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Stability Criteria for yet Another Class of Multidimensional Distributed Systems

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1 Introduction

Recently, in our paper [3] we have established rigorously stability conditions for the ℓ-limited token passing rings, and indicated that the technique can be extended to other disciplines. In this paper, we apply our methodology to derive stability criteria for *time-limited* token passing rings introduced recently by Leung and Eisenberg [6], [7] (cf. Theorem 1). In such a system each station transmits messages for at most an amount of time τ. If the transmission time exceeds τ, the station completes the transmission of the message in progress and sends the token to the next station (the so called nonpreemptive time limited discipline). While we study this system as an important application, the technique can be applied almost without modification to a class of monotonic and contractive policies (cf Theorem 2).

Our approach to the stability of token passing rings follows the idea discussed in our paper [3], and differs from the standard methodology of the Lyapunov test function (cf. [12], [18], [14]). (For other than test function approaches see also [2], [11] [14, 15].) It resembles the general idea of Malyshev's *faces* and *induced Markov chains* [10]. Our method is based on a simple idea of stochastic dominance technique, and application of Loynes [9] stability criteria for an isolated queue. We note that this approach is *not* restricted to token passing rings, and stability of several other distributed systems can be assessed by this methodology (cf. [14, 15]).

We now summarize our main results. We shall analyze the token passing ring with Poisson arrivals with parameter λ_i for the ith station, general distribution of service times $\{S_i^k\}_{k=1}^{\infty}$ and switchover times $\{U_i^k\}_{k=1}^{\infty}$. We consider the gated version of τ-limited token ring. Define $\widetilde{L}_i = \min\{k : \sum_{j=1}^{k} S_i^j \geq \tau_i\}$ and $\ell_i = E\widetilde{L}_i$. Clearly, ℓ_i is the average of the maximum number of customers served during τ_i.

Theorem 1. *Consider a token passing ring consisting of M stations with τ-limited service schedule for the ith station, and Poisson arrivals. Then the system is stable if and only if $\sum_{j=1}^{M}\rho_j < 1$ and*

$$\lambda_j < \frac{\ell_j}{u_0}(1-\rho_0) \quad \text{for all} \quad j \in \mathcal{M} = \{1, \dots M\} ,$$

where $u_0 = \sum_{i=1}^{M} EU_i$ is the average total switchover time, and $\rho_0 = \sum_{i=1}^{M}\rho_i$ with $\rho_i = \lambda_i s_i$ and $s_i = ES_i$ being the average service time at the ith station. ■

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It will be seen that the method of the proof can be applied virtually unchanged to the following class of monotonic and contractive policies. We assume that the number of customers served from queue i when there are n queued messages at the instant of token arrival to queue i is $f_i(n, \mathbf{X})$, where $\mathbf{X}$ is a possibly random quantity that depends on the policy. We assume that $f_i(\cdot, \mathbf{X})$ is a *nondecreasing* function of the number of customers in the ith queue. In addition, the following relation holds

$$f_i(n_1, \mathbf{X}) - f_i(n_2, \mathbf{X}) \leq n_1 - n_2 \qquad if \quad n_1 > n_2 \ .$$

At the kth token arrival to queue i the quantity $\mathbf{X}$ takes the value $\mathbf{A}_k$. The random variables $\{\mathbf{A}_k\}_{k=1}^{\infty}$ are i.i.d, independent of the past arrival times, past service times and past switchover times. Note that $f_i(n, \mathbf{A}_k)$ tends to a random variable, $f_i^*(\mathbf{A}_k)$, when $n \to \infty$. We assume that $f_i^*(\mathbf{A}_k)$ is finite and $\ell_i \stackrel{\text{def}}{=} E f_i^*(\mathbf{A}_k) < \infty$.

We will see in the next section that the time-limited token ring policy falls within the class just described. Other special cases are the ℓ-limited gated discipline and Bernoulli gated disciplines. In the last discipline, $f_i(n, X) = \min\{n, X\}$, where X is a geometrically distributed random variable. The following result follows directly from the arguments that will be presented in this paper.

