Contracts under Asymmetric Information

I

Aristotle, economy (oiko and nemo) and the idea of **exchange values**, subsequently adapted by **Ricardo** and **Marx.** Classical economists.

An **economy** consists of a set of agents each of whom is characterized by her **preferences** and her initial **endowments** (resources).

Walras and Pareto, neo-classical economists and the emergence of rigorous economics equilibrium, von-Neumann, Arrow, Debreu, McKenzie, Nash, Aumann. **Walrasian equilibrium, competitive equilibrium** and **perfect competition.**

Existence and Optimality of equilibrium. Book: Aliprantis et al.

Uncertainty and the **state contingent trade**

model.

II

Asymmetric or Differential Information Economies.

Walrasian Expectations equilibrium (WEE), and **Rational Expectations Equilibrium (REE),** Radner, Lucas, Prescott.

What is the **best** possible contact we can reach when agents are asymmetrically informed?

1. Individual rationality (better off)
2. Efficient
3. Incentive Compatible
4. Existence
5. Implementable as a PBE of an extensive form graph (game tree).

Can we construct in a finite agent economy a contact which has the above properties?

NO

Can we construct a second best contract?

YES

Can we construct an environment or framework where "first" best contracts can be reached?

Yes, under perfect competition – negligible private information.

Book: Glycopantis-Yannelis, **Differential Information Economies**, 2005.

1. Differential information economy (DIE)

We define the notion of a finite-agent economy with differential information for the case where the set of states of nature, Ω and the number of goods, l, per state are finite. I is a set of n players and $\mathbb{R}^l_+$ will denote the set of positive real numbers.

A differential information exchange economy $\mathcal{E}$ is a set

$$\{((\Omega, \mathcal{F}), X_i, \mathcal{F}_i, u_i, e_i, q_i) : i = 1, \ldots, n\}$$

where

1. $\mathcal{F}$ is a σ-algebra generated by a partition of Ω;
2. $X_i : \Omega \rightarrow 2^{\mathbb{R}^l_+}$ is the set-valued function giving the *random consumption set*

of Agent (Player) i, who is denoted by Pi;

3. $\mathcal{F}_i$ is a partition of Ω generating a sub-σ-algebra of $\mathcal{F}$, denoting the *private information* of Pi;

4. $u_i : \Omega \times \mathbb{R}^l_+ \to \mathbb{R}$ is the *random utility* function of Pi; for each $\omega \in \Omega$, $u_i(\omega, .)$ is continuous, concave and monotone;

5. $e_i : \Omega \to \mathbb{R}^l_+$ is the *random initial endowment* of Pi, assumed to be $\mathcal{F}_i$-measurable, with $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$;

6. q_i is an $\mathcal{F}$-measurable probability function on Ω giving the *prior* of Pi. It is assumed that on all elements of $\mathcal{F}_i$ the aggregate q_i is strictly positive. If a common prior is assumed on $\mathcal{F}$, it will be denoted by μ.

We will refer to a function with domain Ω, constant on elements of $\mathcal{F}_i$, as $\mathcal{F}_i$-*measurable*, although, strictly speaking, measurability is with respect to the σ-algebra generated by the partition.

In the first period agents make contracts in the ex ante stage. In the interim stage, i.e., after they have received a signal[1] as to what is the event containing the realized state of nature, they consider the incentive compatibility of the contract.

For any $x_i : \Omega \to \mathbb{R}^l_+$, the *ex ante expected utility* of Pi is given by

$$v_i(x_i) = \sum_{\Omega} u_i(\omega, x_i(\omega))q_i(\omega).$$

Let $\mathcal{G}$ be a partition of (or σ-algebra on) Ω, belonging to Pi. For $\omega \in \Omega$ denote by $E_i^{\mathcal{G}}(\omega)$

[1] A *signal* to Pi is an $\mathcal{F}_i$-measurable function to all of the possible distinct observations specific to the player; that is, it induces the partition $\mathcal{F}_i$, and so gives the finest discrimination of states of nature directly available to Pi.

