The natural mathematics arising in information theory and investment

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Felicity of mathematics

We wish to maximize the growth rate of wealth.

There is a satisfactory theory. The strategy achieving this goal is controversial. (Probably because the strategy involves maximizing the expected logarithm.)

Why is π fundamental? $\pi = C/D$, $\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Recall from physics the statement that the *laws of physics have a strangely felicitous relation with mathematics*. We shall try to establish the reasonableness of the theory of growth optimality by presenting the richness of the mathematics that describes it and by giving a number of problems having growth optimality as the answer.

A theory is natural if it fits and has few “moving parts”. Ideally, it should “predict” other properties.

The new or unpublished statements will be identified.

Outline

- Setup
- Mean variance theory
- Growth optimal portfolios for *stochastic markets*
  - Properties:
    - Stability of optimal portfolio
    - Expected Ratio Optimality
    - Competitive optimality
    - S_n / S_n^* Martingale
    - $S_n^* \doteq e^{nW^*}$ (AEP)
- Growth optimal portolios for *arbitrary markets*
  - Universal portfolios
    - $\hat{S}_n / S_n^* \geq \frac{1}{2\sqrt{n+1}}$ for all x^n
  - Amplification
- Relationship of growth optimality to information theory

Portfolio Selection

Stock X:

$$\mathbf{X} = (X_1, X_2, \ldots, X_m) \sim F(\mathbf{x})$$

$$\mathbf{X} \geq 0$$

$$X_i = \text{ price-relative of stock } i$$

Portfolio $\mathbf{b}$:

$$\mathbf{b} = (b_1, b_2, \ldots, b_m), \quad b_i \geq 0, \quad \sum b_i = 1$$

proportion invested

Wealth Relative S: Factor by which wealth increases

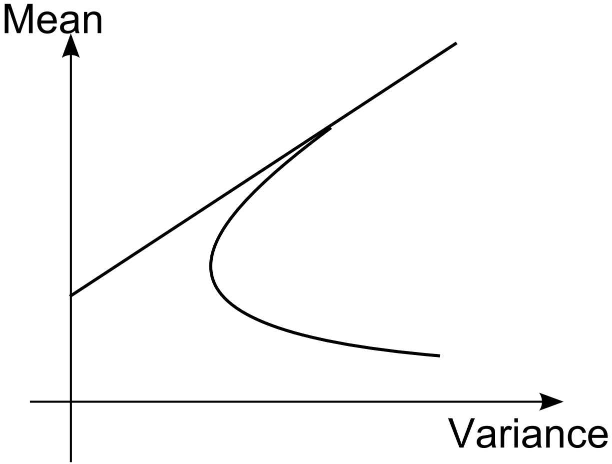
$$S = \sum_{i=1}^{m} b_i X_i = \mathbf{b}^t \mathbf{X}$$

Find the “largest” S.

Mean-Variance Theory.

Markowitz, Tobin, Sharpe, . . .

Choose $\mathbf{b}$ so that (Var S, ES) is undominated. $S = \mathbf{b}^t\mathbf{X}$.



Conflict of mean-variance theory and growth rate.

Portfolio selection:

Maximize growth rate of wealth.

$$S_n(X_1, X_2, \ldots, X_n) \doteq 2^{nW}$$

Efficient portfolio is *not* necessarily growth optimal (E.Thorp)

Consider the stock market process $\{\mathbf{X}_i\}$:

$$\mathbf{X}_i \in \mathbb{R}^m,$$

Portfolios $\mathbf{b}_i(\cdot)$:

$$\sum_{j=1}^{m} b_{ij}(\mathbf{x}^{i-1}) = 1$$

for each time $i = 1, 2, ...$ and for every past $\mathbf{x}^{i-1} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{i-1})$.

Note: $b_{ij} < 0$ corresponds to shorting stock j on day i. Shorting cash is called buying on margin.

Goal: Given a stochastic process $\{\mathbf{X}_i\}$ with known distribution, find portfolio sequence $\mathbf{b}_i(\cdot)$ that “maximizes”