Theorem 2. *Consider a token passing ring that employs a monotonic and contractive service discipline described above. Then, the system is stable if and only if*

$$\lambda_i < \frac{\ell_i}{u_0}(1 - \rho_0) \qquad \text{for all} \quad i \in \mathcal{M}$$

where $\rho_0 < 1$. ■

One policy that does not fall in the previously described category is the *preemptive* τ-limited token ring. In such as system the token interrupts his service immediately after the time limit τ expires and continues servicing the same customer in the next round. While the formal arguments of our methodology can be applied for this system, there are technical difficulties that need to be overcome for a complete proof. We believe that these technical arguments can be provided, but we do not have a complete proof yet, and therefore we express the following conjecture.

Conjecture. *The preemptive* τ*-limited token ring is stable if and only if*

$$\lambda_j < \frac{\tau_j}{s_j u_0}(1 - \rho_0) \quad \text{for all} \quad j \in \mathcal{M} = \{1, \ldots M\} \ ,$$

and $\rho_0 < 1$. ■

2 Preliminary Results

We start with a precise definition of our stochastic model. We shall adopt the following assumptions.

(A1) There are M stations (queues) on a loop, each having infinite capacity buffer.

(A2) The maximum time customers are served during the token visit at a queue is limited to $\tau_i < \infty$ units of time. Only customers that are present at the instant of token arrival can be served. Moreover, we have nonpreemptive discipline, that is, the customer that is in the server when the time limit τ_i is reached, is served to completion before the token moves to the next queue.

(A3) The arrival process A_i^t, $t \in [0, \infty)$ to the ith queue is a Poisson process with parameter $\lambda_i > 0$. Here, A_i^t is the number of arrivals at queue i up to time t. The arrival process at a queue is independent of the arrival processes to other queues.

(A4) The service time process $\{S_i^k\}_{k=1}^{\infty}$ at queue i consists of i.i.d. random variables with mean $s_i = ES_i^1 > 0$. The service time process at a queue is independent of the arrival processes at all queues and independent of the service time processes at other queues.

(A5) The switchover times between i and $i+1 \bmod M$ queue, $\{U_i^k\}_{k=1}^{\infty}$, are i.i.d., independent of the switchover times $\{U_j^k\}_{k=1}^{\infty}$ for $j \neq i$, and independent of the arrival and the service time processes. The average total switchover time is defined as $u_0 = \sum_{i=1}^{M} EU_i^1$. To avoid inessential complications we assume that $P(U_i^n > 0) = 1,\ \ i = 1, \cdots, M$.

Let $\{T_n\}_{n=1}^{\infty}$ be the time instant of the nth visit of the token to a queue. Assumptions (A1)-(A5) imply that the service times of the customers at queue i at instant T_n are i.i.d. independent of the history up to time T_n. This permits us to consider a new model of the system which is stochastically equivalent to the original one and has the advantage that under this new model, many of the arguments that follow become simpler. Specifically, in the new system assumptions (A1)-(A3) and (A5) are the same, while assumption (A4) is replaced with

(A4') Service times are assigned to the customers at queue i upon beginning of service as follows. We consider a doubly infinite sequence of i.i.d random variables $\{S_i^{n,k}\}_{n,k=1}^{\infty}$ with $s_i = ES_i^{n,k} > 0$. The customers that are served during the nth arrival of the token to queue i are assigned the service times $S_i^{n,1}, S_i^{n,2}, \cdots$. The sequence $\{S_i^{n,k}\}_{n,k=1}^{\infty}$ is independent of the sequence $\{S_j^{n,k}\}_{n,k=1}^{\infty}$ for $i \neq j$, independent of the interarrival processes to the queues and of the switchover times.

Next, we need some relationships between the average number of customers served per token visit L_i^n and the average cycle time C_i^n. The former quantity is defined as follows. Let $\widetilde{L}_i^n = \min\{k : \sum_{j=1}^{k} S_i^{n,j} \geq \tau_i\}$. Then $L_i^n = \min\{\widetilde{L}_i^n, N_i^n\}$. The latter quantity is the length of time between the nth and $n+1$st visits of the token to the reference queue i. By EL_i and EC we denote the limiting averages of L_i^n and C_i^n. It turns out that the relations holding for the ℓ-limited case, continue to hold under the more general policies we consider here. Specifically, we have the following result which can be proved along the same lines as in our paper [3].

