Geometric Series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n$$

is a geometric series and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

Thus $\sum_{k=0}^{n} x^k = \Theta(x^n)$ if $x > 1$ (increasing series)

$\sum_{k=0}^{n} x^k = \Theta(1)$ if $x < 1$ (decreasing series)

If the series is infinite and $|x| < 1$, then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

A General Theorem for "Divide and Conquer" Recurrences

Consider recurrences of the form

$T(n) = aT(n/b) + f(n)$ where a and $b > 1$ are integer constants and $f(n)$ is some function.

Theorem 1. ["Master" Theorem] *The solution for the above recurrence $T(n)$ is:*

1) If $af(n/b) = cf(n)$ for some constant $c > 1$ then $T(n) = \Theta(n^{\log_b a})$.

2) If $af(n/b) = f(n)$ then $T(n) = \Theta(f(n) \log_b n)$.

3) If $af(n/b) = cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

Proof.

(We will assume that n is a power of b.)

Draw a "recursion tree" for $T(n)$: $f(n)$ is the root and it has a children each of which is a recursion tree for $T(n/b)$. That is, a recursion tree is a complete a-ary tree where each node at depth i has the value $a^i f(n/b^i)$. The leaves of the tree contains the "base cases" of the recursion. Since we are looking at asymptotic bounds, we can assume without loss of generality (w.l.o.g) that $T(1) = f(1)$. Assuming each level of the tree is full, we have,

$$T(n) = f(n) + af(n/b) + a^2 f(n/b^2) + \ldots + a^L f(n/b^L)$$

where L is the depth of the recursion tree.

$L = \log_b n$ and since $f(1) = \Theta(1)$,

$a^L f(n/b^L) = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a})$

We have three cases:

1) If $f(n)$ is a constant factor smaller than $af(n/b)$ then $T(n)$ is a geometric series with largest term $a^L f(n/b^L) = \Theta(n^{\log_b a})$.

2) If $af(n/b) = f(n)$ then there are $L + 1$ levels each level summing to $f(n)$ and hence $\Theta(f(n) \log_b n)$.

3) If $f(n)$ is a constant factor larger than $af(n/b)$ then $T(n)$ is a geometric series with largest term $f(n)$. Hence $T(n) = \Theta(f(n))$.

$\square$

Examples

1. $T(n) = T(3n/4) + 2n$

Here $af(n/b) = 2(3n/4) = (3/4)f(n)$

Hence $T(n) = \Theta(n)$.

2. $T(n) = 7T(n/2) + \Theta(n^2)$

That is, $T(n) = 7T(n/2) + c_1 n^2$, for some positive constant c_1.

$af(n/b) = 7c_1(n/2)^2 = (7/4)c_1 n^2 = (7/4)f(n)$

Hence, $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$

3. $T(n) = 2T(n/2) + n$

Here $af(n/b) = f(n)$ and hence $T(n) = \Theta(n \log n)$.