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ERGODICITY AND MIXING RATE OF ONE-DIMENSIONAL
CELLULAR AUTOMATA

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Abstract

One-and two-dimensional cellular automata which are known to be fault-tolerant are very complex. On the other hand, only very simple cellular automata have actually been proven to lack fault-tolerance, i.e., to be mixing. The latter either have large noise probability ε or belong to the small family of two-state nearest-neighbor monotonic rules which includes local majority voting.

For a certain simple automaton L called the soldiers rule, this problem has intrigued researchers for the last two decades since L is clearly more robust than local voting: in the absence of noise, L eliminates any finite island of perturbation from an initial configuration of all 0's or all 1's. The same holds for a 4-state monotonic variant of L, K, called two-line voting. We will prove that the probabilistic cellular automata K_ε and L_ε asymptotically lose all information about their initial state when subject to small, strongly biased noise. The mixing property trivially implies that the systems are ergodic.

The finite-time information-retaining quality of a mixing system can be represented by its relaxation time $Relax(\cdot)$, which measures the time before the onset of significant information loss. This is known to grow as $(1/\varepsilon)^c$ for noisy local voting. The impressive error-correction ability of L has prompted some researchers to conjecture that $Relax(L_\varepsilon) = 2^{c/\varepsilon}$. We prove the tight bound $2^{c_1 \log^2 1/\varepsilon} < Relax(L_\varepsilon) < 2^{c_2 \log^2 1/\varepsilon}$ for a biased error model. The same holds for K_ε. Moreover, the lower bound is independent of the bias assumption.

The strong bias assumption makes it possible to apply sparsity/renormalization techniques, the main tools of our investigation, used earlier in the opposite context of proving fault-tolerance.

Contents

Chapter 1

Introduction

The foundations of cellular automata and fault-tolerant computation go back to von Neumann whose seminal works [34, 35] have played a significant role in establishing both research areas. In theoretical fault-tolerant computing, a computing model is defined and its behavior is investigated when certain components of the model are allowed to fail. For example, in the circuit model, given a circuit C of n gates computing a Boolean function, the question is asked whether a circuit C' consisting of m gates each of which gives a wrong output with probability $\varepsilon > 0$ can nevertheless be constructed such that C' computes the same function as C with high probability. The faults may be transient or permanent, the latter frequently leading to degenerate situations which are mathematically less interesting. All of the subsequent discussion including the contribution of this thesis deals with transient faults.

Von Neumann's formulation was done in the context of the Boolean circuit model and a positive answer to the fault-tolerance question was given by von Neumann in [34]. It was later improved by Dobrushin and Ortyukov [6] who showed an upper bound of $m = O(n \log n)$. The logarithmic redundancy factor was proven to be tight in [5]. Pippenger [27] went on to exhibit classes of Boolean functions for which a constant redundancy factor was sufficient, and he gave explicit constructions of fault-tolerant circuits using the notion of compressors. Other significant work on information storage in the context of the circuit model was done by, among others,

Kuznietsov [22] and more recently Sipser and Spielman [29].

Although the circuit model is appealing due to the simple and well-understood nature of fault-tolerant computing, a potential drawback is its reliance on lengthy wires needed for satisfying expansion properties in the construction of compressors. As the number of gates n increases, it quickly becomes more and more difficult to realize such devices feasibly in 3-dimensional space without taking into account the volatility of signals traversing long wires which may induce a level of faultiness matching, if not exceeding, that of the gates. In other words, reflecting back the increased faultiness into the gates, the error probability $\varepsilon(n)$ at every gate is now a function of the size of the system increasing with n. Hence, if possible, it is preferable to have a uniform yet *local* connectivity structure that only requires constant length wires for connected elements. Cellular automata are computing models that satisfy this property.

Cellular automata were introduced by von Neumann in conjunction with investigating the problem of self-reproducing machines [35]. An introduction to self-reproducing machines and its main results can be found in [2]. A cellular automaton is a set of homogeneous finite automata arranged on a d-dimensional lattice where at every time step each element or cell computes its transition based upon the input from its neighbors and its own state. A simple argument shows that cellular automata are capable of simulating Turing machines, hence universal computation. Their parallelism has been exploited, with some success, for arithmetic and matrix computations in the context of VLSI implementations, also known as systolic arrays, pioneered by Kung [20]. Cellular automata have also been investigated as models of "complex systems," but due to their universality, most questions regarding their asymptotic behavior turn out to be undecidable. An interesting collection of papers may be found in [17, 36]. A practical introduction to cellular automata is given in [30].

Related to the topic of cellular automata as models of physical and biological

systems, particle systems have been studied in physics in connection with characterizing their global properties, in particular, with respect to the existence of phase transitions. Starting with Ising's investigation of ferromagnetism in particle systems when assigning Gibbs measures on the space of configurations [18], the question of phase transition as captured by the existence of more than one invariant probability measure has captured the interest of many researchers in the mathematical community also known as interacting particle systems [23, 7, 19]. One of the main goals in this area is to prove whether certain simple particle systems with local interactions are ergodic or not. Two "schools" may be distinguished, one, the American research community which primarily deals with continuous time systems and, two, the Russian (former USSR) community which has primarily investigated discrete time systems. Continuous time systems require the machinery of Markov generators to account for nontrivial existence problems whereas for discrete time systems this is much simpler. However, the analysis of discrete time systems has tended to be much more challenging than their continuous time counterparts (cf. local majority voting [14, 15]) due to the need to track events simultaneously.

Significant progress toward exhibiting nonergodic discrete time media in dimension 2 and higher was made by Toom in the 1970's where several rules were introduced with a proof of nonergodicity [31]. The north-east-center rule, also known as Toom's rule, is a well-known example. Toom's nonergodicity proof was an extension of a general technique called the contour argument, a widely used tool in interacting particle systems which goes back to Peierls' investigation of phase transitions in the two-dimensional Ising model [26]. The search for nonergodic one-dimensional rules was much slower in coming with an interesting candidate rule introduced by Gács, Kurdyumov, and Levin in 1978 called the soldiers rule [9]. Due to a certain self-stabilization property, the authors conjectured that a probabilistic perturbation of the rule was nonergodic. In the continuous time community, a conjecture had arisen which stated that all translation-invariant, finite-range systems with positive

transition rates were ergodic, also known as the positive rates conjecture [14]. This conjecture was disproved in discrete time by Gács in the early 1980's [11] using a complicated hierarchical construction based on some ideas of Kurdyumov [21]. The continuous time version has been recently advanced in [13]. A non-uniform solution (i.e., non-uniform in both space and time) was given earlier by Zirelson [37]. Although the ideas and structures underlying Gács' construction are elegant, the construction itself and its analysis is quite involved. Thus, even though the existence of complicated nonergodic one-dimensional cellular automata has been demonstrated, it is not clear whether more simple nonergodic automata exist or not, in particular, whether the soldiers rule is nonergodic.

In this thesis, we prove that the soldiers rule under a biased error model is mixing, and hence ergodic, and we give tight bounds on the relaxation time which captures the finite time convergence property of the system. We prove the same results for a rule suggested recently by Toom called two-line voting [33]. This rule is similar to the soldiers rule with respect to the deterministic self-stabilization property (also called the eroder property), but is a little easier to handle due to its coordinate-wise monotonicity. For this very reason, we will first prove the results in the case of two-line voting, and then transfer the results to the soldiers rule. The monotonicity property turns out to be inessential to our arguments.

The main technique employed in this thesis is based on the notion of sparsity which was used earlier in the opposite context of proving nonergodicity [11]. In the broader picture, this technology is related to renormalization and scaling in statistical physics and percolation theory [8, 16]. Renormalization has been explored in the interacting particle system context in [3]. The gist of the sparsity technique lies in identifying properties of space-time processes that remain invariant across multiple scales, and exploiting the self-similar or fractal[1] structure to analyze—and in the

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[1]Sparsity and renormalization techniques share the spirit of fractal geometry with respect to modeling/analyzing natural and artificial systems, a paradigm expounded by Mandelbrot [24].

nonergodicity context, to design—the dynamics of the system at hand. In this thesis, we show that our error model induces space-time error patterns which obey a certain self-similar property where errors are distributed "sparsely" in a scale-invariant way with high probability. This, in turn, allows us to define and analyze a self-similar property of space-time configurations—k-blackishness—which can be shown to thrive under k-sparse error conditions where k is a scale parameter. The technical difficulty lies in first identifying a suitable set of scale-invariant properties, estimating their probabilities, and analyzing the dynamics of the system across multiple scales via recursive space-time arguments.

Next, we will give a concise introduction to cellular automata, leading up to the statement of the main results. The remaining chapters will be concerned with their proof.

1.1 Deterministic cellular automata

We define here only one-dimensional nearest-neighbor cellular automata: the generalizations to several dimensions and larger neighborhoods are evident. A *cellular automaton* $\mathrm{CA}(T, m)$ is given by a finite set S of states, a *local transition function* $T : S^3 \to S$ and a set $\mathbb{Z}_m$ of *sites*, or *cells*. When $m = \infty$ then this set is $\mathbb{Z}$, the set of integers. In the finite case, it is the set of remainders mod m. When $m = \infty$, we will omit m from $\mathrm{CA}(T, m)$. When x is a site and r an integer, $x + r$ will be understood mod m. A *space configuration* is a function $\xi : \mathbb{Z}_m \to S$. Let us define the *constant configurations* ζ_s, $s \in S$, where $\zeta_s(x) = s$ for all x. In this thesis, we will restrict ourselves to discrete-time cellular automata. Given a configuration ξ, we will define the configuration $T(\xi)$ as

$$T(\xi)(x) = T(\xi(x - 1), \xi(x), \xi(x + 1)), \quad x \in \mathbb{Z}.$$

A *space-time configuration* is a function $\eta : \mathbb{Z}_m \times \mathbb{Z}_+ \to S$ assigning a state to each site at each nonnegative integer time t. A space-time configuration η is an

orbit of T if $\eta(\cdot, t+1) = T(\eta(\cdot, t))$. By composition, $\eta(\cdot, t) = T^t(\eta(\cdot, 0))$. An orbit η of a deterministic CA is determined by its initial configuration $\eta(\cdot, 0)$ and the transition function. We will omit the words "space" and "space-time" and just refer to a configuration if its meaning is clear from the context.