Proposition 3. *Let the Markov chain $\mathbf{N}^n(i)$ be positive recurrent (ergodic) for some $i \in \mathcal{M}$. Then, $\mathbf{N}^n(j)$ is ergodic for all $j \in \mathcal{M}$, and $\rho_0 = \sum_{j=1}^{M} \rho_j < 1$. In addition,*

$$EL_j = \lambda_j EC, \qquad j \in \mathcal{M} \tag{1}$$

$$EC = \frac{u_0}{1 - \sum_{j=1}^{M} \rho_j}\ , \tag{2}$$

where u_0 is the total average switchover time (cf. assumption (A5)) and $\rho_j = \lambda_j s_j$ is the utilization coefficient for the ith queue.

The next result is one of the key elements of our stability analysis. We state it in a general form, since it is needed to prove Theorem 2. Specifically, in the terminology of [8], we consider the class of monotonic, contractive policies. This amounts to replacing assumption (A2) with the following more general one.

(A2') Let $\mathbf{A}$ denote a sequence of real numbers $\{a_1, a_2, \cdots\}$. Let $f_i(m, \mathbf{A})$ be the number of customers served from queue i when there are m queued messages at the instant of the nth token arrival at queue i and $\{S_i^{n,1}, S_i^{n,2}, \cdots\} = \mathbf{A}$. We assume that for fixed $\mathbf{A}$, $f_i(m, \mathbf{A})$ is a nondecreasing function of m. In addition, for a fixed $\mathbf{A}$, the following relation holds

$$f_i(m_1, \mathbf{A}) - f_i(m_2, \mathbf{A}) \leq m_1 - m_2 \qquad if \quad m_1 > m_2\ . \tag{5}$$

For the case of τ-limited policy we have that $f_i(m, \{S_i^{n,1},\ S_i^{n,2}, \cdots\}) = \min\{m, \tilde{L}_i^n\}$.

Now we are ready to formulate our result. Consider two token passing rings, say θ and Θ. Both satisfy assumptions (A1),(A2'),(A3),(A4'),(A5). The system θ represents our original token passing ring. The system Θ differs only in the switchover times, namely, we assume that the switchover time for Θ is replaced by $\{\Delta_i^k + U_i^k\}_{k=1}^{\infty}$ for $i = 1, \cdots, M$. We assume that for every $i \in \mathcal{M}$ and every $k \geq 0$ we have $\Delta_i^k \geq 0$. We make the following assumption for the process Δ_i^k.

(A6) The random variable Δ_i^k is independent of the service times, switchover times and the Poisson increments of the arrival processes to all stations after time $T_{M(k-1)+(i+1)} - U_i^k$ (see Fig.1).

Proposition 4. *Let $\widetilde{\mathbf{N}}^n(\theta)$ and $\widetilde{\mathbf{N}}^n(\Theta)$ denote the process of queue lengths at times T_n in systems θ and Θ respectively. Then, under the above assumptions, and under the condition that the token starts from the same queue, say queue number one, and with the same number of initial customers in both systems, the following holds*

$$\widetilde{\mathbf{N}}^n(\theta) \leq_{st} \widetilde{\mathbf{N}}^n(\Theta), \tag{6}$$

where $\leq_{st}$ means stochastically smaller.

Proof. The proof is along the lines of Theorem 4 our paper [3]. For completeness in the presentation, we provide some details. To avoid cumbersome notation we present the proof only for $M = 2$ users.

We define some new variables. For a system θ let T_n^θ and D_n^θ denote the instances of the nth visit and the nth token departure from any queue respectively. Let also J_n^θ denote the queue number visited at the nth visit of the token to any queue. Finally, let $L_i^n(\theta)$ be the number of customers served from queue i at the nth visit of the token to any queue. Clearly, for our two station system $L_1^n(\theta) = 0$ for n even, and $L_2^n(\theta) = 0$ for n odd. In a similar manner we define respective quantities in the Θ system.