the element of $\mathcal{G}$ containing ω; in the particular case where $\mathcal{G} = \mathcal{F}_i$ denote this just by $E_i(\omega)$. Pi's conditional probability for the state of nature being ω', given that it is actually ω, is then

$$q_i\big(\omega' | E_i^{\mathcal{G}}(\omega)\big) = \begin{cases} 0 & : \quad \omega' \notin E_i^{\mathcal{G}}(\omega) \\ \frac{q_i(\omega')}{q_i\left(E_i^{\mathcal{G}}(\omega)\right)} & : \quad \omega' \in E_i^{\mathcal{G}}(\omega). \end{cases}$$

The *interim expected utility* function of Pi, $v_i(x|\mathcal{G})$, is given by

$$v_i(x|\mathcal{G})(\omega) = \sum_{\omega'} u_i(\omega', x_i(\omega'))q_i\big(\omega' | E_i^{\mathcal{G}}(\omega)\big),$$

which defines a $\mathcal{G}$-measurable random variable.

Denote by $L_1(q_i, \mathbb{R}^l)$ the space of all equivalence classes of $\mathcal{F}$-measurable functions $f_i : \Omega \rightarrow \mathbb{R}^l$; when a common prior μ is assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$.

L_{X_i} is the set of all $\mathcal{F}_i$-measurable selections from the random consumption set of Agent i, i.e.,

$$L_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \; x_i : \Omega \rightarrow \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega) \; q_i\text{-}a.e.\}$$

and let $L_X = \prod_{i=1}^{n} L_{X_i}$.

Also let

$$\bar{L}_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \; x_i(\omega) \in X_i(\omega) \; q_i\text{-}a.e.\}$$

and let $\bar{L}_X = \prod_{i=1}^{n} \bar{L}_{X_i}$.

An element $x = (x_1, \ldots, x_n) \in \bar{L}_X$ will be called an *allocation*. For any subset of players S, an element $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ will also be called an allocation, although strictly speaking it is an allocation to S.

In case there is only one good, we shall use the notation $L^1_{X_i}$, L^1_X etc. When a common prior is also assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$.

Finally, suppose we have a coalition S, with members denoted by i. Their pooled information $\bigvee_{i \in S} \mathcal{F}_i$ will be denoted by $\mathcal{F}_S$ [2]. We assume that $\mathcal{F}_I = \mathcal{F}$.

Is it possible for agents to write incentive compatible and efficient or Pareto optimal contracts? Let us answer this question by considering a simple two agents example.

Example 0.1 There are two Agents, 1 and 2, and three equally probable states of nature denoted by a, b, c and one good per state denoted

[2] The "join" $\bigvee_{i \in S} \mathcal{F}_i$ denotes the smallest σ-algebra containing all $\mathcal{F}_i$, for $i \in S$.

by x. The utility functions, initial endowments and private information sets are given as follows:

$$u_1(w, x_1) = \sqrt{x_1}, \qquad \text{for } w = a, d, c$$
$$u_2(w, x_2) = \sqrt{x_2}, \qquad \text{for } w = a, b, c$$
$$e_1(a, b, c) = (10, 10, 0), \qquad \mathcal{F}_1 = \{\{a, b\}, \{c\}\}$$
$$e_2(a, b, c) = (10, 0, 10), \qquad \mathcal{F}_2 = \{\{a, c\}, \{b\}\}.$$

Notice that a "fully", pooled information, Pareto optimal, (i.e. a weak fine core outcome) is

$$\begin{aligned} x_1(a, b, c) &= (10, 5, 5) \\ x_2(a, b, c) &= (10, 5, 5). \end{aligned} \qquad (1)$$