A configuration ξ is *invariant* with respect to a transition rule T if $T(\xi) = \xi$. Two configurations ξ_1, ξ_2 are called *equivalent* with respect to T if there is a $t \geq 0$ such that $T^t(\xi_1) = T^t(\xi_2)$. In [9], a configuration ξ is called *attractive* if every other configuration ζ with $\{ x : \xi(x) \neq \zeta(x) \}$ finite, is equivalent to it. We add the requirement that ξ be invariant. We call a transition function *conservative* if it has non-equivalent periodic attractive configurations.

Toom defined conservative transition rules with state set $\{0, 1\}$ whose attractive configurations are the constant configurations ζ_0, ζ_1. These rules can also be viewed as monotonic Boolean functions. An example is the north-east-center voting rule also known as Toom's rule. In [9] some conservative one-dimensional cellular automata were constructed. The simplest example, denoted by L and called the *soldiers rule* or GKL rule after the originators Gács, Kurdyumov, and Levin, is described as follows. There are two states labeled -1 and 1. Imagine that each cell is a soldier with his/her nose pointing left (-1) or right (1). The rule is not nearest-neighbor: it uses neighbors up to a distance of 3. A compact description is given by

$$L(\xi)(x) = \mathrm{Maj}(\xi(x), \xi(x + \xi(x)), \xi(x + 3\xi(x))). \tag{1.1.1}$$

Thus, at each step, each soldier sets its state to the majority of the current state and the states of its first and third neighbors found in the direction determined by where his nose is pointing.

Theorem 1.1.2 *The configurations ζ_{-1} and ζ_1 are attractive. Moreover, if ξ is any configuration which differs from ζ_1 (or ζ_{-1}) on an interval of size n, then the perturbation is corrected within a space-time rectangle of size at most $2n \times (2n - 2)$.*

This result in not too difficult to prove and a proof can be found in [4]. The reader may

find it interesting to verify that the configurations ζ_{-1} and ζ_1 are indeed attractive and why this is so. Figure 1.1.1 (bottom) shows a finite black island being erased within an initial configuration of otherwise all white cells. The grayish region is actually a space-time triangle of alternating -1 and 1, where the alternation occurs both in space and time. At time 0, an alternating region is induced at the right boundary of the black island which propagates with speed 1 in both directions. When the left front of this growing region meets the stationary left boundary of the black island, it gives rise to a growing front of the white region which travels with speed 3 eventually catching up with the front of the alternating region.

Remark 1.1.3   In [25] and a number of related papers, genetic algorithms have been used to generate rules with apparently similar properties.

In this thesis, we study the behavior of the GKL rule and a rule recently suggested by Toom which is similar to L in its important properties, but a little easier to handle due to its monotonicity. The new rule is called *two-line voting* [33] and is denoted by K. The new set of states is $S = \{0, 1\}^2$ and we represent a state as a 2-element bit array indexed by $-1, 1$. The *bottom bit* of state s is $s(-1)$, the *top bit* is $s(1)$. For $j \in \{-1, 1\}$, the rule K is defined as follows:

$$K(\xi)(x)(j) = \text{Maj}(\xi(x)(-j), \xi(x-j)(j), \xi(x-2j)(j)).$$

In words, a bottom (top) bit turns into the majority of its top (bottom) counterpart, and its two nearest right (left) neighbors. Note that the state of the bit itself does not participate in the vote. All the top bits of a configuration ξ can be taken together as the *top track* $\xi(\cdot)(1)$ and similarly for the *bottom track* $\xi(\cdot)(-1)$.

Theorem 1.1.4 (eroder)   *The configurations ζ_{00} and ζ_{11} are attractive. Moreover, if ξ is any finite perturbation which differs from ζ_{00} or ζ_{11} on an interval of size n, then the perturbation is corrected within a space-time rectangle of size at most $(4n + 4) \times (3n/2 + 2)$.*
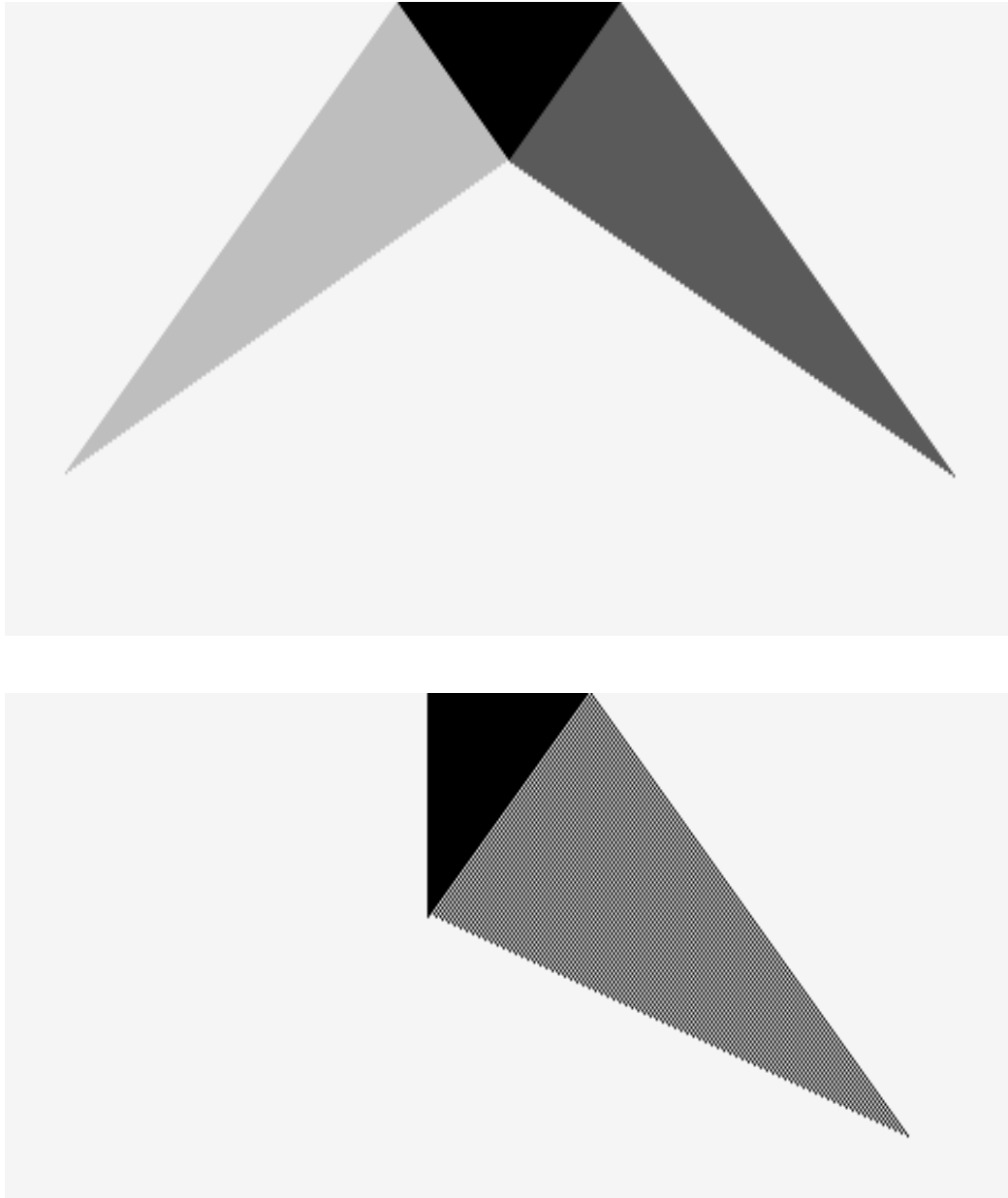
Figure 1.1.1: Space-time snapshots of deterministic eroder property: two-line voting (top) and GKL rule (bottom). Time flows "downwards."

Proof. Let η be an orbit. Without loss of generality (by translation invariance), suppose that $\eta(\cdot, 0)$ differs from ζ_{00} only over the segment $-n/2 \leq x < n/2$. With time, 0's slide into the interval $[-n/2, n/2)$ from the right with speed 1 on the bottom track and from the left on the top track. Once the front of the bottom 0's meets the front of the top 0's, the 0's begin to extend outward on both tracks with speed 2, catching up with the 1's that were sliding out with speed 1. Each of these claims can be verified by inspection of K. Also, note that this argument does not depend on the values of $\eta(x, 0)$ for $-n/2 \leq x < n/2$. The size of the error-correction space-time rectangle $(4n+4) \times (3n/2+2)$ is attained when the perturbation is "worst-case," i.e., an island of all 1's, and this is easily checked by simple calculation. By symmetry, an analogous argument holds for ζ_{11}. ∎

Figure 1.1.1 (top) shows a finite island of all black being corrected or erased within a sea of all white. The two shades of gray represent the states 10 and 01, respectively, with state 01 (the top row is 1) encoded as the darker shade. Notice that with K both boundaries are nonstationary.

Remark 1.1.5   Two-line voting can be viewed as "hardwiring" the asymmetry of the GKL rule (i.e., the dependence on a cell's state in determining its relevant neighborhood—left or right) by introducing two additional states. As Leonid Levin has pointed out, whereas in the GKL rule the decision from which neighborhood to take the majority vote is determined by the *state* of the cell, in two-line voting it is possible to interprete the top element of a site i as corresponding to the location $2i + 1$ on $\mathbb{Z}$ and the botton bit being located at $2i$, and by switching at every other time step, the GKL rule can be seen to be embedded in two-line voting whereby the "parity" is now the determining factor.