$$S_n = \prod_{i=1}^{n} \mathbf{b}_i^t(\mathbf{X}^{i-1})\mathbf{X}_i$$

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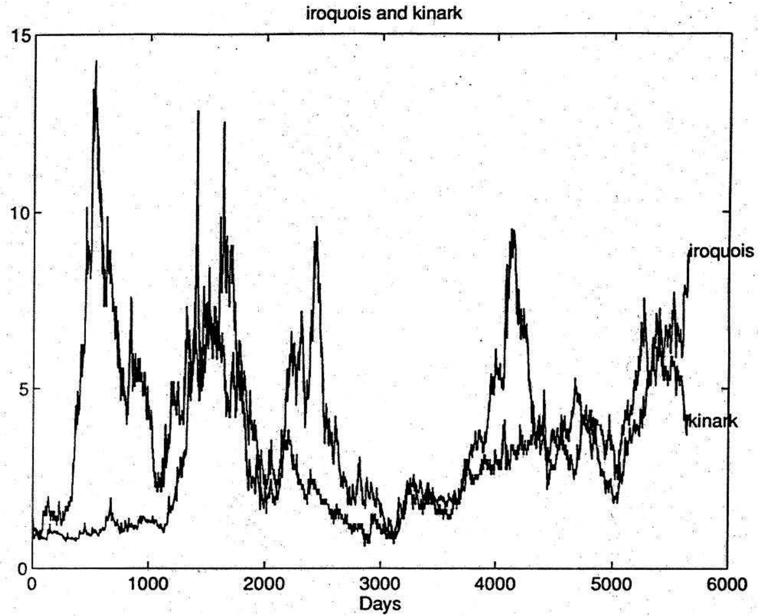

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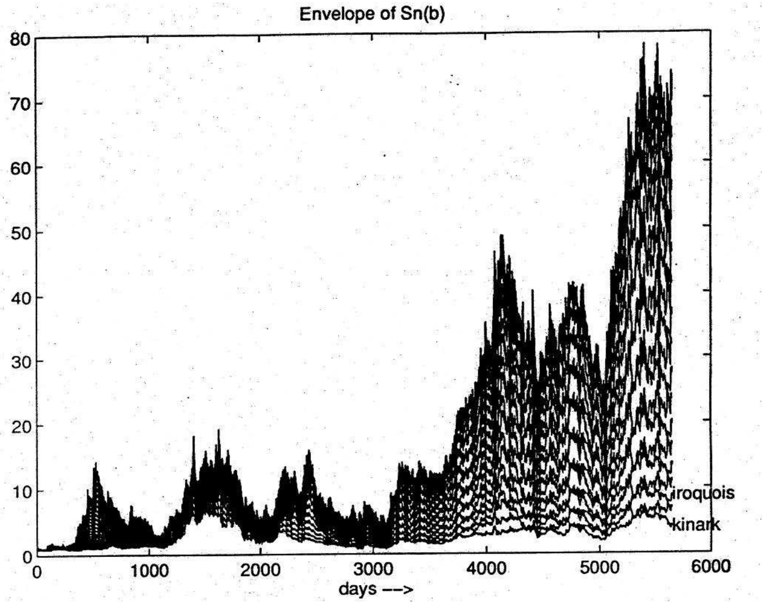
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$\mathbf{X}_1, \mathbf{X}_2, \ldots$ i.i.d. $\sim F(\mathbf{x})$

Wealth at time n:

$$\begin{aligned} S_n &= \prod_{i=1}^{n} \mathbf{b}^t \mathbf{X}_i \\ &= 2^{(n \frac{1}{n} \sum \log \mathbf{b}^t \mathbf{X}_i)} \\ &= 2^{n(E \log \mathbf{b}^t \mathbf{X} + o(1))}, \quad \text{a.e.} \end{aligned}$$

Definition: Growth rate

$$\begin{aligned} W(\mathbf{b}, F) &= \int \log \mathbf{b}^t \mathbf{x} \;\; dF(\mathbf{x}) \\ W^* &= \max_{\mathbf{b}} W(\mathbf{b}, F) \\ S_n &\doteq 2^{nW^*} \quad . \end{aligned}$$

Example

Cash vs. Hot Stock

$$\mathbf{X} = \begin{cases} (1,2), & \text{prob } \frac{1}{2} \\ \left(1, \frac{1}{2}\right), & \text{prob } \frac{1}{2} \end{cases} \qquad \mathbf{b} = (b_1, b_2)$$

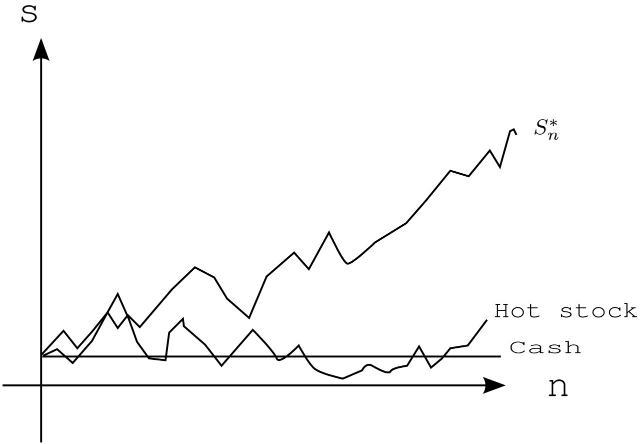
$$E \log S = \frac{1}{2} \log(b_1 + 2b_2) + \frac{1}{2} \log(b_1 + \frac{1}{2} b_2)$$