However, this outcome is *not* incentive compatible because if the realized state of nature is a, then Agent 1 has an incentive to report that it is state c, (notice that Agent 2 cannot distinguish state a from state c) and become better off. In particular, Agent 1 will keep her initial endowment in the event $\{a, b\}$ which is 10 units and receive another 5 units from Agent 2, in state c, (i.e., $u_1(e_1, (a) + x_1(c) - e_1(c)) = u_1(15) > u_1(x, (a)) = 10$) and becomes better off. Obviously Agent 2 is worse off. Similarly, Agent 2 has an incentive to report b when he observes $\{a, c\}$

This example demonstrates that "*full or ex post Pareto optimality*" *is not necessarily compatible with incentive compatibility.*

The following example will illustrate the role of the *private information measurability of an allocation.*

Example 0.2 There are two Agents, 1 and 2, two goods denoted by x and y and two equally probable states denoted by $\{a, b\}$. The agents' characteristics are:

$$u_1(w, x_1, y_1) = \sqrt{x_1 y_1}, \qquad \text{for } w = a, b$$
$$u_2(w, x_2, y_2) = \sqrt{x_2 y_2}, \qquad \text{for } w = a, b$$
$$e_1(a, b) = ((10, 0), (10, 0)), \quad \mathcal{F}_1 = \{a, b\}$$
$$e_2(a, b) = ((10, 8), (0, 10)), \quad \mathcal{F}_2 = \{\{a\}, \{b\}\}.$$

The feasible allocation below is Pareto optimal (interim, ex post and ex ante).

$$((x_1(a), y_1(a)), (x_1(b), y_1(b))) = ((5, 2), (5, 5))$$
$$((x_2(a), y_2(a)), (x_2(b), y_2(b))) = ((15, 6), (5, 5)). \quad (2)$$

However, the allocation in (2) above is not incentive compatible because if b is the realized state of nature Agent 2 can report state a and become better off, i.e.,

$$\begin{aligned} u_2(e_2(b) \;+\; & (x_2(a), y_2(a)) - e_2(a)) \\ &= u_2((0, 10) + (15, 6) - (10, 8)) \\ &= u_2(5, 8) > u_2(x_2(b), y_2(b)) = u_2(5, 5). \end{aligned}$$

Notice that the allocation in (2) is *not* $\mathcal{F}_1$-measurable (i.e., measurable with respect to the private information of Agent 1). Hence, *an individually rational, efficient (interim, ex ante, ex post) without the* $\mathcal{F}_i$*-measurability* ($i = 1, 2$) *condition need not be incentive compatible.*

Observe that one can restore the incentive compatibility simply by making the allocation in (2) above $\mathcal{F}_i$-measurable for each $i, (i = 1, 2)$. In particular, the $\mathcal{F}_i$-measurable allocation below is incentive compatible, and private information ($\mathcal{F}_i$-measurable) Pareto optimal.

$$(x_1(a), y_1(a)), (x_1(b), y_1(b)) = ((5, 5), (5, 5))$$
$$(x_2(a), y_2(a)), (x_2(b), y_2(b)) = ((15, 3), (5, 5)).$$

The importance of the measurability condition in restoring incentive compatibility and of course guaranteeing the existence of an optimal contract is obvious in the above example and this approach was introduced by Yannelis (1991).

Example 0.3 Consider a three person differential information economy, with Agents 1, 2, 3, two goods denoted by x, y, and the three equal probable states are denoted by a, b, c. The agents' utility functions, random initial endowments and private information sets are as follows:

$$u_i(x_i, y_i) = \sqrt{x_i y_i}, \qquad i = 1, 2, 3,$$

$$e_1(a, b, c) = ((20, 0), (20, 0), (20, 0)),$$

$$\mathcal{F}_1 = \{a, b, c\}$$

$$e_2(a, b, c) = ((0, 10), (0, 10), (0, 5)),$$

$$\mathcal{F}_2 = \{\{a, b\}, \{c\}\}$$

$$e_3(a, b, c) = ((10, 10), (10, 10), (20, 30)),$$

$$\mathcal{F}_2 = \{\{a\}, \{b\}, \{c\}\}.$$

The allocation below is individual incentive compatible but *not* coalitional.