1.2 Probabilistic cellular automata

1.2.1 Transition probabilities

A probability distribution μ over the set of all configurations $S^{\mathbb{Z}}$ is determined by its values over the cylinder sets

$$\mu(s_{-n}, s_{-n+1}, \ldots, s_n) \;\equiv\; \mu\{\, \xi : \xi(-n) = s_{-n}, \ldots, \xi(x_n) = s_n \,\}$$

for all n and all possible tuples $(s_{-n}, \ldots, s_n) \in S^{2n+1}$. A sequence μ_t of distributions is said to converge (weakly) to a distribution μ if for all vectors $(s_{-n}, \ldots, s_n)$, the numbers $\mu^t(s_{-n}, s_{-n+1}, \ldots, s_n)$ converge to $\mu(s_{-n}, s_{-n+1}, \ldots, s_n)$. When it does not lead to confusion we will denote μ by "Pr". Probability measures over $S^{\mathbb{Z}^2}$ are defined analogously. A *probabilistic cellular automaton* $\mathrm{PCA}(\mathcal{P}, m)$ is defined by a *transition matrix* which is an array of nonnegative numbers

$$(\mathcal{P}(s \mid u, v, w))_{s,u,v,w \in S}$$

with $\sum_s \mathcal{P}(s \mid u, v, w) = 1$. A random space-time configuration η is an *orbit* of this automaton if from time t to time $t + 1$, each cell x makes a transition to state s independently of all the others with probability

$$\mathcal{P}(s \mid \eta(x - 1, t) = u, \eta(x, t) = v, \eta(x + 1, t) = w), \qquad s, u, v, w \in S.$$

Note that a deterministic transition function T defines a special transition matrix with $\mathcal{P}(s \mid u, v, w) = 1$ for $s = T(u, v, w)$, and 0 otherwise. If we have $\mathcal{P}(s \mid u, v, w) \geq 1 - \varepsilon$ for $s = T(u, v, w)$ then we will say that $\mathcal{P}$ is an ε-*perturbation* of T, and $\mathrm{PCA}(\mathcal{P})$ is an ε-perturbation of $\mathrm{CA}(T)$. For a random orbit η, let μ^t denote the distribution of $\eta(\cdot, t)$. Then μ^t obeys the following recursive definition:

$$\mu^{t+1}(s_{-n}, \ldots, s_n) \;=\; \sum_{(r_{-n-1}, \ldots, r_{n+1})} \mu^t(r_{-n-1}, \ldots, r_{n+1}) \prod_{i=-n}^{n} \mathcal{P}(s_i \mid r_{i-1}, r_i, r_{i+1}).$$

The above definition can also be written as $\mu^{t+1} = P\mu^t$ where P is a linear operator giving $\mu^t = P^t \mu^0$. A probabilistic cellular automaton is a discrete-time Markov process. If the set of cells is finite then the PCA defines a finite-state Markov chain.

1.2.2 Ergodicity and mixing

A distribution μ over configurations is called *invariant* if $P\mu = \mu$. We call a PCA *ergodic* if it has only one invariant distribution. An ergodic PCA is *mixing* if for every probability measure μ, the sequence $P^t\mu$ of measures converges to the invariant distribution. Let us call

$$\min_{u,v,w,s} \mathcal{P}(s \mid u, v, w)$$

the *noise lower bound*. The cellular automaton is called *noisy* if the noise lower bound is positive. It is a textbook result that noisy finite Markov chains are mixing. However, there are examples of noisy infinite cellular automata that are not even ergodic.

For a mixing probabilistic cellular automaton, it is justified to say that the automaton cannot remember even a single bit of information for an unbounded time. Indeed, since the distribution over the configurations converges to the invariant one, all information about the initial distribution is eventually lost. On the other hand, a nonergodic cellular automaton clearly keeps at least one bit of information since it can be started up in two different initial invariant measures and this difference is preserved forever.

Remark 1.2.1 At this time, it is not known whether for infinite noisy cellular automata ergodicity implies mixing. In the interacting particle systems literature, ergodicity is defined to include the requirement of mixing.

1.2.3 Relaxation time

Since finite noisy cellular automata are all mixing, it needs some justification that we want to investigate the problem at all—in practice, many systems are finite. Let us introduce the notion of relaxation time. Define the following distance between two

distributions:

$$d_n(\mu, \nu) \;=\; \sum_{\mathbf{s} \in S^{2n+1}} |\mu(\mathbf{s}) - \nu(\mathbf{s})|.$$

Clearly, $0 \le d_n(\mu, \nu) \le 2$. Let $\mathcal{L}(m, S)$ denote the set of all possible distributions over configurations where m is the finite or infinite space size. Let

$$D_n(t, P, m) \;=\; \sup_{\nu, \mu \in \mathcal{L}(m,S)} d_n(P^t \nu, P^t \mu). \tag{1.2.2}$$

Let δ_ξ be the distribution that assigns probability 1 to the configuration ξ. It is easy to see that the supremum in D_n is already achieved over measures μ, ν of the form δ_ξ. It is also easy to see that P is mixing over a space of size m iff $D_n(t, P, m) \searrow 0$. The *relaxation time* is defined as

$$Relax(n, \varrho, P, m) = \min\{\, t : D_n(t, P, m) < \varrho \,\}.$$

It is obviously an increasing function of n (defined only for $n \le (m-1)/2$) and a decreasing function of the accuracy parameter ϱ. We may omit m from the arguments if it is ∞, i.e., $Relax(n, \varrho, P) \equiv Relax(n, \varrho, P, \infty)$. We will also omit P if it is clear from the context. Relaxation measures the rate of information loss: it shows how long we have to wait until it is guaranteed that no matter what initial configuration we started from, on segments of length $2n + 1$, the distribution comes to within ϱ of the unique invariant distribution. The following fact is easy to prove.

Fact 1.2.3 *For all $n < (m-1)/2$, ϱ with $Relax(n, \varrho, P) < (m-1)/2 - n$, we have*

$$Relax(n, \varrho, P, m) \le Relax(n, \varrho, P).$$

This implies that if P is mixing over the infinite space then increasing the finite space size m does not increase the relaxation time significantly for any fixed n. In each segment of length $2n + 1$ of any finite space, information is being lost during the evolution at least as fast as in the infinite space. On the other hand, if P is not mixing over the infinite space then this is not necessarily so. Indeed, in the known

examples of infinite non-ergodic PCA [32],[11] and the ones derived from them, for any fixed n and ϱ, the relaxation time grows exponentially with the size m of the finite-space version.

Remarks 1.2.4

1. In finite Markov chains, speed of convergence to equilibrium can be measured by the second-largest eigenvalue λ_2 of a certain matrix; i.e., the relaxation time can be estimated by $1/(1 - \lambda_2)$ (see, e.g., [1]). In certain reversible Markov chains arising in combinatorial optimization, the quantity called "conductance" was found useful for estimating $1 - \lambda_2$ [28]. The exponential growth in m of the relaxation time of the above systems means therefore that their λ_2 is exponentially close to 1.

2. Even when $Relax(n, \delta, P)$ is finite, if it grows very fast with n this may make information storage practical even in mixing cellular automata.

1.3   A brief history

Sufficient criteria for mixing in terms of noise lower bounds were introduced in a number of papers in the early 1970's by Dobrushin, Shlossman and others: see [23] for a reference. Nonergodic noisy cellular automata were first constructed by Andre Toom in the mid 1970's [32]. These automata had dimension ≥ 2. Toom's conservative 2-dimensional north-east-center voting is one such example. The difficult part of Toom's result is the theorem stating that for a monotonic transition rule F with state set $\{0, 1\}$, an attractive configuration ζ_s remains attractive (in a certain probabilistic sense) under any sufficiently small perturbation of F.

There is no one-dimensional two-state monotonic rule with ζ_0, ζ_1 as attractive configurations, and therefore Toom's theorem cannot be used to exhibit a one-dimensional nonergodic noisy cellular automaton. The soldiers rule mentioned above

is conservative but not monotonic. The two-line voting rule is conservative, monotonic but has more than two states. Moreover, it is really monotonic only as a function of the individual bits in its states (i.e., monotonic in the sense of the natural partial ordering of its states). Still, these rules appear to have better error-correction properties than local majority voting. This has prompted some efforts to test the rule experimentally. Unless the error probability is very large it seems practically impossible to wait until a random orbit breaks away from ζ_s. Nonetheless, the perturbed rule is believed by many researchers to be ergodic since if a large island of opposite bits is created artificially, then its tendency to disappear is negligible.

For a while, it had been conjectured that all one-dimensional noisy cellular automata are mixing, also known as the positive rates conjecture. The conjecture could be based on the following reasoning. Suppose, for simplicity, that ζ_u and ζ_v are two initial configurations that we want to remember where $u \neq v$. Suppose that we started in ζ_u. Then, in the very first step, with probability ε^{2n+1} an island of size $2n + 1$ of v's arises around site 0. Now, the cells well inside the island will behave as if they came from ζ_v while the cells at the boundary will have no way of determining whether they should behave as if they came from ζ_u or from ζ_v. The rules L and K essentially send "signals" in both directions outward from the meeting point of two islands. If a signal reaches the other end of the island it starts a higher-speed process that catches up with the signal running in the other direction. Random errors, however, will stop such signals within an essentially constant distance. Apparently, therefore, with all simple rules the boundary of such a big island will just fluctuate randomly like a symmetric random walk. This fluctuation takes so long to erase the island (infinite expected time) that in the meantime many other islands arise. Figure 1.3.1 shows the fluctuating boundaries[2] of K_ε (top) and L_ε (bottom), respectively.

