Purdue University
CS 381

S. Hambrusch- C. Guerra
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Sample Final Exam Questions

1.) (i) Let Π_1 and Π_2 be two decision problems. Assume $\Pi_1 \leq_P \Pi_2$. If there exists a polynomial time algorithm for Π_1, does this imply that there exists a polynomial time algorithm for Π_2?
(ii) Let Π_{dec} be an NP-complete decision problem and let Π_{opt} be its corresponding optimization problem. Assume Π_{opt} can be solved in polynomial time. What does this imply for Π_{dec}? What does this imply for the class of NP-complete problems and the class NP, respectively?
(iii) Does there exist a polynomial time algorithm to determine whether an undirected graph contains a clique of size 3? Explain.
(iv) If an NP-complete problem can be solved deterministically in $O(n^3)$ time, can every problem in class NP be solved in $O(n^3)$ time?
(v) Let G be an connected, undirected, and weighted n-vertex graph. Each one of its m edges has weight 2. Describe and analyze an algorithm that finds a minimum spanning tree of G. Your algorithm should be faster than Kruskal's algorithm.

2.) For each of the problems listed below state the asymptotic running time of the best algorithm you know. If you think the problem is NP-complete, state so (you do not need to give a running time). You do not need to give details about the algorithm. If you wish to make any comments, please limit them to two lines.

1. Sorting n integers $a_1, a_2, \ldots, a_n$ with $0 \leq a_i \leq n^2 \log^2 n$.
2. Determining the $\frac{n}{5}$-th largest element in an unsorted set of size n.
3. In a directed, weighted graph $G = (V, E)$ with positive weights and $|V| = n$ and $|E| = m$, determine the shortest path between a given pair of vertices.
4. In an n-node rooted tree T, determine the number of leaves whose parent has more than one child.
5. Given a boolean formula in conjunctive normal form (i.e., $C_1 \wedge C_2 \wedge \ldots \wedge C_k$, where every C_i contains an arbitrary number of literals $\vee$-ed together), determine whether there exists a truth assignment to the variables satisfying the formula.

6. Given a boolean formula in disjunctive normal form (i.e., $C_1 \vee C_2 \vee \ldots \vee C_k$, where every C_i contains an arbitrary number of literals $\wedge$-ed together), determine whether there exists a truth assignment to the variables satisfying the formula.

3.) Assume $G = (V, E)$ is an undirected, connected, weighted graph, $|V| = n$, $|E| = m$, represented by adjacency lists. Weights can be positive as well as negative. For each problem listed below, give the asymptotic time bound of the best algorithm you know. Give a 2-sentence explanation of your solution.
(i) Determine whether G contains at least 10 edges of cost ≥ 100.
(ii) Determine whether G is a tree.
(iii) Find a spanning tree of G having minimum cost.
(iv) Given two vertices u and v, does there exist a path from u to v?
(v) Given two vertices u and v, determine the length of the longest path from u to v?

4.) Let $G = (V, E)$ be a directed graph with integer weights on its edges, $|V| = n, |E| = m$. Let s and t be two vertices in G and k be an integer. Consider the problem of determining whether the cost of the longest path from s to t is at least k. (In case there exists no path between s and t, return the answer "no".)

What is the complexity of this problem for the three classes of graphs stated below? Either describe an efficient polynomial time algorithm or show that the problem is NP-complete.
(i) G is a rooted tree.
(ii) G is an acyclic graph.
(iii) G is an arbitrary graph.

5.) A point $p_i = (x_i, y_i)$ dominates point $p_j = (x_j, y_j)$ if $x_i \geq x_j$ and $y_i \geq y_j$. Given n points in arbitrary order, describe and analyze an efficient algorithm which determines the points *not* dominated by any other point.

6.) (i) Let A be a set of n numbers given in unsorted order. Consider the problem of determining $x \in A$, $y \in A$ such that $|x - y| \geq |w - z|$ for all $w \in A$, $z \in A$. State an efficient algorithm and its running time.
(ii) Let A be a set of n numbers given in unsorted order. Consider the problem of determining $a_i \in A$, $a_j \in A$ such that $|a_i - a_j| \leq |a_q - a_r|$ for all $q \neq r$. Describe an efficient algorithm for determining a_i and a_j.

7.) The partition problem is defined a follows: Given a set A, determine whether the elements of A can be partitioned into two sets so that the sum of the elements in one set is equal to that of the elements in the other set. This problem is known to be NP-complete.

Consider the following 1-processor scheduling problem: Given are n jobs and job i has length l_i, penalty p_i, and deadline d_i associated with it. The n jobs are to be scheduled on the processor. If job i is not completed by time d_i, a penalty of p_i occurs.

In the Min_Pen problem we are given l_i, p_i, d_i, $1 \leq i \leq n$, and a quantity P and are to determine whether there exists a schedule such that the sum of the arising penalties is at most P. Show that the Min_Pen problem is NP-complete.