$$((x_1(a), y_1(a)), (x_1(b), y_1(b)), (x_1(c), y_1(c))) = ((10, 5), (10, 5), (12.5, 7.5)) \tag{3}$$

$$((x_2(a), y_2(a)), (x_2(b), y_2(b)), (x_2(c), y_2(c))) = ((10, 5), (10, 5), (2.5, 2.5)) \tag{4}$$

$$((x_3(a), y_3(a)), (x_3(b), y_3(b)), (x_3(c), y_3(c))) = ((10, 10), (10, 10), (25, 25)). \tag{5}$$

Notice that only Agent 3 can cheat Agents 2 and 3 in state a or b, by announcing b and a respectively, but has no incentive to do so. Hence, allocation (3) is individual incentive compatible. However, Agents 2 and 3 can form a coalition and when state c occurs they report to Agent 1 state b. Thus, Agent 1 gets $(10, 5)$ instead of $(12.5, 7.5)$ and Agents 2 and 3 distribute among themselves 2.5 units of each good, and clearly are better off.

Example 0.4 Consider a three person economy, with Agents 1, 2, 3, one good denoted by x, and three equally probable states denoted by a, b, c. The agents' utility function, initial endowments, and private information sets are as follows:

$$u_i = \sqrt{x_i}, \qquad i = 1, 2, 3$$

$$e_1(a, b, c) = (5, 5, 0)), \qquad \mathcal{F}_1 = \{\{a, b\}, \{c\}\}$$

$$e_2(a, b, c) = (5, 0, 5), \qquad \mathcal{F}_2 = \{\{a, c\}, \{b\}\}$$

$$e_3(a, b, c) = (0, 0, 0), \qquad \mathcal{F}_2 = \{\{a\}, \{b\}, \{c\}\}.$$

The allocation below is $\mathcal{F}_i$-measurable ($i = 1, 2, 3$) and cannot be improved upon by any $\mathcal{F}_i$-measurable, and feasible redistributions of the initial endowments of any coalition (this is the *private core*, Yannelis (1991)):

$$\begin{aligned} x_1(a, b, c) &= (4, 4, 1) \\ x_2(a, b, c) &= (4, 1, 4) \\ x_3(a, b, c) &= (2, 0, 0). \end{aligned} \tag{6}$$

Notice that the allocation in (4) is incentive compatible in the sense that Agent 3 is the only one who can cheat Agents 1 and 2 if the realized state of nature is a. However, Agent 3 has no incentive to misreport state a since this is the only state she gets positive consumption, and in any case one of Agents 1 or 2 will be able to tell the lie. Neither is it possible, as it can be easily seen, to form a coalition, profitable to both members, and misreport the state they have observed. Finally, notice that if Agent 3 had "bad" information, i.e., $\mathcal{F}_3' = \{a, b, c\}$, then, in a private core allocation, she gets zero consumption in each state. Thus, *advantageous information* is taken into account.

2. Cooperative equilibrium concepts: Core

Definition 3.1. An allocation $x \in L_X$ is said to be a *private core allocation* if

(i) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$ and

(ii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

Definition 3.2. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *WFC allocation* if

(i) each $x_i(\omega)$ is F_I-measurable;

(ii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$, for all $\omega \in \Omega$;

(iii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\mathcal{F}_S$-measurable for all $i \in S$, $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

3. Noncooperative equilibrium concepts: Walrasian expectations equilibrium and REE

A *price system* is an $\mathcal{F}$-measurable, non-zero function $p : \Omega \rightarrow \mathbb{R}^l_+$ and the *budget set* of Agent i is given by

$$B_i(p) = \{x_i : x_i : \Omega \rightarrow \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable}$$
$$x_i(\omega) \in X_i(\omega) \text{ and } \sum_{\omega \in \Omega} p(\omega) x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) e_i(\omega)\}.$$

Definition 4.1. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in L_X$ is an allocation, is a *Walrasian expectations equilibrium* if

(i) for all i the consumption function maximizes v_i on $B_i(p)$

(ii) $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ (free disposal), and

(iii) $\sum\limits_{\omega \in \Omega} p(\omega) \sum_{i=1}^n x_i(\omega) = \sum\limits_{\omega \in \Omega} p(\omega) \sum_{i=1}^n e_i(\omega)$.