The positive rates conjecture in discrete time was refuted in [11] by a complex

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[2]Boundary processes that partition a medium into two stable regions are sometimes called *interfaces* in physics.
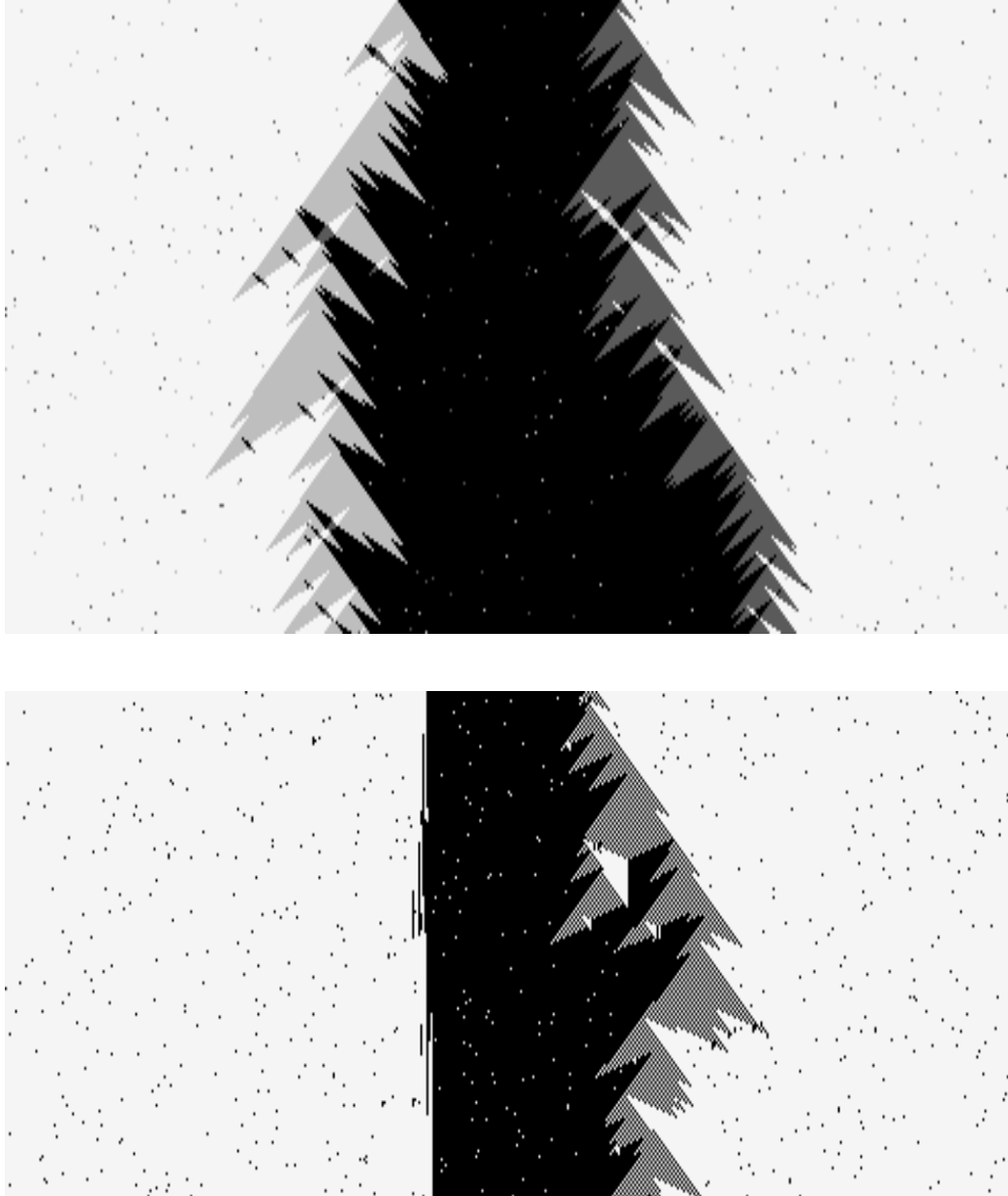
Figure 1.3.1: Space-time snapshots of sample path of ε-perturbation: two-line voting (top) and GKL rule (bottom). Time flows "downwards."

construction supporting a certain hiearchical pattern. (See also [12] for a hierarchy supporting 2-dimensional computation.) Some form of hierarchical behavior seems necessary for non-ergodic one-dimensional cellular automata to overcome the volatility of simple conservative rules in the presence of random errors outlined above. It is not even clear, however, how "hierarchical behavior" should be defined in general. The simplest one-dimensional rules that seem to have some error-correction capability are the local majority vote rules. In the continuous-time context, Gray proved that these rules are mixing [14]. The discrete time case is technically more difficult but Gray has outlined a proof of the mixing property for local majority voting in [15].

1.4   New results

This thesis is a contribution in the direction of showing that one-dimensional rules, without some form of hierarchical behavior, fail to conserve information. We define a special perturbation $K_{\varepsilon,\beta}$ of the rule K, for $0 \leq \beta \leq 1$ as follows. Let

$$\eta(x, t+1)(j) = K(\eta(x-2,t), \eta(x-1,t), \eta(x,t), \eta(x+1,t), \eta(x+2,t))(j) \quad (1.4.1)$$

with probability $1 - \varepsilon$. With probability $\beta\varepsilon$ the bit $\eta(x,t+1)(j)$ turns into 0 (or stays there if it was 0 to begin with), and it is set to 1 with probability $(1-\beta)\varepsilon$. Thus, the smaller $\beta > 0$, the greater the bias in favor of 1's.

Similarly for $L_{\varepsilon,\beta}$ where $\eta(x, t+1)$ obeys the deterministic transition L with probability $1 - \varepsilon$; with probability $\beta\varepsilon$ it sets its state to 0 and with probability $(1-\beta)\varepsilon$ $\eta(x, t+1)$ is set to 1.

Theorem 1.4.2   *There exist $0 < \beta_* < \beta^* < 1$ such that $\forall\beta \in [0, \beta_*) \cup (\beta^*, 1]$, $\forall\varepsilon \in (0, 1]$, $K_{\varepsilon,\beta}$ and $L_{\varepsilon,\beta}$ are mixing.*

The strong bias is a weak point since it does not preclude the possibility that for $\beta \approx 1/2$, $K_{\varepsilon,\beta}$ and $L_{\varepsilon,\beta}$ are nonergodic. On the other hand, the mixing property holds for any positive error probability $\varepsilon > 0$, and the bias assumption makes it possible

to apply "renormalization" methods used earlier in the opposite context of proving fault-tolerance [10]. Referring back to the informal argument above on why simple noisy automata should be mixing, the boundaries of a large island of 1's that arise randomly will not just fluctuate but will expand with a certain speed, eventually taking over the whole space making it look "blackish." Thus it corresponds to a random walk with drift.

Even though the PCA considered here are mixing there is a way to express the fact that they keep information much better than local voting. To measure these finer differences we use relaxation time. It is easy to see that if V is a local voting rule and V_ε any of its ε-perturbations, then $Relax(n, \varrho, V_\varepsilon) = \Omega((1/\varepsilon)^2)$. It is possible to deduce from the proofs in [14], [15] that in continuous time, if V_ε is any ε-perturbation of a local voting rule V with noise lower bound ε, then $Relax(n, \varrho, V_\varepsilon) = O((1/\varepsilon)^2)$. Widening the neighborhood will increase the relaxation time to $(1/\varepsilon)^c$ for some $c > 2$ but its logarithm will still be linear in $\log(1/\varepsilon)$. Experiments in [4] let the authors conjecture that for some ε-perturbations L_ε of L, the log relaxation time is $\Omega(1/\varepsilon)$.

Theorem 1.4.3   *There exist $0 < \beta_* < \beta^* < 1$, $\varepsilon_0 > 0$ such that $\forall \beta \in [0, \beta_*) \cup (\beta^*, 1]$, $\forall \varepsilon < \varepsilon_0$,*

$$\log Relax(n, \varrho, K_{\varepsilon,\beta}) = \Theta(\log^2(1/\varepsilon)).$$

Similarly for $Relax(n, \varrho, L_{\varepsilon,\beta})$.

Thus, the log relaxation time grows indeed faster than $\log(1/\varepsilon)$. However, with biased errors, it grows much slower than $1/\varepsilon$.

Remarks 1.4.4

1. We actually prove that the lower bound holds for any $0 < \beta < 1$. The expression of the upper bound is also independent of β but not its proof: after showing that all-black islands of a certain size arise in time $2^{c\log^2(1/\varepsilon)}$ with constant probability, we need to make use of the mixing property (which depends on

β) to be able to say how those islands behave subsequently. We note that the relaxation time result depends on ε being small.

2. The dependence of the relaxation time on n is of auxiliary interest and it will be investigated elsewhere. We believe the dependence is logarithmic but the lower bound argument is yet incomplete.

The rest of the thesis is organized as follows. In the next chapter, we will outline the structure of the proof of the mixing property (Theorem 1.4.2) in the two-line voting case. This is followed by Chapter 3 which gives the proofs of the various components. Chapter 4 proves the lower bound and upper bound of the relaxation time. Chapter 5 gives the analogous results and proofs for the GKL rule, first with respect to mixing followed by the lower and upper bounds on the relaxation time. We conclude with a discussion of our results and future work.

Chapter 2

Proof structure of mixing property: two-line voting

2.1 Coupling

Let P be the operator of a PCA. According to what was said after the introduction of D_n in (1.2.2), the mixing property is equivalent to saying that for any two configurations ξ_1, ξ_2, the expression

$$d_n(P^t \delta_{\xi_1}, P^t \delta_{\xi_2}) \tag{2.1.1}$$

converges to 0 as $t \to \infty$ for all n. If η_i is a random orbit with $\eta_i(\cdot, 0) = \xi_i$ then the distribution of η_i is $P^t \delta_{\xi_i}$ and (2.1.1) compares these two distributions over the interval $[-n, n]$ of sites. A joint distribution for the processes η_1, η_2 will be called a *coupling* of η_1 and η_2. Couplings obviously exist, e.g., we can make the two processes independent. Let

$$D_x'(t) = \sup_{\xi_1, \xi_2} \inf \Pr(\eta_1(x, t) \neq \eta_2(x, t))$$

where the infimum goes over all couplings of orbits η_1 and η_2 with $\eta_i(\cdot, 0) = \xi_i$. It is easy to see that $\lim_{t \to \infty} D_x'(t) = 0$ implies mixing. Intuitively, this condition says that there are random orbits η_i with the initial configurations ξ_i and a joint distribution,

such that with time, the probability of $\eta_1(x,t) = \eta_2(x,t)$ converges to 1. Thus, not only do the distributions become more and more equal but the sample paths of the random orbits become equal, too.

Let us return to the probabilistic cellular automaton $K_{\varepsilon,\beta}$. We will define not just two random orbits but for each initial configuration ξ, we define an orbit η^ξ with $\eta^\xi(\cdot,0) = \xi$, with a joint distribution for all these random orbits simultaneously by a method called *basic coupling* (see, e.g., [15]). First, for all x,t, $j = -1, 1$, we independently toss a coin $E_{x,t,j}$ which is 0 with probability $1 - \varepsilon$ and 1 with probability ε. This is followed by a coin $B_{x,t,j}$ which is 1 with probability $1 - \beta$ and 0 with probability β. Now, for each $\eta = \eta^\xi$ we proceed as follows. Suppose that $\eta(\cdot,t)$ is defined up to t. We will define it for $t + 1$. If $E_{x,t+1,j} = 1$ then the definition uses (1.4.1). Otherwise, $\eta(x, t + 1)(j) = B_{x,t+1,j}$. It is easy to check that this is indeed a coupling and due to the monotonicity of the rule K, the following relation holds: if $\xi_1 \leq \xi_2$ (pointwise) then $\eta^{\xi_1} \leq \eta^{\xi_2}$. Therefore we have

$$\eta^{\zeta_{00}}(x,t) \leq \eta^\xi(x,t) \leq \eta^{\zeta_{11}}(x,t).$$

In other words, all other processes η^ξ are "squeezed" between the processes $\eta^{\zeta_{00}}$ and $\eta^{\zeta_{11}}$ with the constant initial configurations ζ_{00}, ζ_{11}. Put $\eta_0 = \eta^{\zeta_{00}}$, $\eta_1 = \eta^{\zeta_{11}}$. It follows that for mixing, it is sufficient to establish

$$\lim_{t \to \infty} \Pr(\eta_0(x,t) = \eta_1(x,t)) = 1.$$

In words: we will establish that the process starting from all 0's becomes equal, with large probability, to the process starting from all 1's.