$$\mathbf{b}^* = (\frac{1}{2}, \frac{1}{2})$$

$$W^* = \frac{1}{2} \log \frac{9}{8}$$

$$S_n^* \doteq \left(\frac{9}{8}\right)^{n/2} \doteq (1.06)^n$$

Live off fluctuations



Calculation of optimal portfolio

$$\mathbf{X} \sim F(\mathbf{x})$$

Log Optimal Portfolio $\mathbf{b}^*$:

$$\max_{\mathbf{b}} E \log \mathbf{b}^t \mathbf{X} = W^*$$

Log Optimal Wealth:

$$S^* = \mathbf{b}^{*t} \mathbf{X}$$

$$\frac{\partial}{\partial b_i} E \ln \mathbf{b}^t \mathbf{X} = E \frac{X_i}{\mathbf{b}^t \mathbf{X}}$$

Kuhn-Tucker conditions:

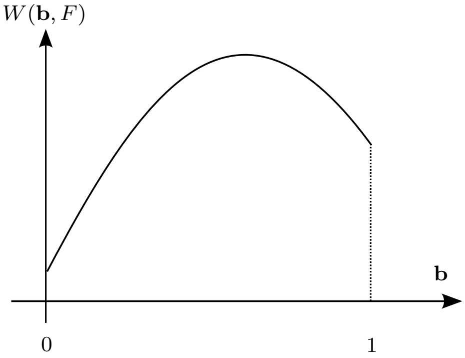
$$\mathbf{b}^* : E \frac{X_i}{\mathbf{b}^{*t}\mathbf{X}} \begin{array}{ll} = 1, & b_i^* > 0 \\ \leq 1, & b_i^* = 0 \end{array}$$

Consequence: $ES/S^* \leq 1,$ for all S.

Theorem $E \ln \frac{S}{S^*} \leq 0, \forall S \quad \Leftrightarrow \quad E \frac{S}{S^*} \leq 1, \forall S$

Theorem $W(\mathbf{b}, F)$ is concave in $\mathbf{b}$ and linear in F.

Let $\mathbf{b}_F$ maximize $W(\mathbf{b}, F)$ over all portfolios $\mathbf{b} : \sum_{i=1}^{m} \mathbf{b}_i = 1$.
$W^*(F) = W(\mathbf{b}_F, F)$



Theorem $W^*(F)$ is convex in F.

Question: Let $W(\mathbf{b}) = \int \ln \mathbf{b}^t \mathbf{x} \, dF(\mathbf{x})$. Is $W(\mathbf{b})$ a transform?

$\mathbf{b}^*$ is a stable point

Let $\mathbf{b} = (b_1, b_2, ..., b_m)$ denote the proportion of wealth in each stock.

The proportions held in each stock at the end of the trading day are

$$\widetilde{\mathbf{b}} = (\frac{b_1 X_1}{\mathbf{b}^t \mathbf{X}}, \frac{b_2 X_2}{\mathbf{b}^t \mathbf{X}}, ..., \frac{b_m X_m}{\mathbf{b}^t \mathbf{X}})$$

Then $\mathbf{b}$ is log optimal if and only if

$$\mathbf{b} = E\widetilde{\mathbf{b}}$$

i.e. $b_i = E\frac{b_i X_i}{\mathbf{b}^t \mathbf{X}}, \quad i = 1, 2, ..., m,$ i.e. the expected proportions remain unchanged.

This is the counterpart to Kelly gambling.

Generalization to arbitrary stochastic processes $\{\mathbf{X}_n\}$

$\mathbf{X}_n$: arbitrary stochastic process:

$$\text{Wealth from } \mathbf{b}_i(\cdot): \quad S_n = \prod_{i=1}^{n} \mathbf{b}_i^t \mathbf{X}_i, \quad \mathbf{b}_i = \mathbf{b}_i(\mathbf{X}^{i-1})$$

$$\text{Let } S_n^* = \prod_{i=1}^{n} \mathbf{b}_i^{*t} \mathbf{X}_i, \quad \mathbf{b}_i^* = \mathbf{b}_i^*(\mathbf{X}^{i-1})$$

where $\mathbf{b}_i^*$ is **conditionally log optimal**. Thus

$$\mathbf{b}_i^*(\mathbf{X}^{i-1}) : \max_{\mathbf{b}} E\{\ln \mathbf{b}^t \mathbf{X}_i | \mathbf{X}^{i-1}\}$$

Optimality for arbitrary stochastic processes $\{\mathbf{X}_n\}$

Theorem For any market process $\{X_i\}$,

$$E\{S_{n+1}/S^*_{n+1}|\mathbf{X}^n\} \le S_n/S^*_n.$$

S_n/S^*_n is a nonnegative super martingale with respect to $\{\mathbf{X}_n\}$

$$S_n/S^*_n \longrightarrow Y, \text{ a.e.}$$

$$EY \le 1.$$

Corollary:

$$\Pr\{\sup_n \frac{S_n}{S^*_n} \ge t\} \le 1/t,$$

by Kolmogorov's inequality. So S_n cannot *ever* exceed S^*_n by factor t with probability greater than $1/t$. Same as fair gambling.