Next we turn our attention to the notion of REE. We shall need the following. Let $\sigma(p)$

be the smallest sub-σ-algebra of $\mathcal{F}$ for which a price system $p : \Omega \rightarrow \mathbb{R}^l_+$ is measurable and let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ denote the smallest σ-algebra containing both $\sigma(p)$ and $\mathcal{F}_i$. We shall also condition the expected utility of the agents on $\mathcal{G}$ which produces a random variable.

Definition 4.2. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is an allocation, is a *REE* if

(i) for all i the consumption function $x_i(\omega)$ is $\mathcal{G}_i$-measurable;

(ii) for all i and for all ω the consumption function maximizes $v_i(x_i|\mathcal{G}_i)(\omega)$ subject to the budget constraint at state ω,

$$p(\omega)x_i(\omega) \leq p(\omega)e_i(\omega);$$

(iii) $\sum_{i=1}^{n} x_i(\omega) = \sum_{i=1}^{n} e_i(\omega)$ for all $\omega \in \Omega$.

REE is an interim concept because we condition on information from prices as well. An REE is said to be *fully revealing* if $\mathcal{G}_i = \mathcal{F} = \bigvee_{i \in I} \mathcal{F}_i$ for all $i \in I$. Although in the definition we do not allow for free disposal, we comment briefly on such an assumption in the context of Example 5.2.

Example 5.1 Consider the following three agents economy, $I = \{1, 2, 3\}$ with one commodity, i.e. $X_i = \mathbb{R}_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$.

The endowments and information partitions of the agents are given by

$$e_1 = (5, 5, 0), \qquad \mathcal{F}_1 = \{\{a, b\}, \{c\}\};$$

$$e_2 = (5, 0, 5), \qquad \mathcal{F}_2 = \{\{a, c\}, \{b\}\};$$

$$e_3 = (0, 0, 0), \qquad \mathcal{F}_3 = \{\{a\}, \{b\}, \{c\}\}.$$

$u_i(\omega, x_i(\omega)) = x_i^{\frac{1}{2}}$ and every player has the same prior distribution $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$.

The redistribution

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}$$

is a private core allocation, where the ith line refers to Player i and the columns from left to right to states a, b and c.

If the private information set of Agent 3 is the trivial partition, i.e., $\mathcal{F}_3' = \{a, b, c\}$, then no trade takes place and clearly in this case he gets zero utility. Thus the private core is sensitive to information asymmetries. On the other hand in a Walrasian expectations equilibrium or a REE Agent 3 will always receive zero quantities as he has no initial endowments, irrespective of whether her private information partition is the full one or the trivial one.

4. Incentive compatibility

There are alternative formulations of the notion of incentive compatibility. The basic idea is that an allocation is incentive compatible if no coalition can misreport the realized state of nature and have a distinct possibility of making its members better off.

Suppose we have a coalition S, with members denoted by i, and the complementary set $I \setminus S$ with members j. Let the realized state of nature be ω^*. Each member $i \in S$ sees $E_i(\omega^*)$. Obviously not all $E_i(\omega^*)$ need be the same, however all Agents i know that the actual state of nature could be ω^*.

Consider a state $\omega^{'}$ such that for all $j \in I \setminus S$ we have $\omega^{'} \in E_j(\omega^*)$ and for at least one $i \in S$ we have $\omega^{'} \notin E_i(\omega^*)$. Now the coalition S decides that each member i will announce that she has seen her own set $E_i(\omega^{'})$ which, of course, contains a lie. On the other hand we have that $\omega^{'} \in \bigcap_{j \notin S} E_j(\omega^*)$.