Remark 2.1.2   With a slight abuse of notation, we will use η^ξ to denote both the random process proper (i.e., random orbit) as well as its sample path. The interpretation should be clear from the context.

2.2 Spreading of agreement and blackishness

It is not too difficult to prove the following fact (see, e.g., [14]):

Proposition 2.2.1 *If there is a $\delta_1 > 0$ such that $\forall n \in \mathbb{N}$, $\exists t_0 > 0$, $\forall t > t_0$*

$$\Pr(\eta_0(x,t) = \eta_1(x,t), \; -n \leq x \leq n) \; > \; \delta_1, \qquad\qquad (2.2.2)$$

then $K_{\varepsilon,\beta}$ is mixing.

We will prove (2.2.2) in the following way. Let $\mathcal{E}_0(n,t)$ be the event that $\eta_0(x,t) = 11$ for $x \in [-n, n]$, and let $\mathcal{E}_1(n,t) = \bigcup_{t' \leq t} \mathcal{E}_0(n,t')$.

Lemma 2.2.3 *There are $c_0, \delta_2, \varepsilon_0 > 0$, such that $\forall t \geq 0$, $\forall \varepsilon < \varepsilon_0$,*

$$\Pr(\eta_0(x,t) = \eta_1(x,t), \; -t/2 \leq x \leq t/2 \mid \mathcal{E}_1(c_0 \varepsilon^{-1/2}, t)) > \delta_2.$$

This lemma says that if we start from a sufficiently large island of 11 then the (probable) equality of η_0, η_1 will spread with speed $1/2$ in both directions into the cone

$$\{\, (x,t) : -t/2 \leq x \leq t/2, \, t \geq 0 \,\}.$$

The lemma implies the sufficient (and necessary) condition for mixing of Proposition 2.2.1 with $\delta_1 = \delta_2 \Pr(\mathcal{E}_1(c_0 \varepsilon^{-1/2}, T))$ where it is sufficient that T satisfy

$$T \; \geq \; \varepsilon^{-2c_0 \varepsilon^{-1/2}}$$

since $\Pr(\mathcal{E}_1(c_0 \varepsilon^{-1/2}, T)) \nearrow 1$ as $T \to \infty$ and $\mathcal{E}_1(c_0 \varepsilon^{-1/2}, T)$ has constant probability lower bound for $T \geq \varepsilon^{-2c_0 \varepsilon^{-1/2}}$. Since the cone of agreement spreads with speed $1/2$ in both directions, the time lower bound in Proposition 2.2.1 is given by

$$t_0 > T + 2n.$$

Remark 2.2.4   We note that δ_1 need not be constant for Proposition 2.2.1 to hold. It is sufficient that $\delta_1(\varepsilon)$ be a function of ε. However, in Section 4.2 we will show a much tighter bound on T in connection with establishing an upper bound on the relaxation time.

The reason that the two processes become equal is, alas, simple: they both become "largely" 11 (i.e., η_0 becomes largely 11 and by the monotonicity of basic coupling, η_1 has even more 1's). The precise notion of "largely 11" is called k-black. This concept is similar to the ones used in [11], [10] and [12]. The definition will be given later; let us just mention that as k increases the level of "blackishness" decreases. Let us define a *trapezoid* with parameters y, z, u, v, q as the set

$$\{\,(x,t) : u \leq t \leq v, y - q(t - u) < x < z + q(t - u)\,\}.$$

For a trapezoid R with these parameters and $b \geq 0$, let $R(b)$ be the trapezoid with parameters $y + b, z - b, u + b, v, q$. Let R_i be given by $(y_i, z_i, u_i, v_i, q_i)$ for $i = 1, 2$. We will say that (R_1, b_1) is *forward-linked* to (R_2, b_2) if

$$\mathbb{Z} \times [-\infty, u_2 + b_2] \cap R_2 \subset R_1(b_2/2).$$

Lemma 2.2.3 will be implied by the following two lemmas, both of which depend on a sequence b_k and sequence of trapezoids R_k extending into the future which will be defined later for $k = 0, 1, \ldots$ in such a way that

(i) (R_k, b_k) is forward-linked to (R_{k+1}, b_{k+1}),

(ii) $\{\,(x,t) : -t/2 \leq x \leq t/2\,\} \subset \bigcup_k R_k(b_k)$.

Lemma 2.2.5 (agreement) *If η_0, η_1 are k-black on trapezoid R_k, then we have $\eta_1(x,t) = \eta_0(x,t)$ for all $(x,t) \in R_k(b_k)$.*

Lemma 2.2.6 (stacked blackishness) *There are $c_0, \delta > 0$ such that for all $\varepsilon > 0$, $i \in \{0, 1\}$,*

$$\Pr\left(\bigwedge_{k=0}^{\infty} \eta_i \restriction R_k \text{ is } k\text{-black} \,\middle|\, \mathcal{E}_0(c_0 \varepsilon^{-1/2}, 0)\right) > \delta_2.$$
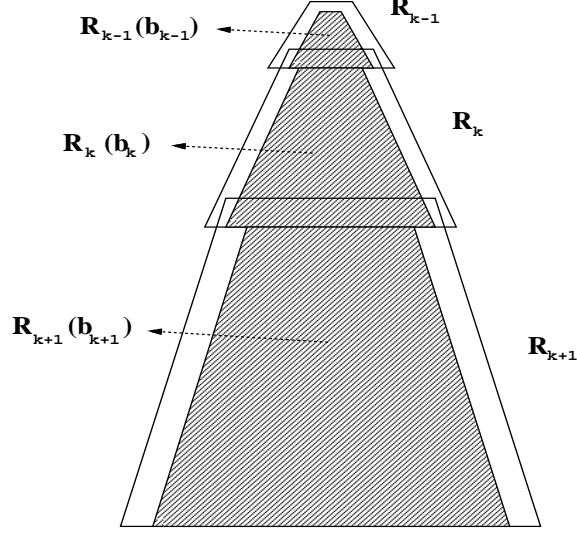
Figure 2.2.1: Forward-linked trapezoids R_{k-1}, R_k, R_{k+1} in space-time. The enclosed, shaded trapezoids represent regions of agreement $R_{k-1}(b_{k-1})$, $R_k(b_k)$, $R_{k+1}(b_{k+1})$.

In words, R_k forms a sequence of trapezoids in which blackishness prevails at level k, whereas the sequence $R_k(b_k) \subset R_k$ is a further restriction on which the coupled processes η_0, η_1 agree. Figure 2.2.1 shows trapezoids R_{k-1}, R_k, R_{k+1} forward-linked in space-time, with their enclosed trapezoids $R_{k-1}(b_{k-1})$, $R_k(b_k)$, $R_{k+1}(b_{k+1})$ shown shaded. Figure 2.2.1 is only a schematic depiction and is not drawn to scale. Forward-linkedness assures that the space-time region of agreement expands without interruption, and it also provides an overlap needed in the proof of Lemma 2.2.6.

Lemma 2.2.5 is proved by induction in Subsection 3.2. Lemma 2.2.6 will be proved by showing how blackishness is propagated from R_k to R_{k+1}. For this, a sequence $S_k \supset R_k$ of space-time squares and a property of the error pattern called "k-sparsity with black-bias" is used which is defined later. Intuitively, k-sparsity (with black-bias) means that there are many errors introducing 1's and few errors introducing 0's. The degree of the bias is determined by the parameter β; however, β does not contribute to the *definition* of sparsity which is purely a combinatorial property. The strictness of this condition is governed by the parameter k which becomes weaker with
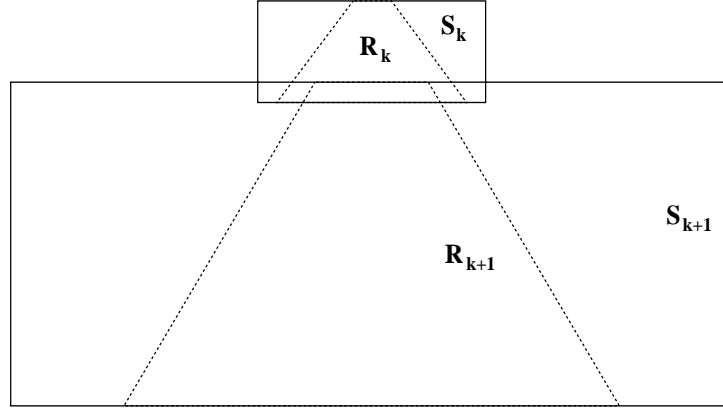
Figure 2.2.2: k-sparse and $(k + 1)$-sparse space-time squares S_k, S_{k+1} enclosing forward-linked k-black and $(k + 1)$-black trapezoids R_k, R_{k+1}.

the increase in k. k-sparsity of S_k will allow us to conclude that $\eta_i \upharpoonright R_k$ is k-black, $i = 0, 1$.

Remarks 2.2.7

1. When we say that a space-time set A is k-sparse, we mean that it is k-sparse with respect to the error process defined in Section 3.1.

2. The symbol "$\upharpoonright$" denotes *restriction*. That is, given a function $f : A \to B$ and $C \subseteq A$, $f \upharpoonright C$ is the partial function taking on the same values as f but being defined only on C.

The following two lemmas will finish the proof.

Lemma 2.2.8 (stacked sparsity)   *There exists $\beta < 1/2$ such that for all $\varepsilon > 0$,*

$$\Pr\left(\bigwedge_{k=0}^{\infty} S_k \text{ is k-sparse with black-bias } \beta \right) > \delta_2.$$

Lemma 2.2.9 (inheritance)

(a) *There is a c_0 such that if $\mathcal{E}_0(c_0\varepsilon^{-1/2}, 0)$ holds and S_0 is 0-sparse then $\eta_i \upharpoonright R_0$ is 0-black, $i = 0, 1$.*

(b) *For all $k \in \mathbb{N}$, $i \in \{0, 1\}$, if $\eta_i \upharpoonright R_k$ is k-black and S_{k+1} is $(k+1)$-sparse then $\eta_i \upharpoonright R_{k+1}$ is $(k+1)$-black.*

Lemma 2.2.8 is proved by upper-bounding the probability that S_k is not sparse. It is a consequence of Lemma 3.1.2 proved in Subsection 3.1. The proof of Lemma 2.2.9 is the main technical task. Part (a) will be implied by Lemma 3.2.4 and part (b) is proved in Subsection 3.2 by reducing it to Lemma 3.2.2.