Theorem If $\{X_i\}$ is ergodic, then $\frac{1}{n}\log S^*_n \longrightarrow W$, a.e.

3. Value of Side Information

Theorem: Believe that $X \sim g$, when in fact $X \sim f$. Loss in growth rate:

$$\Delta(f\|g) = E_f \log \frac{\mathbf{b}_f^t \mathbf{X}}{\mathbf{b}_g^t \mathbf{X}} \quad \leq \quad D(f\|g) = \int f \log \frac{f}{g}.$$

Mutual information: $\quad I(X;Y) = \sum p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$

Value of side information:

$$W(\mathbf{X}) = \max_{\mathbf{b}} E \ln \mathbf{b}^t \mathbf{X}, \quad W(\mathbf{X}|\mathbf{Y}) = \max_{\mathbf{b}(\cdot)} E \ln \mathbf{b}^t(\mathbf{Y})\mathbf{X}$$

$$\begin{array}{rcl} W(X) & \rightarrow & W(X|Y) \\ \mathbf{b}^* & & \mathbf{b}^*(y) \\ \Delta(X;Y) & = & \text{Increase in growth rate for market } X. \end{array}$$

Theorem: (A.Barron ,T.C.)

$$\Delta(X;Y) \leq I(X;Y).$$

Cash: 1

Stock: $X_i = \begin{cases} 1+u, & \text{w.p. } p \\ 1-d, & \text{w.p. } q \end{cases}$

Option: Pay c dollars today for option to buy at time n the stock at price K.

$$c \to \begin{cases} (X_n - K), & X_n \geq K \\ 0, & X_n < K \end{cases}$$

Black, Scholes idea:
Replicate option by buying and selling X_i, at times $i = 1, 2, ..., n$.
Example: Option expiration date $n = 1$. Strike price K. Initial wealth $= c$.

$$c_1 + c_2 X = (X - K)^+. \quad c = c_1 + c_2.$$

If it takes c dollars to replicate option, then c is a correct price for the option.

Growth optimal approach:

$$\left(1, X, \frac{(X-K)^+}{c}\right)$$

Best portfolio without option:

$$max_{b_1+b_2=1} E \ln(b_1 + b_2 X)$$

Growth optimal wealth:

$$X^* = b_1^* + b_2^* X$$

Add option:

$$\max_b E \ln\left((1-b)X^* + b\frac{(X-K)^+}{c}\right)$$

$$\frac{d}{db} E \ln\left((1-b)X^* + \frac{b(X-k)^+}{c}\right)\Bigg|_{b=0} = E\frac{\frac{(X-K)^+}{c} - X^*}{X^*} \geq 0,$$

$$\text{or} \quad E\frac{(X-K)^+}{X^*} \geq c.$$

Critical price:

$$c^* = E\frac{(X-K)^+}{X^*}.$$

But this is the same critical option price c^* as the Black Scholes theory.
Note: c^* does not depend on probabilities, only on u and d.

AEP

$$X_1, X_2, ..., X_n \;\; i.i.d. \sim p(x),$$

$$\frac{1}{n} \log \frac{1}{p(X_1, X_2, ..., X_n)} \to H.$$

AEP for markets

Wealth:

$$\begin{aligned} S_n &= \prod_{i=1}^{n} \mathbf{b}^t \mathbf{X}_i. \\ \frac{1}{n} \log S_n &\to W. \end{aligned}$$

Proof:

$$\frac{1}{n} \log S_n = \frac{1}{n} \log \prod_{i=1}^{n} \mathbf{b}^t \mathbf{X}_i = \frac{1}{n} \sum_{i=1}^{n} \log \mathbf{b}^t \mathbf{X}_i \to W.$$

$$\begin{aligned} p(X_1, X_2, ..., X_n) &\doteq 2^{-nH} \\ S_n(X_1, X_2, ..., X_n) &\doteq 2^{nW} \end{aligned}$$

$$\begin{aligned} \mathbf{b} &= (b_1, b_2, ..., b_m), \\ \mathbf{X} &= (0, 0, ..., 0, \underbrace{m}, 0, ..., 0), \text{ with probability } p_i, \\ \mathbf{b}^* &= (p_1, p_2, ..., p_m) \quad \text{Kelly gambling} \end{aligned}$$

Proof:

$$\begin{aligned} W &= E \log S \\ &= \sum_{i=1}^{m} p_i \log b_i m \\ &= \log m + \sum_i p_i \log \frac{b_i}{p_i} + \sum_i p_i \log p_i \\ &\leq \log m - H(p_1, ..., p_m), \end{aligned}$$

with equality if and only if $b_i = p_i$, for $i = 1, 2, ..., m$.