We will construct from the system Θ a token passing ring $\bar{\theta}$, which is stochastically equivalent to the system θ and for which we have that $\widetilde{\mathbf{N}}^n(\bar{\theta}) \leq \widetilde{\mathbf{N}}^n(\Theta)$. Figure 1 should help to understand our construction. Assume $\widetilde{N}_i^1(\bar{\theta}) = \widetilde{N}_i^1(\Theta)$ for $i = 1, 2$. The service times in system $\bar{\theta}$ are assigned

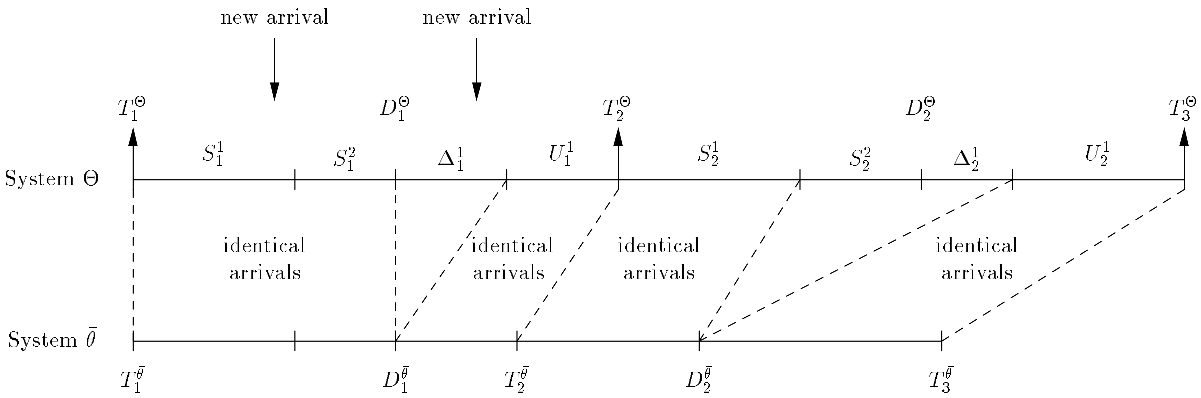

Fig. 1. Illustration to the proof of Theorem 4

from the same sequences $S_i^{n,k}$ as in Θ (according to assumption (A4').) Also, the same functions $f_i(m, \mathbf{A})$, $i = 1, 2$ are used in both systems. Therefore, the decision to switch to queue 2 will occur at the same time, namely $D_1^{\bar{\theta}} = D_1^\Theta$. The switchover time for $\bar{\theta}$ becomes now U_1^1, and of course $T_2^{\bar{\theta}} \leq T_2^\Theta$ since $\Delta_1^1 \geq 0$ (see Fig. 1).

The arrivals in the system $\bar{\theta}$ in $[D_1^{\bar{\theta}}, T_2^{\bar{\theta}})$ *are now assumed to be identical* to the arrivals in $[D_1^\Theta + \Delta_1^1, T_2^\Theta)$ in Θ system. Therefore clearly $\widetilde{N}_i^2(\bar{\theta}) \leq \widetilde{N}_i^2(\Theta)$ for $i = 1, 2$. The arrivals to system $\bar{\theta}$ in $[T_2^{\bar{\theta}}\ ,\ T_2^{\bar{\theta}} + S_2^1 + \cdots + S_2^{L_2^1(\bar{\theta})})$ are taken to be identical to the arrivals in $[T_2^\Theta\ ,\ T_2^\Theta + S_2^1 + \cdots + S_2^{L_2^1(\bar{\theta})})$. Note that this can be done since by (A2') $L_2^1(\bar{\theta}) \leq L_2^1(\Theta)$. Observe also that $T_2^{\bar{\theta}} + S_2^1 + \cdots + S_2^{L_2^1(\bar{\theta})} = D_2^{\bar{\theta}}$ (Fig. 1).