Chapter 3

Sparsity and blackishness

3.1 Sparsity

Let us look at the independent coin tosses $E_{x,t,j}, B_{x,t,j}$ defined in Subsection 2.1 generating the random orbits η^ξ. We will say there is an *error at* (x,t) if $E_{x,t,j} = 1$ for some j. It is a *bad error* at j if $E_{x,t,j} = 1 \wedge B_{x,t,j} = 0$, and a *good error* at j if $E_{x,t,j} = B_{x,t,j} = 1$. That is, a bad error sets its bit to 0, while a good error for j sets its bit to 1. Note that an error does not necessarily result in a state that is different from the one dictated by the deterministic cellular automaton rule. We will say that the set of errors is "sparse" (with black-bias) if the bad errors are few and the good errors are plentiful. More importantly, space-time regions where this condition is violated must be "well-separated," and this property must be preserved in a scale-invariant way. Let c_1, c, d be positive constants to be defined later. Let

$$
\begin{aligned}
W_\ell &= [0,\ell) \times [0,\ell), \\
r_k &= c_1 \varepsilon^{-1/2} c^k (k!)^2,
\end{aligned}
$$

where $k = 0, 1, 2, \ldots$ and $\ell > 0$.

Definition 3.1.1 (sparsity) The error process $(E_{x,t,j}, B_{x,t,j})_{x,t,j}$ is 0-*sparse* with black-bias over the space-time set A if there are no bad errors in A and for all squares

B of the form $(s,t) + W_{r_0}$, if $B \subset A$, then there exists a good error site for some $j \in \{-1, 1\}$ in B.

For $k \geq 1$, the error process is *k-sparse* with black-bias over A if for all squares B of the form $(s,t) + W_{r_k}$ there exists (s', t') such that $A \cap B \setminus (s', t') + W_{3r_{k-1}}$ is $(k-1)$-sparse. We will call the error process *strictly k-sparse* if it is k-sparse but not $(k-1)$-sparse.

Lemma 3.1.2 (sparsity) $\forall c > 1, \exists C_1 > 0, 0 < \beta_0 < 1/2$ *such that* $\forall 0 < \beta < \beta_0,$ $0 < \varepsilon \leq 1, c_1 > C_1, k \in \mathbb{N}$, *the following holds. Let* η *denote an* ε*-perturbation of* K, $K_{\varepsilon,\beta}$. *Let* $A = \bigcup_{i \in [1,N]} (a_i, b_i) + W_{r_k}$, $a_i, b_i \in \mathbb{Z}$, *where* $\eta \restriction A$ *is defined. Let* q_k *be the probability that* η *is not* k*-sparse on* A. *Then*

$$q_k < N\gamma^{2^{k-1} + (k+1)/2}$$

where $0 < \gamma < 1$ *is a constant depending only on* c.

Proof. The proof goes by induction on k. Let $k = 0$. For any window $W^i = (a_i, b_i) + W_{r_0}$, consider the probability that η is not 0-sparse on W^i. Since $r_0 = c_1 \varepsilon^{-1/2}$,

$$\Pr(\eta \text{ is not 0-sparse on } W^i) \leq 2(c_1 \varepsilon^{-1/2})^2 \varepsilon\beta + (1 - \varepsilon(1 - \beta))^{2(c_1 \varepsilon^{-1/2})^2}$$

$$\leq 2c_1^2\beta + e^{-2c_1^2(1-\beta)} < \gamma,$$

where the last inequality holds for c_1 sufficiently large and β sufficiently small. Hence $q_0 < N\gamma$ which equals the probability bound with $k = 0$.

Assume the relation holds for $k > 0$. For $i, j \in \{0, 1\}$, define partition $\mathcal{P}_{i,j}$ of A as follows:

$$\mathcal{P}_{i,j} = \{ A \cap r_{k+1}(2s + i, 2t + j) + W_{2r_{k+1}} : s, t \in \mathbb{Z} \}.$$

Each $(a_l, b_l) + W_{r_{k+1}}$ is intersected by at most four elements of $\mathcal{P}_{i,j}$, hence $|\mathcal{P}| \leq 16N$ where $\mathcal{P} = \bigcup_{i,j} \mathcal{P}_{i,j}$. Suppose there exists $B = (a, b) + W_{r_{k+1}}$ such that η is not $(k+1)$-sparse on $A \cap B$, $A \cap B \neq \emptyset$. Since $A \cap B \subset V$ for some $V \in \mathcal{P}$, η is not

$(k + 1)$-sparse on V. Thus,

$$q_{k+1} \leq 16Nq' \tag{3.1.3}$$

where $q' = \Pr(\eta$ is not $(k + 1)$-sparse on $V)$.

Partition V as before but with r_k in place of r_{k+1}. Denote the four partitions by $\mathcal{R}_{i,j}$, $i, j \in \{0, 1\}$, and let $\mathcal{R} = \bigcup_{i,j} \mathcal{R}_{i,j}$. Since $r_{k+1}/r_k = c(k + 1)^2$, $|\mathcal{R}_{i,j}| \leq (c(k + 1)^2 + 1)^2$. Consider the event $\mathcal{E}$: there exist $U, U' \in \mathcal{R}$, $U \cap U' = \emptyset$, such that η is not k-sparse on U and U'.

Claim $q' \leq \Pr(\mathcal{E})$.

Proof: We will prove the contrapositive: $\neg\mathcal{E} \implies \eta$ is $(k+1)$-sparse on V. Suppose $\neg\mathcal{E}$. That is, if $U, U' \in \mathcal{R}$ and η is not k-sparse on U and U', then $U \cap U' \neq \emptyset$. In particular, if we fix U, then all elements of $\mathcal{R}$ on which η is not k-sparse must intersect with U as well as with each other. By definition of $\mathcal{R}_{i,j}$, $\forall U, U' \in \mathcal{R}$ with $U \cap U' \neq \emptyset$, there exists $(a, b) \in \mathbb{Z}^2$ such that $U, U' \subset (a, b) + W_{3r_k}$. $(a, b) + W_{3r_k}$ covers all elements of $\mathcal{R}$ on which η is not k-sparse, and by containment, any r_k-window in V on which η is not k-sparse. Hence η is k-sparse on $V \setminus (a, b) + W_{3r_k}$ implying η is $(k + 1)$-sparse on V. $\blacktriangle$

By independence, $\Pr(\eta$ is not k-sparse on U and $U') \leq q''^2$ where $U, U' \in \mathcal{R}$, $U \cap U' = \emptyset$, and $q'' = \Pr(\eta$ is not k-sparse on $A \cap (s, t) + W_{2r_k})$. Since the total number of disjoint pairs in $\mathcal{R}$ is strictly less than $|\mathcal{R}|^2$,

$$q' \leq \Pr(\mathcal{E}) < 16(c(k + 1)^2 + 1)^4 q''^2.$$

Using (3.1.3) and the inductive hypothesis on q'',

$$\begin{aligned} q_{k+1} &\leq 16Nq' < 16^2 N(c(k + 1)^2 + 1)^4 4^2 q_k^2 \\ &\leq 16^2 N(c(k + 1)^2 + 1)^4 \gamma^{2(2^{k-1} + (k+1)/2)} \\ &= 16^3(c(k + 1)^2 + 1)^4 \gamma^{k/2} N \gamma^{2^k + (k+2)/2}. \end{aligned}$$

Clearly, for γ sufficiently small, $16^3(c(k+1)^2+1)^4\gamma^{k/2} < 1$, $k \geq 1$, which completes the proof. ∎

Let us define the sequences R_k, b_k, and S_k for $k \in \mathbb{N}$. First, $b_k = dr_k$ where $d = 100$ is a constant used in the definition of k-black in Section 3.2. The trapezoid $R_k = (y_k, z_k, u_k, v_k, q_k)$ is given by

$$z_k - y_k = (50 + 2d)r_k,$$

$$v_k - u_k = (80 + 3d)\,r_{k+1},$$

$$q_k = 1/2 + 1/(k+2.5).$$

Based on where R_0 is located in space-time, R_k is placed such that it is forward-linked to R_{k-1} for $k \geq 1$. In particular, for $k \geq 0$, if $y_k = -(25 + d)r_k$ and $z_k = (25 + d)r_k - 1$ (i.e., centered around site 0), then $y_{k+1} = -(25 + d)r_{k+1}$, $z_{k+1} = (25 + d)r_{k+1} - 1$, and $u_{k+1} = v_k - b_{k+1}$.

Given R_k, S_k is the space-time rectangle of size

$$(400 + 15d)r_{k+1} \times (80 + 3d)r_{k+1}$$

such that $R_k \subset S_k$ is centered within S_k (see Figure 2.2.2). Since $r_{k+1}/r_k = c(k+1)^2$, S_k can be expressed as the disjoint union of $(400+15d)(80+3d)c^2(k+1)^4$ space-time windows W_{r_k}.

Proof of Lemma 2.2.8. We will show that Lemma 3.1.2 $\implies$ Lemma 2.2.8. To lower-bound $\Pr(\bigwedge_{k=0}^{\infty} S_k$ is k-sparse with bias $\beta)$, let us upper-bound the probability of its

complement event $\Pr(\exists k : S_k$ is not k-sparse with bias $\beta)$.

$$
\begin{aligned}
\Pr(\exists k :\ S_k \text{ is not } k\text{-sparse with bias } \beta) \ &\leq\ \sum_{k=0}^{\infty} \Pr(S_k \text{ is not } k\text{-sparse with bias } \beta) \\
&<\ \sum_{k=0}^{\infty} N_0(k+1)^4 \gamma^{2^{k-1}+k/2+1/2} \\
&=\ \sum_{k=0}^{\infty} N_0 \gamma^{2^{k-1}+k/2+1/2-\alpha\log(k+1)} \\
&<\ \frac{N_0\gamma}{1-\gamma}\ <\ 1-\delta_2
\end{aligned}
$$

where $\alpha = -4/\log\gamma$, $N_0 = (400+15d)(80+3d)c^2$, and we have used Lemma 3.1.2 with $N = N_0(k+1)^4$ for each $k \geq 0$. For γ sufficiently small, $\alpha < 1/2$, and the last two inequalities hold. ∎

3.2 Blackishness

Whereas sparsity describes the combinatorial structure of errors occurring in a sample path of an ε-perturbation, blackishness is a property of the sample path capturing the fact that sparse errors are "corrected" locally in space-time, preserving the blackishness property.