Conservation law

$$W + H = \log m$$

Comparisons

Information Theory	**Investment**
Entropy Rate $H = -\sum p_i \log p_i$	**Doubling Rate** $W^* = \max_{\mathbf{b}} E \log \mathbf{b}^t \mathbf{X}$
AEP $p(X_1, X_2, ..., X_n) \doteq 2^{-nH}$	$S^*(X_1, X_2, ..., X_n) \doteq 2^{nW^*}$
Universal Data Compression $l^{**}(X_1, X_2, ..., X_n) \doteq nH$	**Universal Portfolio Selection** $S^{**}(X_1, X_2, ..., X_n) \doteq 2^{nW^*}$

$$W^* + H \leq \log m$$

$\mathbf{X} \sim F(\mathbf{x})$. Consider the two-person zero sum game:

Player 1: Portfolio $\mathbf{b}_1$. Wealth $S_1 = W_1 \mathbf{b}_1^t \mathbf{X}$.

Player 2: portfolio $\mathbf{b}_1$. Wealth $S_2 = W_2 \mathbf{b}_2^t \mathbf{X}$.

Fair randomization: $EW_1 = EW_2 = 1, W_i \geq 0$.

Payoff: $\Pr\{S_1 \geq S_2\}$

$$V = \max_{\mathbf{b}_1, W_1} \min_{\mathbf{b}_2, W_2} \Pr\{S_1 \geq S_2\}$$

Theorem (R.Bell, T.C.) The value V of the game is $1/2$. Optimal strategy for player 1 is $\mathbf{b}_1 = \mathbf{b}^*$, where $\mathbf{b}^*$ is the log optimal portfolio. $W_1 \sim \mathrm{unif}[0, 2]$.

Comment: $\mathbf{b}^*$ is both long run and short run optimal.

Market sequence

$$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$$

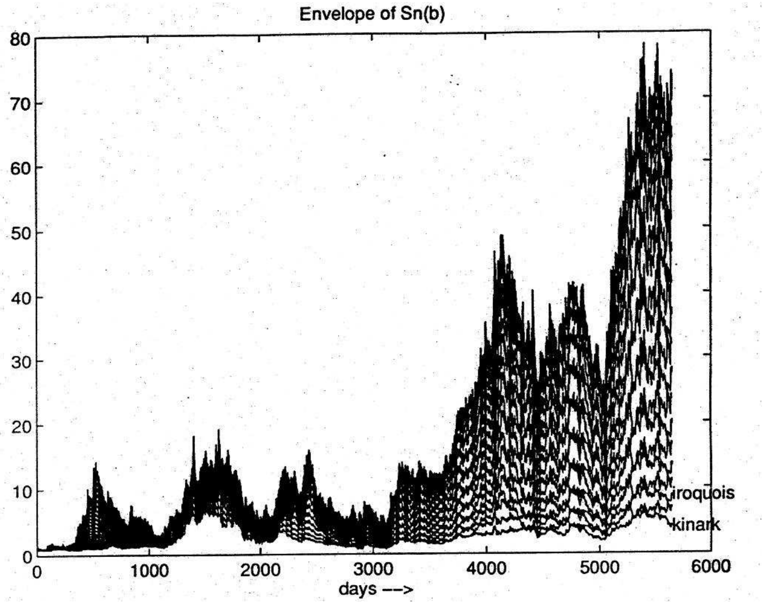
$$S_n(\mathbf{b}) = \prod_{i=1}^{n} \mathbf{b}^t \mathbf{x}_i$$

$$S_n^* = \max_{\mathbf{b}} S_n(\mathbf{b}) = \prod_{i=1}^{n} \mathbf{b}^{*t} \mathbf{X}_i.$$

Investor:

$$\hat{\mathbf{b}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{i-1})$$

$$\hat{S}_n = \prod_{i=1}^{n} \hat{\mathbf{b}}_i^t \mathbf{x}_i$$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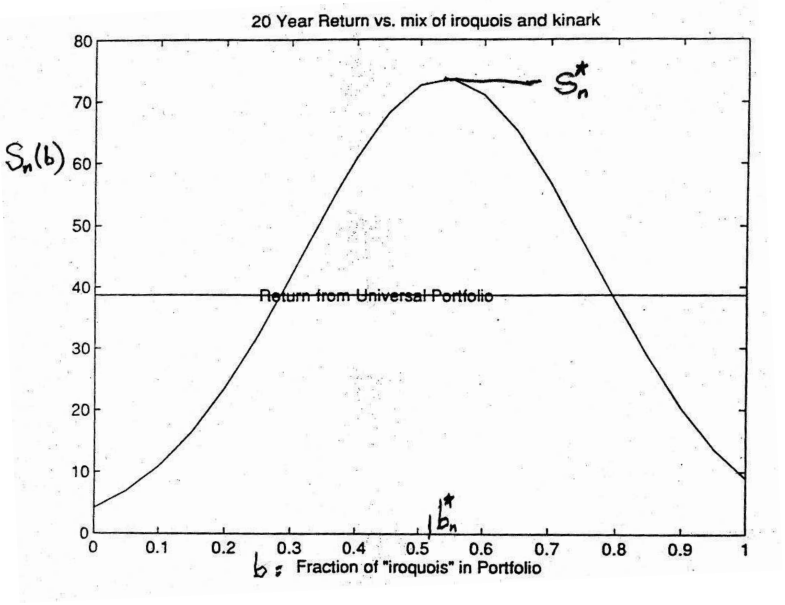

20 Year Return vs. mix of iroquois and kinark
$S_n(b)$
S_n^*
Return from Universal Portfolio
b_n^*
b: Fraction of "iroquois" in Portfolio
0
10
20
30
40
50
60
70
80
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Minimax regret for horizon n is defined as

$$R_n^* = \min_{\hat{b}(\cdot)} \max_{\mathbf{x}^n, \mathbf{b}} \frac{\prod_{i=1}^n \mathbf{b}^t \mathbf{x}_i}{\prod_{i=1}^n \hat{\mathbf{b}}_i(\mathbf{x}^{i-1}) \mathbf{x}_i} = \min_{\hat{\mathbf{b}}} \max_{\mathbf{x}^n} \frac{S_n^*}{\hat{S}_n}$$

Theorem: (Erik Ordentlich, T.C.)

$$R_n^* = \frac{1}{V_n},$$

where $V_n = \sum \binom{n}{n_1, \ldots, n_m} 2^{-nH(\frac{n_1}{n}, \ldots, \frac{n_m}{n})}$

Note: For $m = 2$ stocks,

$$\begin{aligned} V_n &= \sum_{k=0}^n \binom{n}{k} 2^{-nH(\frac{k}{n})} \sim \sqrt{\frac{2}{\pi n}} \\ V_n &\leq \frac{2}{\sqrt{n+1}} \end{aligned}$$

Corollary: For $m = 2$ stocks, there exists $\hat{\mathbf{b}}_i(\mathbf{x}^{i-1})$ such that

$$\hat{S}_n \geq \frac{2 S_n^*}{\sqrt{n+1}}, \qquad \text{for every sequence } \mathbf{x}_1, \ldots, \mathbf{x}_n.$$

Portfolio $\hat{\mathbf{b}}_i(\mathbf{X}_{i-1})$:
Invest

$$\hat{\mathbf{b}}(j^n) = \frac{1}{V_n}\left(\frac{n_1(j^n)}{n}\right)^{n_1(j^n)}\left(\frac{n_2(j^n)}{n}\right)^{n_2(j^n)}\cdots\left(\frac{n_m(j^n)}{n}\right)^{n_m(j^n)}$$

in “plunging” strategy j^n and let it ride, where $j^n \in \{1, 2, ..., m\}^n$.

Example For horizon $n = 2$. For $m = 2$.

$$\begin{aligned} X_1 &= (X_{11}, X_{12}) \\ \hat{\mathbf{b}}_1 &= (\tfrac{1}{2}, \tfrac{1}{2}) \\ \hat{\mathbf{b}}_2(\mathbf{X}_1) &= \left(\frac{\frac{4}{5}X_{11}+\frac{1}{5}X_{12}}{X_{11}+X_{12}}, \frac{\frac{1}{5}X_{11}+\frac{4}{5}X_{12}}{X_{11}+X_{12}}\right) \end{aligned}$$

$$\begin{aligned} \hat{b}(11) &= 4/10 \\ \hat{b}(12) &= 1/10 \\ \hat{b}(21) &= 1/10 \\ \hat{b}(22) &= 4/10 \end{aligned}$$

Stock $\mathbf{x} \in \mathcal{R}_+^m$, requires $\mathbf{b} \in \mathcal{R}_+^m$, so that $\mathbf{b}^t\mathbf{x} \geq 0$.

$$\text{Let } \mathcal{X}(\alpha) = \{\mathbf{x} \in \mathcal{R}^m : \; x_i \geq \alpha, \; \sum_{i=1}^{m} x_i = 1\}$$

$$\mathcal{B}(\alpha) = \{\mathbf{b} \in \mathcal{R}^m : \; \sum_{i=1}^{m} b_i = 1, \; \mathbf{b}^t\mathbf{x} \geq 0, \; \forall \mathbf{x} \in \mathcal{X}(\alpha)\}$$

$\mathcal{B}(\alpha)$ is polar cone to $\mathcal{X}(\alpha)$: $\mathcal{B}(\alpha) = \mathcal{X}^{\perp}(\alpha)$.