To complete the description of the system $\bar{\theta}$ we have to specify the arrivals in $[D_2^{\bar{\theta}}\ ,\ D_2^{\bar{\theta}} + U_2^1)$. These are taken to be exactly the arrivals in $[D_2^{\Theta} + \Delta_2^1\ ,\ D_2^{\Theta} + \Delta_2^1 + U_2^1)$ in the dominant system Θ (see Fig. 1). Note from the construction that

$$\widetilde{N}_1^3(\bar{\theta}) = \widetilde{N}_1^2(\bar{\theta}) + A_{[T_2^{\bar{\theta}}, T_3^{\bar{\theta}}]} \leq \widetilde{N}_1^2(\Theta) + A_{[T_2^{\Theta}, T_3^{\Theta}]} = \widetilde{N}_1^3(\Theta)$$

and also by (5), $\widetilde{N}_2^3(\bar{\theta}) = \widetilde{N}_2^2(\bar{\theta}) - L_2^2(\bar{\theta}) \leq \widetilde{N}_2^2(\Theta) - L_2^2(\Theta) = \widetilde{N}_2^3(\Theta)$. We can now repeat exactly the same procedure to construct $\bar{\theta}$ in the interval $[T_n^{\bar{\theta}}, T_{n+1}^{\bar{\theta}})$, $n \geq 3$, in the same manner as it was constructed in the interval $[T_2, T_3)$. By construction the service times and switchover times of system $\bar{\theta}$ are identically distributed to the corresponding variables of system θ and are independent of the interarrival process. In addition, assumption (A6) and the fact that the servicing policy is nonanticipative assures that the times $T_{n+1}^{\Theta} - U_{J_n}^{k_n}$ are stopping times for the Poisson arrival processes to all stations. The independence of the increments of the Poisson process implies now, that the constructed interarrival process in system $\bar{\theta}$ is Poisson with rate λ_i for queue i. Moreover, by construction $\widetilde{\mathbf{N}}^n(\bar{\theta}) \leq \widetilde{\mathbf{N}}^n(\Theta)$. Since $\bar{\theta}$ is stochastically equivalent to θ, we have that the distribution of $\widetilde{\mathbf{N}}^n(\theta)$ is identical to the distribution of $\widetilde{\mathbf{N}}^n(\bar{\theta})$. This completes the proof of the proposition. ■

3 Main Results

We start by showing a preliminary result. As in Section 2 consider a doubly infinite sequence of i.i.d random variables $\{S^{n,k}\}_{n,k=1}^{\infty}$ and recall the definition $\tilde{L}^n = \min\{k : \sum_{j=1}^{k} S^{n,j} \geq \tau\}$, $0 \leq \tau < \infty$. Consider further a queue with vacations such that upon the nth arrival of the server to the queue, $\tilde{L}^n$ customers (dummy if necessary) are served and then the server goes for a vacation. The service times of the customers served during the nth visit of the server are the random variables $S^{n,k}$, $k = 1, \cdots, \tilde{L}^n$. Let $\{C^n\}_{n=1}^{\infty}$ be the process of cycle times (time intervals between successive visits to the queue). It is assumed that the processes $\{C^n, \tilde{L}^n\}_{n=1}^{\infty}$ are jointly *stationary and ergodic* (no independence is required). The arrival process A^t to this queue is a Poisson process with parameter λ, independent of the processes $\{C^n, \tilde{L}^n\}_{n=1}^{\infty}$. Let N^n represent the queue length at the beginning of the nth cycle. By X^n we denote the number of customers arrived during the nth cycle. Since A^t is Poisson and independent of the processes $\{C^n, \tilde{L}^n\}_{n=1}^{\infty}$, the processes $\{X^n, \tilde{L}^n\}_{n=1}^{\infty}$ are jointly stationary and ergodic, and $EX = \lambda EC$ where $EC = EC^1$. Clearly, the process of queue lengths at the instants of the visits of the server to the queue satisfies the following recurrence

$$N^{n+1} = \max\{N^n + X^n - \widetilde{L}^n\ ,\ X^n\}\ ,\quad n = 1, 2, \cdots \tag{7}$$

Let $\ell = E\widetilde{L}^n$. We prove the following stability result.