The idea is that if all members of $I \setminus S$ believe the statements of the members of S then each $i \in S$ expects to gain. For *coalitional Bayesian incentive compatibility* (CBIC) of an allocation we require that this is not possible.

Definition 7.1. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$, with or without free disposal, is said to be TCBIC if it is not true that there exists a coalition S, states ω^* and $\omega^{'}$, with ω^* different from $\omega^{'}$ and $\omega^{'} \in \bigcap_{i \notin S} E_i(\omega^*)$ and a random, net-trade vector, $z = (z_i)_{i \in S}$ among the members of S,

$$(z_i)_{i \in S}, \sum_S z_i = 0$$

such that for all $i \in S$ there exists $\bar{E}_i(\omega^*) \subseteq Z_i(\omega^*) = E_i(\omega^*) \cap (\bigcap_{j \notin S} E_j(\omega^*))$, for which

$$\begin{aligned} \sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i) q_i\big(\omega | \bar{E}_i(\omega^*)\big) \\ > \sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, x_i(\omega)) q_i\big(\omega | \bar{E}_i(\omega^*)\big). \qquad (7) \end{aligned}$$

Notice that $e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i(\omega) \in X_i(\omega)$ is not necessarily measurable. The definition implies that no coalition can hope that by misreporting a state, every member will become better off if they are believed by the members of the complementary set.

We now provide a *characterization* of TCBIC:

Proposition 7.1. Let $\mathcal{E}$ be a one-good DIE, and suppose each agent's utility function, $u_i = u_i(\omega, x_i(\omega))$ is monotone in the elements of the vector of goods x_i, that $u_i(., x_i)$ is $\mathcal{F}_i$-measurable in the first argument, and that an element $x = (x_1, \ldots, x_n) \in \bar{L}^1_X$ is a feasible allocation in the sense that $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)\ \forall\omega$. Consider the following conditions:

(i) $x \in L^1_X = \prod_{i=1}^{n} L^1_{X_i}$.

and

(ii) x is TCBIC.

Then (i) is equivalent to (ii).

5. Bayesian learning with cooperative solution concepts

Let $T = \{1, 2, ...\}$ denote the set of time periods and $\sigma(e_i^t, u_i^t)$ the σ-algebra that the random initial endowments and utility function of Agent i generated at time t. At any given point in time $t \in T$, the private information of Agent i is defined as:

$$\mathcal{F}_i^t = \sigma\left(e_i^t, u_i^t, (x^{t-1}, x^{t-2}, ...)\right) \tag{8}$$

where x^{t-1}, x^{t-2}, ... are past periods private core allocations.

Relation (22) says that at any given point in time t, the private information which becomes available to Agent i is $\sigma(e_i^t, u_i^t)$ together with the information that the private core allocations generated in all previous periods. In this scenario, the private information of Agent i in period $t+1$ will be $\mathcal{F}_i^t$ together with the information the private core allocation generated at at period t, i.e. $\sigma(x^t)$. More explicitly, the assumption is that the private information of Agent i at time $t+1$ will be $\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma(x^t)$, which denotes the "join", that is the smallest σ-algebra containing $\mathcal{F}_i^t$ and $\sigma(x^t)$.

Therefore for each Agent i we have that

$$\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq F_i^{t+2} \subseteq \dots . \tag{9}$$

Relation (23) represents a learning process for Agent i and it generates a sequence of differential information economies $\{\mathcal{E}^t : t \in T\}$ where now the corresponding private information sets are given by $\{\mathcal{F}_i^t : t \in T\}$.