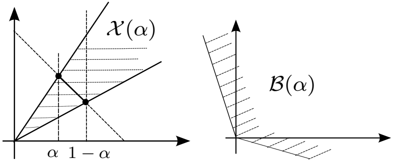
$\mathcal{B}(\alpha)$ allows short selling and buying on margin.

Thus $\mathbf{x} \in \mathcal{X}(\alpha)$, $\mathbf{b} \in \mathcal{B}(\alpha)$ yields $S = \mathbf{b}^t\mathbf{x} \geq 0$.
Let $\Omega = \mathcal{R}_+^m$, $\mathcal{X}(\alpha) = A\Omega$, $\mathcal{B}(\alpha) = A^{-1}\Omega$.

$$A = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix} \quad A^{-1} = \frac{1}{2\alpha - 1}\begin{pmatrix} \alpha & -(1-\alpha) \\ -(1-\alpha) & \alpha \end{pmatrix}$$

$\mathbf{b} \in \Omega$,$\mathbf{X} \in \Omega$. $\quad \tilde{\mathbf{b}} = A^{-1}\mathbf{b} \in \mathcal{B}(\alpha)$, $\quad \tilde{\mathbf{X}} = A\mathbf{X} \in \mathcal{X}(\alpha)$.

$$\tilde{\mathbf{b}}^t\tilde{\mathbf{X}} = \mathbf{b}^t\left(A^{-1}\right)^t A\mathbf{X} = \mathbf{b}^t\mathbf{X}$$



Theorem (Acceleration (Erik Ordentlich, T.C., to appear))

$m = 2$ stocks. The short selling investor can come within factor $V_n(\alpha)$ of the best long-only investor given hindsight:

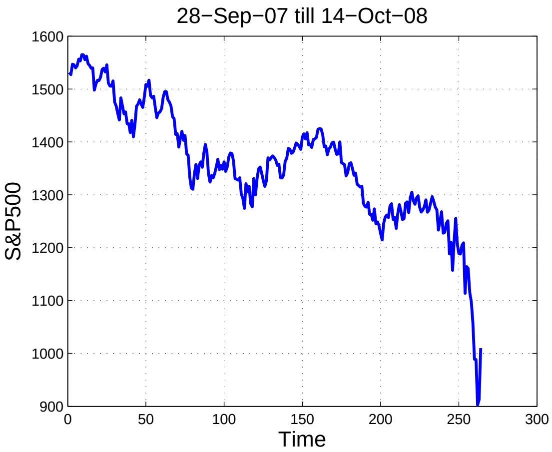
$$\max_{\widehat{\mathbf{b}}_i(\cdot)\in\mathcal{B}(\alpha)} \min_{\substack{\mathbf{x}\in\mathcal{X}^n(\alpha),\\ \mathbf{b}\in\mathcal{B}(0)}} \frac{\prod_{i=1}^n \widehat{\mathbf{b}}_i^t \mathbf{x}_i}{\prod_{i=1}^n \mathbf{b}^t \mathbf{x}_i} = V_n(\alpha),$$

where $[x] = x$ rounded off to interval $[\alpha, \bar{\alpha}]$.

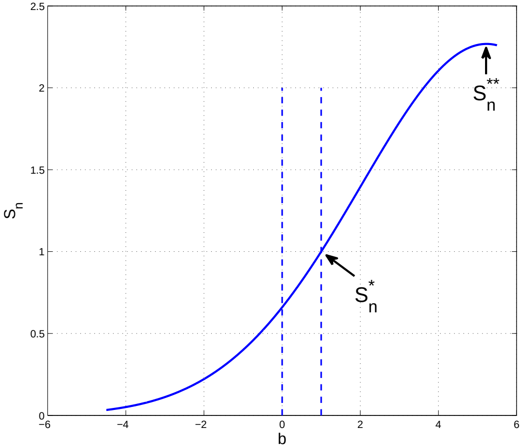
$$V_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{k}{n}\right]^k \left[\frac{n-k}{n}\right]^{n-k}$$

Note: $V_n(\alpha) \nearrow$. $\quad V_n(0) \sim \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$. $\quad V_n(\frac{1}{2}) = 1$.

Accelerated Performance



Accelerated Performance

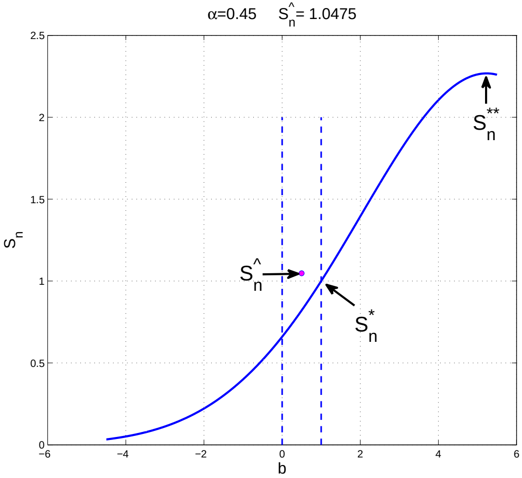


9/28/07 – 10/14/08, $n = 263$.