Lemma 5. *Consider the queueing system just described. If $\lambda EC < \ell$, then the queue is stable.*

Proof. Since the processes $\{X^n, \tilde{L}^n\}_{n=1}^{\infty}$ are stationary we can assume without loss of generality that they are defined for $n < 0$ as well. By telescoping the recurrence (7) we immediately obtain for $n \geq 2$, $N^n = \max_{1 \leq r \leq n-1}\{X^r + \sum_{k=r+1}^{n-1} \overline{X}^k\}$ where $\overline{X}^k = X^k - \widetilde{L}^k$, provided $N^1 = 0$. Arguing as in Loynes [9] we have that N^n converges in distribution to an honest random variable if and only if

$$\limsup_{r \to \infty} \{X^{-r} + \sum_{k=1}^{r} \overline{X}^{-k}\} < \infty\ . \tag{8}$$

Observe now that $X^{-r} + \sum_{k=1}^{r-1} \overline{X}^{-k} = r(\sum_{k=1}^{r} X^{-k}/r - \frac{r-1}{r} \sum_{k=1}^{r-1} \tilde{L}^{-k}/(r-1))$ for $r = 2, 3, \cdots$. Since by the ergodicity of the sequences X^n, $\tilde{L}^n$ we have that $\lim_{r\to\infty} \sum_{k=1}^{r} X^{-k}/r = \lambda EC$ and $\lim_{r\to\infty} \sum_{k=1}^{r} \tilde{L}^{-k}/r = \ell$, the condition $\lambda EC < \ell$ assures the validity of (8). The assumption that $N^1 = 0$ can be removed as in [9]. ■

Now we are ready to prove our main result described in Theorem 1. In the next proposition we show that the conditions of the theorem are sufficient.

Proposition 6. *The Markov chain $\mathbf{N}^n(i)$ representing the queue lengths in the token passing ring when it visits queue $i \in \mathcal{M}$ is ergodic if*

$$\lambda_j < \frac{\ell_j}{u_0}(1-\rho_0) \qquad \text{for all} \quad j \in \mathcal{M} \tag{9}$$

where $\rho_0 = \sum_{j=1}^{M} \rho_j$ and $\ell_i = E\widetilde{L}_i$.

Proof. We use mathematical induction. For $M = 1$ the proof is simple since it directly follows from Foster's criterion (cf. [12], [18]).

Now we assume that the theorem is true for $M-1$ and prove that it can be extended to the M queue case. Let $(\mathcal{U},\, \mathcal{S})$, $\mathcal{U} \neq \emptyset$, be a partition of the set $\mathcal{M}$ of M queues into *persistent* and *nonpersistent* queues. A persistent queue in $\mathcal{U}$ transmits $\tilde{L}_i^n$ messages during the nth visit of the token by transmitting dummy packets if necessary. A nonpersistent queue in $\mathcal{S}$ operates in the normal fashion. Note that the cardinality $|\mathcal{S}|$ of $\mathcal{S}$ is not larger than $M-1$. Let $\overline{\mathbf{N}}^n(i) = \{\overline{N}_1^n(i), \cdots, \overline{N}_M^n(i)\}$ be the queue lengths when the token visits the ith queue for the nth time in the $(\mathcal{U},\, \mathcal{S})$ system. Observe that the modified system differs from the original token ring system only in the switchover time from a persistent queue to the successor of that queue in the ring. Specifically, if $i \in \mathcal{U}$, then the switchover times become $\overline{U}_i^k = \Delta_i^k + U_i^k$ where Δ_i^k is the time needed to service the dummy messages at node i (if any), and Δ_i^k satisfies condition (A6) of Section 2. Therefore, according to Proposition 4, if $\mathbf{N}^1(1) = \overline{\mathbf{N}}^1(1)$, then $\mathbf{N}^n(j) \leq_{st} \overline{\mathbf{N}}^n(j)$, for all n, $j \in \mathcal{M}$.