Definition 3.2.1 (blackishness)   A space-time configuration η is 0-*black* over $A \subseteq \mathbb{Z} \times \mathbb{N}$ if $\eta(x,t) = 11$ for all $(x,t) \in A$. For $k \geq 1$, η is k-*black* over A if for all squares B of the form $(s,t) + W_{r_k - 2dr_{k-1}}$ there exists (s',t') such that η is $(k-1)$-black over $A \cap B \setminus (s',t') + W_{dr_{k-1}}$. We will call $\eta \restriction A$ *strictly* k-black if $\eta \restriction A$ is k-black but not $(k-1)$-black.

Notice that a slightly smaller window size $r_k - 2dr_{k-1}$ is used for separating non-k-black squares in space-time. In essence, we will show that the effects of $3r_{k-1}$-size bad errors are corrected and contained within a space-time window of size dr_{k-1}.
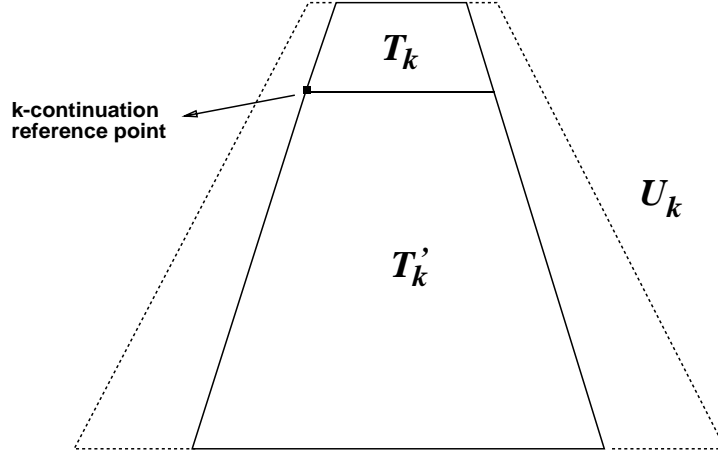
Figure 3.2.1: Depiction of Expansion Lemma with trapezoids $\mathcal{T}_k$, $\mathcal{T}'_k$, U_k and k-continuation reference point.

Lemma 3.2.2 (expansion)   *Let $\beta_k = 1/2 + 1/(k+2.5)$. Let $\mathcal{T}_k = (0, w_k, 0, h_k, \beta_k)$ be a trapezoid where $w_0 \geq 50r_0$, $h_0 = 280r_0$, and $w_k \geq r_k$, $h_k = 3r_k$ for $k > 0$. Let*

$$\mathcal{T}'_k = (-\beta_k h_k, w_k + \beta_k h_k, h_k, h_k + h, \beta_k)$$

where $h \geq 0$. Let $U_k = (-2r_k, w_k + 2r_k, 0, h_k + h, 1)$. Then, $\forall \xi \in S^{\mathbb{Z}}$,

$$\eta^\xi \upharpoonright \mathcal{T}_k \text{ is } k\text{-black} \ \wedge \ U_k \text{ is } k\text{-sparse} \quad \Longrightarrow \quad \eta^\xi \upharpoonright (\mathcal{T}_k \cup \mathcal{T}'_k) \text{ is } k\text{-black}.$$

The Expansion Lemma is the main technical lemma. It states that k-sparse errors are not able to impede the expansion of a sufficiently large k-black region. We will say that η^ξ is a *k-continuation* at $(-\beta_k h_k, h_k)$ with *width* $w_k + 2\beta_k h_k$ and *extension* h. Notice that this completely determines $\mathcal{T}_k$, $\mathcal{T}'_k$, and U_k. We will call $\mathcal{T}_k$ the *context* of the k-continuation. A pictorial depiction is given in Figure 3.2.1.

The proof of the Expansion Lemma goes by induction on k. First, we will prove a simple fact which will be referred to frequently in later proofs.

Proposition 3.2.3 (speed-of-light)   *Let $\xi(x) = \xi'(x)$ for $0 \leq x < \ell$ where $\ell > 0$, $\xi, \xi' \in S^{\mathbb{Z}}$. Then*

$$\eta^\xi(s, t) = \eta^{\xi'}(s, t), \qquad (s, t) \in A,$$

where $A = \{\,(s,t) : 2t \leq s < \ell - 2t,\, t > 0\,\}$.

Proof. Note that by coupling $\eta^\xi, \eta^{\xi'}$ share the same errors. Let $A_t = \{\,(s,t) : 2t \leq s < \ell - 2t\,\}$, $t > 0$. Clearly,

$$A_{t+1} \subseteq A_t$$

with $A_t = A_{t+1}$ iff $A_t = \emptyset$. Let t_0 be the minimum t such that $A_t = \emptyset$. The proof goes by induction on t. Assume $t = 1$. If $A_1 = \emptyset$ we are done. For all $(s,1) \in A_1$,

$$\eta^\xi(x,0) = \eta^{\xi'}(x,0) \qquad \text{for } (x,0) \in \mathcal{N}(s,1),$$

where $\mathcal{N}(s,1) = \{\,(s-2,0),(s-1,0),(s,0),(s+1,0),(s+2,0)\,\}$. Hence, $\eta^\xi(s,1) = \eta^{\xi'}(s,1)$. Assume the statement holds for $1 < t < t_0 - 1$. By the same argument, $\eta^\xi, \eta^{\xi'}$ must agree on A_{t+1}. ∎

The proposition states that a signal cannot travel faster than the neighborhood size, i.e., "speed-of-light" (SOL) which is 2 in the case of two-line voting and 3 in the case of the GKL rule.

Lemma 3.2.4 (bootstrap) *Let φ be a sample path of $K_{\varepsilon,\beta}$ such that $\varphi(s,0) = 11$ for $s \in [0,\ell)$ where $\ell \geq 50r_0$. Let the trapezoid $(-2r_0, \ell + 2r_0, 0, h, 1)$ be 0-sparse. Then $\varphi \restriction \mathcal{K}$ is 0-black where*

$$\mathcal{K} = \{\,(s,t) : 14r_0 - t \leq s < \ell - 14r_0 + t,\, 0 \leq t < h\,\}.$$

The Bootstrap Lemma says that a black island of sufficient size expands with speed 1 in both directions when subject to 0-sparse errors. We will prove Lemma 2.2.9 using the Expansion Lemma and the Bootstrap Lemma.

Proof of Lemma 2.2.9. Lemma 3.2.4 $\implies$ Lemma 2.2.9 (a). Consider a sample path φ of $K_{\varepsilon,\beta}$ with $\mathcal{E}_0(c_0\varepsilon^{-1/2}, 0)$ where $c_0 = (80 + 2d)c_1$. Let $\mathcal{K}(\ell)$ denote the trapezoid in Lemma 3.2.4 with width ℓ, height $h = (80 + 3d)r_1$, centered at 0. Since

$R_0 = (y_0, z_0, u_0, v_0, 0.9)$ where

$$z_0 - y_0 = (50 + 2d)\, c_1 \varepsilon^{-1/2},$$

$$v_0 - u_0 = (80 + 3d)\, r_1,$$

$R_0 \subset \mathcal{K}(c_0 \varepsilon^{-1/2})$. S_0 is 0-sparse and its width $(400 + 15d)r_1$ satisfies the width requirement in the supposition of Lemma 3.2.4:

$$(50 + 2d)r_0 + 4(80 + 3d)r_1 < (400 + 15d)r_1.$$

Hence, $\varphi \upharpoonright \mathcal{K}(c_0 \varepsilon^{-1/2})$ is 0-black which by containment implies $\varphi \upharpoonright R_0$ is 0-black.

Lemma 3.2.2 $\implies$ Lemma 2.2.9 (b). Since R_k being k-black trivially implies it is $(k+1)$-black, the implication holds if (R_k, b_k) being forward-linked to (R_{k+1}, b_{k+1}) satisfies the supposition of Lemma 3.2.2. Let $k = 0$. Part (a) has shown that $\varphi \upharpoonright R_0$ is 0-black. By the definition of forward-linkedness, it is easily checked that φ is a 1-continuation at (y_1, u_1) with width $z_1 - y_1$ and extension $v_1 - u_1$. Hence, $\varphi \upharpoonright R_1$ is 1-black. Since $v_1 - u_1 = (80 + 3d)r_2$ and the expansion factor is at least $1/2$ in both directions, the width of R_1 at time v_1 is at least $(80 + 3d)r_2$. The previous argument applies to any $k > 0$ which carries the induction step. ∎

3.3 Spreading of blackishness

Proposition 3.3.1 *Let φ be an orbit of K with initial condition $\varphi(0,0) = 11$, $\varphi(s,0) = 01$, for $s \neq 0$. Let $\theta^l, \theta^r : \mathbb{N} \to \mathbb{Z}$ denote the endpoint processes of the maximum interval $[\theta^l(t), \theta^r(t)]$ such that*

$$\varphi(s,t) = \begin{cases} 11 & \text{if } s \in [\theta^l(t), \theta^r(t)], \\ 01 & \text{otherwise.} \end{cases}$$

Then, $\theta^l(t) = -2t$ and $\theta^r(t) = -t$.

Proof. Let $t = 0$. Clearly, $\theta^l(0) = \theta^r(0) = 0$. Assume the relations hold for $t > 0$. Consider the space-time points $(-2t - 2, t + 1)$, $(-2t - 1, t + 1)$, and $(-t, t + 1)$. By the

action of K, $\varphi(-2t-2, t+1) = \varphi(-2t-1, t+1) = 11$, and $\varphi(-t, t+1) = 01$. All other sites remain unchanged. Hence, $\theta^l(t+1) = -2(t+1)$ and $\theta^r(t+1) = -(t+1)$. ∎

We will refer to this deterministic expansion process as a *left-moving black cone*. By symmetry, if 01 is replaced by 10, the statement holds with $\theta^l(t) = t$, $\theta^r(t) = 2t$. We will call this a *right-moving black cone*. It may help to think of all-01 or all-10 space configurations as being "unstable" in the sense that even a single good (bad) error will give rise to a left (right)-moving black (white) island traveling with speed 2 at the front and trailing with speed 1 in the back.