S_n^*: Wealth of best *long-only* constant rebalanced portfolio in hindsight.

S_n^{**}: Wealth of best *short selling and margin* constant rebalanced portfolio in hindsight.

Accelerated Performance



9/28/07 – 10/14/08, $n = 263$.

S_n^*: Wealth of best *long-only* constant rebalanced portfolio in hindsight.

S_n^{**}: Wealth of best *short selling and margin* constant rebalanced portfolio in hindsight.

$\hat{S}_n$: Wealth of universal portfolio.

General Market

$\mathbf{X} \sim F(\mathbf{x})$

$\mathbf{b}^* : \quad E\frac{b_i^* X_i}{\mathbf{b}^{*t}\mathbf{X}} = b_i^*$

$W^* = E\mathbf{b}^{*t}\mathbf{X}$

Wrong distribution $G(\mathbf{x})$:

$\Delta(F||G) = \int \frac{\mathbf{b}_F^t \mathbf{x}}{\mathbf{b}_G^t \mathbf{x}} dF(\mathbf{x})$

Side information $(\mathbf{X}, \mathbf{Y}) \sim f(\mathbf{x}, \mathbf{y})$:

$\Delta = \int \ln \frac{\mathbf{b}_{f(\mathbf{x}|\mathbf{y})}^t \mathbf{x}}{\mathbf{b}_{f(\mathbf{x})}^t \mathbf{x}} f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$

Horse Race Market

$\mathbf{X} = m\mathbf{e}_i, \quad p_i$

$b_i = p_i$ Kelly gambling

$W^* = \log m - H(p)$, $H =$entropy

$\Delta = \sum p_i \ln \frac{p_i}{g_i} = D(p||g)$, relative entropy

$\Delta = \sum p(x,y) \ln \frac{p(x,y)}{p(x)p(y)}$
$= I(X;Y)$, mutual information

Comparisons

General Market

Asymptotic growth rate

$\{\mathbf{X}_i\}$ stationary:

$W^* = \max_{\mathbf{b}} E\{\ln \mathbf{b}^t \mathbf{X}_0 | \mathbf{X}_{-\infty}^{-1}\}$

AEP for ergodic processes:

$\frac{1}{n} \log S_n^* \to W^*, a.e.$

$S_n^* \doteq 2^{nW^*}$

Horse Race Market

$$W^* = \log m - H(X_0 | X_{-\infty}^{-1})$$
$$= \log m - H(\mathcal{X}), \quad H(\mathcal{X}) = \text{entropy rate}$$

$-\frac{1}{n} \log p(X^n) \to H(\mathcal{X}), a.e.$

$p(X^n) \doteq 2^{-nH}$

Universal portfolio (individual sequences):

General Market	**Horse Race Market**
$\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in \mathbb{R}_+^m$	$\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in \{e_1, ..., e_m\}$
$S_n(\mathbf{b}, \mathbf{x}^n) = \prod_{i=1}^n \mathbf{b}^t \mathbf{x}_i$	$S_n(\mathbf{b}, \mathbf{x}^n) = \prod_{i=1}^m b_i^{n_i(\mathbf{x}^n)}$
$\widehat{S_n}(\hat{\mathbf{b}}^n, \mathbf{x}^n) = \prod_{i=1}^n \hat{\mathbf{b}}^t(\mathbf{x}^{i-1})\mathbf{x}_i$	$\widehat{S_n}(\hat{\mathbf{b}}^n, \mathbf{x}^n) = \hat{\mathbf{b}}(\mathbf{x}^n)$
V_n	V_n

Same cost of universality for both.

$$
\begin{aligned}
V_n &= \min_{\hat{\mathbf{b}}(\cdot)} \max_{\mathbf{b}, \mathbf{x}^n} \frac{\widehat{S_n}(\hat{\mathbf{b}}^n, \mathbf{x}^n)}{S_n(\mathbf{b}, \mathbf{x}^n)} \\
&= \sum \binom{n}{n_1, ..., n_m} 2^{-nH(\frac{n_1}{n}, \dots, \frac{n_m}{n})}
\end{aligned}
$$

Concluding remarks

Growth optimal portfolios have many properties:

- Long run optimality
- Martingale property
- Competitive optimality
- Asymptotic equipartition property
- Universal achievability
- Black-Scholes
- Amplification
- Relationship with information theory

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