Note now that the queues in $\mathcal{S}$ constitute a token passing ring with $|\mathcal{S}|$ stations satisfying conditions (A1)-(A3),(A4'),(A5) of Section 2, whose operation is independent of the interarrival processes in the persistent queues. The total average switchover time $\overline{u}_0$ to this ring is equal to $\overline{u}_0 = u_0 + \sum_{i\in\mathcal{U}} \ell_i s_i$. Let the queue lengths in such a system be denoted as $\{\overline{\mathbf{N}}_{\mathcal{S}}^n(i)\}_{i\in\mathcal{S}}$. Clearly, $\overline{\mathbf{N}}_{\mathcal{S}}^n$ is a Markov chain, and since $|\mathcal{S}| \leq M-1$ we can apply the induction hypothesis. Hence, for $i \in \mathcal{S}$, $\overline{\mathbf{N}}_{\mathcal{S}}^n$ is ergodic if

$$\lambda_i < \frac{\ell_i}{u_0 + \sum_{i\in\mathcal{U}} \ell_i s_i}\left(1 - \sum_{i\in\mathcal{S}} \rho_i\right) \qquad i \in \mathcal{S}\ . \tag{10}$$

Assume now that (10) holds, and consider a queue in $\mathcal{S}$, say queue 1 and let $C_{\mathcal{S}}^n(1)$ be the process of cycle lengths (successive visits to queue 1). The process $(\mathbf{N}_{\mathcal{S}}^n(1), C_{\mathcal{S}}^n(1))$ is a Markov chain, as easy to notice. Since $\mathbf{N}_{\mathcal{S}}^n(1)$ is ergodic, it is easy to see that also the process $\{\overline{\mathbf{N}}_{\mathcal{S}}^n(1), C_{\mathcal{S}}^{n-1}(1)\}_{n=2}^{\infty}$ is stationary and ergodic (for details see [3]). Now we are in position to use use Lemma 5 which implies that the process $\overline{N}_i^n(1)$, $i \in \mathcal{U}$ is stable provided that

$$\lambda_i < \frac{\ell_i}{EC_{\mathcal{S}}^1(1)} = \frac{\ell_i}{u_0 + \sum_{i\in\mathcal{U}} \ell_i s_i}\left(1 - \sum_{i\in\mathcal{S}} \rho_i\right) \qquad i \in \mathcal{U}\ , \tag{11}$$

where the equality in (11) follows from the fact that by Proposition 4,

$$EC_{\mathcal{S}}^1(1) = \frac{\overline{u}_0}{1 - \sum_{i\in\mathcal{S}} \rho_i} = \frac{u_0 + \sum_{j\in\mathcal{U}} \ell_j s_j}{1 - \sum_{j\in\mathcal{S}} \rho_j}\ . \tag{12}$$

Since the process $\overline{N}_i^n(1)$, $i \in \mathcal{S}$ is stable by construction, it follows from the stochastic dominance that the irreducible, aperiodic Markov chain $\mathbf{N}^n(1)$ is substable and therefore, ergodic. The fact that $\mathbf{N}^n(j)$ is ergodic for all $j \in \mathcal{M}$ follows from Proposition 4.

Putting everything together, from (10) and (11) we finally have that the Markov chain $\mathbf{N}^n(j)$ is ergodic for every $j \in \mathcal{M}$ if

$$\lambda_i < \frac{\ell_i}{u_0 + \sum_{i \in \mathcal{U}} \ell_i s_i} \left(1 - \sum_{i \in \mathcal{S}} \rho_i \right) \qquad i \in \mathcal{M} \ . \tag{13}$$

Since (13) holds for every partition $\mathcal{P} = (\mathcal{S},\mathcal{U})$ of the set $\mathcal{M}$ such that $\mathcal{S} \neq \mathcal{M}$, we conclude that the sufficient condition for stability of the system is $\mathcal{R} = \bigcup_{\mathcal{S} \subset \mathcal{M}} \mathcal{R}_{\mathcal{S}}$ where $\mathcal{R}_{\mathcal{S}} = \{\lambda = (\lambda_1, \dots, \lambda_M) :$ condition (13) holds$\}$. Finally, to complete the proof we need to show that $\bigcup_{\mathcal{S} \subset \mathcal{M}} \mathcal{R}_{\mathcal{S}} = \{\lambda = (\lambda_1, \dots, \lambda_M) : \lambda_i < \frac{\ell_i}{u_0}(1 - \sum_{i=1}^{M} \rho_i) \ \ i \in \mathcal{M}\}$. This requires only algebraic manipulations which are almost identical to the ones in [3], therefore we omit them here. ■

The fact that the conditions of Theorem 1 are also necessary can be proved along the lines of the corresponding proof in [3].

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