{n-1}, \\ t_n(x;\omega) &= \frac{u_n}{\omega^n - 1} \end{aligned}\right\} & \quad n = 2, 3, \dots\,, \end{aligned} \tag{17}$$

which needs to be initialized by

$$u_1 = x - 1. \tag{18}$$

To interpolate the common logarithm $\log x$, the initial terms t_1 and u_1 must be multiplied by $\log \omega$.

The "obstacles" referred to above have to do with the fact that for values of ω larger than, but close to 1, the quantities $\log \omega \cdot t_n(x;\omega)$ become extremely large before eventually converging to zero as $n \to \infty$, at least when $x \le 10$ is relatively large. This is illustrated in Table 1 below, which shows $\max_{n\ge1} |\log \omega \cdot t_n(x;\omega)|$ for selected values of x and ω.

$x\backslash\omega$	1.1	1.05	1.025	1.0125	1.00625
2	.41	.42	.43	.43	.43
6	$.11\times10^{4}$	$.78\times10^{7}$	$.81\times10^{15}$	$.17\times10^{32}$	$.15\times10^{65}$
10	$.19\times10^{8}$	$.24\times10^{16}$	$.74\times10^{32}$	$.13\times10^{66}$	$.82\times10^{132}$

Table 1: Largest values of $|\log \omega \cdot t_n(x;\omega)|$

Yet, for each fixed ω, the series $S(x;\omega) = \sum_{n=1}^{\infty} t_n(x;\omega)$ converges. Because of the enormous amount of internal cancellation that may take place in this series, however, the computation must be performed in appropriately high precision.

$x\backslash\omega$	1.1	1.05	1.025	1.0125	1.00625
2	$.17\times10^{-12}$	$.14\times10^{-23}$	$.95\times10^{-46}$	$.18\times10^{-88}$	$.20\times10^{-174}$
6	$.24\times10^{-8}$	$.43\times10^{-15}$	$.22\times10^{-28}$	$.43\times10^{-55}$	$.11\times10^{-107}$
10	$.43\times10^{-4}$	$.12\times10^{-6}$	$.17\times10^{-11}$	$.54\times10^{-21}$	$.76\times10^{-40}$
d	40	50	60	100	200
n	100	200	400	800	1500

TABLE 2: Errors achievable by the interpolation process of §5

This again is illustrated in Table 2, showing the errors achievable in symbolic/variable-precision computation with d decimal digits and n terms of the series. It should, perhaps, be emphasized that for each fixed ω, increasing d and n beyond the values shown, will not reduce the errors any further; all it does is compute $S(x;\omega)$, and with it, $\log\omega \cdot S(x;\omega) - \log x$, more accurately. This is why we called the errors "achievable".

This somewhat bizarre behavior of the interpolation process, on reflection, is not entirely unexpected: For one, x in (9) had already to be restricted to the interval $(0,2)$. For another, when $\omega > 1$ is very close to 1, then all $x_r = \omega^r$ initially are almost equal to 1. If they were all equal to 1, then the interpolation series would be Taylor's expansion of $\log x$ at 1, which diverges if $x > 2$. Our interpolation process, for x much larger than 2, thus behaves, initially, as if it would diverge, and only when n becomes large and the points x_r begin to spread out, does it turn around and take on a more reasonable, eventually convergent, demeanor. While there may be some theoretical interest in this kind of approximation process, it has little practical merit because of the excessive computing effort required. (The last five columns of Table 2 take respectively 96, 187, 382, 741, and 1493 seconds to compute on the Sun Ultra5 Workstation.)