Example 10.1 Consider the following DIE with two agents $I = \{1, 2\}$ three states of nature $\Omega = \{a, b, c\}$ and goods, in each state, the quantities of which are denoted by x_{i1}, x_{i2}, were i refers to the agent. The utility function are given by $u_i(\omega, x_{i1}, x_{i2}) = x_{i1}^{\frac{1}{2}} x_{i2}^{\frac{1}{2}}$, and states are equally probable, i.e. $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. Finally the measurable endowments and the private information of the agents is given by

$$e_1^t = \big((10, 0), (10, 0), (0, 0)\big),\ \mathcal{F}_1 = \big\{\{a, b\}, \{c\}\big\}$$
$$e_2^t = \big((10, 0), (0, 0), (10, 0)\big),\ \mathcal{F}_2 = \big\{\{a, c\}, \{b\}\big\}.$$

The structure of the private information of the agents implies that the private core allocation, (x_1^t, x_2^t), in $t = 1$ consists of the initial endowments.

Notice also that the information generated in Period 2 is the full informa $\sigma(x_1^t, x_2^t) = \big\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \emptyset\big\}$. It follows that the private information of each agent in periods $t \geq 2$ will be

$$\mathcal{F}_1^{t+1} = F_1^t \vee \sigma(x_1^t, x_2^t) = \big\{\{a\}, \{b\}, \{c\}\big\};$$
$$\mathcal{F}_2^{t+1} = F_2^t \vee \sigma(x_1^t, x_2^t) = \big\{\{a\}, \{b\}, \{c\}\big\}.$$

Now in $t = 2$ the agents will make contracts on the basis of the private information sets in (25). It is straightforward to show that a private core allocation in period $t \geq 2$ will be

$$x_1^{t+1} = \big((5, 5), (10, 0), (0, 0)\big);$$
$$x_2^{t+1} = \big((5, 5), (0, 0), (0, 10)\big).$$

Notice that the allocation in (26) makes both agents better off than the one given in (24). In other words, by refining their private information using the private core allocation they have observed, the agents realized a Pareto improvement.

Of course, in a generalized model with more than two agents and a continuum of states, unlike the above example, there is no need that the full information private core will be reached in two periods. The main objective of learning is to examine the possible convergence of the private core in an infinitely repeated DIE. In particular, let us denote the one shot limit full information economy by $\bar{\mathcal{E}} = \{(X_i, u_i, \bar{\mathcal{F}}_i, e_i, q_i : i = 1, 2, ..., n)\}$ where $\bar{\mathcal{F}}_i$ is the pooled information of Agent i over the entire horizon, i.e. $\bar{\mathcal{F}}_i = \bigvee_{i=1}^{\infty} \mathcal{F}_i^t$.

The questions that learning addresses itself to are the following:

(i) If $\{\mathcal{E}^t : t \in T\}$ is a sequence of DIE and x^t is a corresponding private core or value allocation, can we extract a subsequence which converges to a limit full information private core allocation for $\bar{\mathcal{E}}$?

(ii) Is the answer to (i) above affirmative, if we allow for bounded rationality in the sense that x^t is now required to be an approximate, ϵ-private core allocation for $\mathcal{E}^t$, but nonetheless it converges to an exact private core allocation for $\bar{\mathcal{E}}$?

(iii) Given a limit full information private core allocation say $\bar{x}$ for $\bar{\mathcal{E}}$, can we construct a sequence of ϵ-private core allocation x^t in $\mathcal{E}^t$ which converges to $\bar{x}$? In other words, can we construct a sequence of bounded rational plays, such that the corresponding ϵ-private core allocations converge to the limit full information private core allocation.

The above questions have been affirmatively answered in Koutsougeras - Yannelis (1999).

It should be noted that in the above framework it may be the case that in the limit incomplete information may still prevail. In other words, it could be the case that

$$\bar{\mathcal{F}}_i = \bigvee_{i=1}^{\infty} \mathcal{F}_i^t \subset \bigvee_{i=1}^{n} \mathcal{F}_i^t.$$

Hence in the limit a private core allocation may not be a fully revealing allocations of the same kind. However, if learning in each period reaches the complete information in the limit, i.e. $\bar{\mathcal{F}}_i \supset \bigvee_{i=1}^{n} \mathcal{F}_i^t$ the private core is indeed fully revealing.