Nevertheless, if x is restricted to the interval $[1,5]$, the process is not entirely impractical, since when $\omega = 1.1$, for example, there holds $|\log\omega \cdot t_n(x;\omega)| < 72.2$, and with $n = 100$ terms, one is still able to obtain values of $\log x$, $1 \le x \le 5$, accurate to about 10 decimal digits, even in 14-digit computation. For values of x in the interval $(5,10]$, one applies the process to $x/2$ and adds $\log 2$ to the result. Better yet, in today's age of technology and binary computer arithmetic, we may restrict x to the interval $[1, 2]$, in which case $|\log \omega \cdot t_n(x; \omega)| < 1$ and $\omega = 1.1$, $n = 20$ generally yields an accuracy of 10 or more decimal digits (9 digits near $x = 1$), while $\omega = 1.05$, $n = 15$ yields 11 or more correct digits.

7. There is still another way in which EULER's ideas can in principle be salvaged and made workable. To begin with, choose as interpolation abscissae $x_r = 10^{\omega/(r+1)}$, $\omega > 0$, $r = 0, 1, 2, \ldots$, so that

$$x_r \in (1, 10^\omega] \quad \text{for all } r = 0, 1, 2, \ldots \; . \tag{19}$$

It is known, in fact (cf., e.g., [4, p. 83]), that for the function f and (arbitrary) abscissae of interpolation, all lying in a finite interval $[a, b]$, the interpolation series converges to $f(x)$ for any x in $[a, b]$, provided f has infinitely many in $[a, b]$ continuous derivatives and, moreover, there holds

$$\lim_{k \to \infty} \frac{(b-a)^k}{k!} M_k = 0, \tag{20}$$

where M_k denotes an upper bound of $|f^{(k)}|$ on $[a, b]$. This easily follows from Cauchy's formula [4, (2.12)] for the error of interpolation. It can also be shown (*ibid.*, p. 84), that (20) is indeed true if f is analytic in a disk with center at the middle of the interval $[a, b]$ and radius $r > \frac{3}{2}(b-a)$.

In our case $f(x) = \log x$, one has $f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}/\ln 10$, and (20) is equivalent to $|(b/a) - 1| < 1$. More precisely, one has at least geometric convergence with ratio q if

$$\left| \frac{b}{a} - 1 \right| \le q < 1. \tag{21}$$

Choosing $q = \frac{1}{2}$, one obtains for the interval (19), where $b/a = 10^\omega$,

$$\omega \le \log \tfrac{3}{2} = 0.17609\ldots \; . \tag{22}$$

Thus, in the interval (19), when ω is given by (22), the interpolation series converges (to the correct value) at least geometrically with ratio 1/2.

Now if x is given with $1 \le x < 10$, one determines the integer $k_0 \ge 0$ such that

$$10^{k_0 \omega} \le x < 10^{(k_0+1)\omega}. \tag{23}$$

This can easily be achieved (on a computer) by means of a small routine like (in pseudocode)

```
k0=0;
while x ≥ 10^((k0+1)ω)
   k0=k0+1;
end
```

If then one puts $t = 10^{-k_0\omega}x$, there holds $1 \le t < 10^{\omega}$, and one computes $\log t$ as above, whereupon $\log x = k_0\omega + \log t$.

Here too, however, not everything works as expected. It transpires (apparently because of the crowding of the interpolation abscissae in the lower part of the interval $(1, 10^{\omega}])$, that the algorithm described eventually succumbs to the detrimental effects of rounding errors. The latter progressively affect the computation of the divided differences (no longer explicitly known) to the point of rendering them completely meaningless. In computation with 36 decimal places (quadruple precision in Fortran), for example, the error of the interpolation polynomial is seen to decrease monotonically with increasing degree, but only up to a degree n of about $n = 18$; thereafter, the error increases rapidly. Nevertheless, it is still possible, in this precision, to obtain at least 10 good decimals, generally, however, many more, even as many as 35.

8. We have tried to understand and, following his own ideas, to rehabilitate EULER's unsuccessful computation of the logarithm, but do not want to leave behind the impression that the resulting computational schemes would be competitive with newer methods of approximation theory (see, e. g., [5]). These modern methods, however, are products of the 20th century.

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