Proof of Lemma 3.2.4.   We need to show that $(a, b) \in \mathcal{K} \implies \varphi(a, b) = 11$. First, define four boundary processes $\theta_1^l, \theta_2^l, \theta_1^r, \theta_2^r : \mathbb{N} \to \mathbb{Z} \cup \{\infty\}$ as follows:

$$\theta_1^l(t) = \max\{\, s \in \mathbb{Z} \,:\, \varphi(i, t) = 00,\ i \le s \,\},$$

$$\theta_1^r(t) = \min\{\, s \in \mathbb{Z} \,:\, \varphi(i, t) = 00,\ i \ge s \,\},$$

and $[\theta_2^l(t), \theta_2^r(t)]$ is the maximum interval containing the point $(\ell/2, t)$ such that

$$\varphi(s, t) = 11 \qquad \text{for } s \in [\theta_2^l(t), \theta_2^r(t)].$$

Clearly, $\theta_1^l(0) \le 1$, $\theta_2^l(0) \le 0$, $\theta_2^r(0) \ge \ell - 1$, and $\theta_1^r(0) \ge \ell$. If no errors occur, it can be easily checked by induction

$$\theta_1^l(t) \le -1 - t, \quad \theta_2^l(t) \le t,$$

$$\theta_2^r(t) \ge \ell - 1 - t, \quad \theta_1^r(t) \ge \ell + t,$$

for $t < \ell/2$. Moreover, $\varphi(s, t) \in \{10, 11\}$ for $\theta_1^l(t) < s < \theta_2^l(t)$, and $\varphi(s, t) \in \{01, 11\}$ if $\theta_2^r(t) < s < \theta_1^r(t)$.

Let $(a, b) \in \mathcal{K}$. Without loss of generality (by symmetry), we may consider only those points with $a \ge 0$. If $a \in [\theta_2^l(t), \theta_2^r(t)]$, then $\varphi(a, b) = 11$ and we are done. Let us define a cone T emanating from (a, b) and going in backwards in time:

$$T = \{\, (s, t) \,:\, -(t - b) + a \le s \le -2(t - b) + a \,\}.$$

If $\varphi(s,t) \in \{01, 11\}$ for all $(s,t) \in T$ and $\varphi(s,t) = 11$ for at least one $(s,t) \in T$, then by Proposition 3.3.1, this gives rise to at least one left-moving black cone emanating from T, call it C, and $(a,b) \in C$. Let us consider $T' \subset T$ given by

$$T' = \{\, (s,t) \in T \,:\, s < \theta_1^r(t) \,\}.$$

It follows that $\varphi(s,t) \in \{01, 11\}$ for all $(s,t) \in T'$, and it suffices to show that $\varphi(s,t) = 11$ for some $(s,t) \in T'$.

Claim There exists (x,y) such that $(x,y) + W_{r_0} \subset T'$.

Proof: Let (s^*, t^*) be the intersection point of the two lines

$$s = -(t-b) + a, \tag{3.3.2}$$

$$s = t + \ell, \tag{3.3.3}$$

and let (s_*, t_*) be the intersection point of (3.3.3) and

$$s = -2(t-b) + a. \tag{3.3.4}$$

A straightforward calculation yields $(s^*, t^*) = ((a+b+\ell)/2, (a+b-\ell)/2)$ and $(s_*, t_*) = ((a+2b+2\ell)/3, (a+2b-\ell)/3)$. The Euclidean distance between (s_*, t_*) and (s^*, t^*) is $d = |b+\ell-a|/3\sqrt{2}$. Since (3.3.2) and (3.3.3) are perpendicular and the slope of (3.3.4) is -1, a simple argument shows that for a $r_0 \times r_0$ square to fit into T', it suffices that $d \geq 2r_0$. This yields the condition

$$6\sqrt{2}r_0 \leq |b+\ell-a|.$$

Clearly, this is satisfied for $(s,t) \in \mathcal{K}$ since $(s,t) \in \mathcal{K}$ implies $14r_0 < t + \ell - s$. ▲

Since $\varphi \restriction (x,y) + W_{r_0}$ is 0-sparse, there is at least one good error in $(x,y) + W_{r_0}$ which completes the proof. ∎

Before proceeding with the proof of the main lemma, we will prove the Error-Correction Lemma which states that a white island sandwiched between two sufficiently large blackish islands is erased within a space-time square of a certain size in
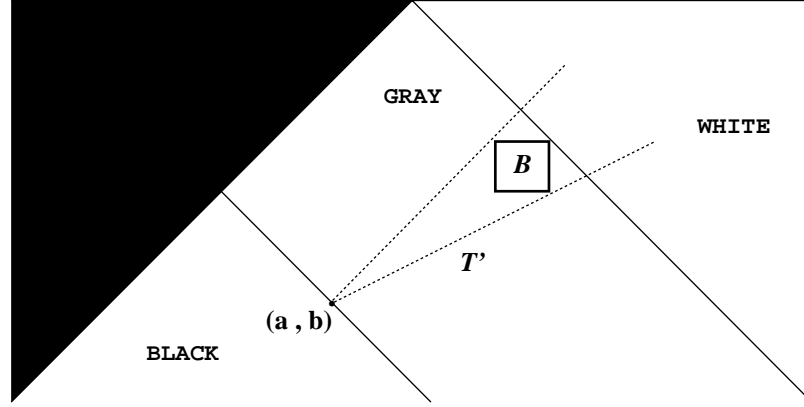
Figure 3.3.1: The backward 01/11-cone T' emanating from (a, b) and a 0-sparse $r_0 \times r_0$ space-time square B contained in T'.

the presence of k-sparse errors.

Lemma 3.3.5 (error-correction)   *Let φ be a k-continuation at $(0,0)$ with width ℓ and extension h, and let φ be a k-continuation at $(a,0)$ with width ℓ' and extension h. Let $a > \ell$ and $h = a - \ell + dr_k$. Let*

$$\mathcal{L} = \{ (s,t) : -\beta_k t < s < \beta_k t + a + \ell', \ 0 \le t < h \},$$

$$B = (\ell - r_k, 0) + W_{a-\ell+2r_k}.$$

Then $\varphi \restriction \mathcal{L} \setminus B$ is k-black.

Fact 3.3.6   *For each $k \ge 0$, Lemma 3.2.2 $\Longrightarrow$ Lemma 3.3.5.*

Proof.   Let

$$\mathcal{K} = \{(s,t) : -\beta_k t < s < \beta_k t + \ell, \ 0 \le t < h\},$$

$$\mathcal{K}' = \{(s,t) : -\beta_k t + a < s < \beta_k t + a + \ell, \ 0 \le t < h\}.$$

By Lemma 3.2.2, φ is k-black on $\mathcal{K}$ and $\mathcal{K}'$.

Figure 3.3.2: Error-correction Lemma in action: error region sandwiched between the k-black trapezoids K, K' is eaten up. $\varphi \upharpoonright (K \cup K') \setminus B$ is k-black.

Let (x, y) be the intersection point of the two lines
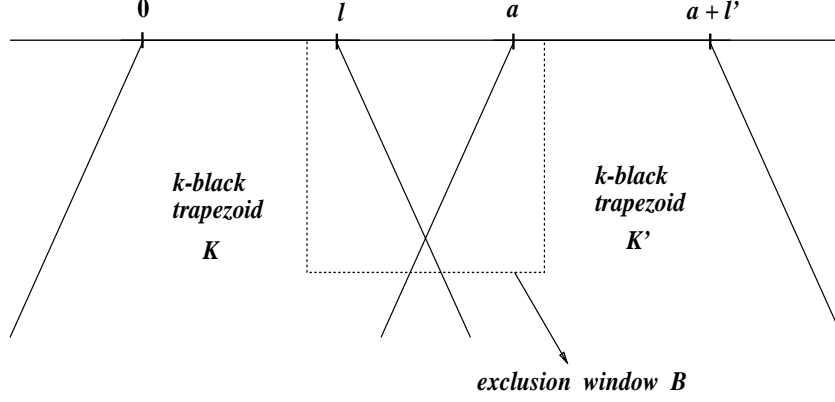
$$
\begin{aligned}
s &= t/2 + \ell, \\
s &= -t/2 + a.
\end{aligned}
$$

By simple calculation, $(x, y) = ((\ell + a)/2, a - \ell)$. Since

$$
\{ (s, t) : t/2 + \ell < s < -t/2 + a \} \subset B
$$

for $t \geq 0$, and

$$
(s, t) + W_{r_k} \subset \mathcal{L} \setminus B \quad \Longrightarrow \quad (s, t) + W_{r_k} \subset \mathcal{K} \ \vee \ (s, t) + W_{r_k} \subset \mathcal{K}', \qquad (3.3.7)
$$

noting that $\beta_k > 1/2$, $\varphi \upharpoonright \mathcal{L} \setminus B$ is k-black. Implication (3.3.7) holds since the size of B was chosen $2r_k$ larger than necessary to cover the correction process. ∎

Figure 3.3.2 is a depiction (not drawn to scale) of the error-correction process facilitated by Lemma 3.3.5. We will be particularly interested in the case $a - \ell \leq 15r_k$ which accounts for the maximum spreading effect of a $3r_k \times 3r_k$ error window under the speed-of-light 2 given by $6r_k + 3r_k + 6r_k$. The exclusion window that covers the error effect is contained in $(\ell - 2r_k, 0) + W_{17r_k}$. To cover the k-sparse error itself, we

may use a window of size $20r_k \times 20r_k$.

Remarks 3.3.8

1. Lemma 3.3.5, although implied by Lemma 3.2.2 for each level k, is itself used in the proof of the latter. Hence, Lemma 3.3.5 is proven conjointly with Lemma 3.2.2 in the induction.

2. In the proof of Fact 3.3.6 we have used a window size r_k in showing the k-black property even though the definition of blackishness requires only a window size of $r_k - 2dr_{k-1}$. The stronger property is proved mainly out of convenience. The definition of k-black, however, does require the smaller window size $r_k - 2dr_{k-1}$.

Definition 3.3.9 (cover) Let $k \geq 1$. Let $\varphi : A \to S$ be k-black where $A = \bigcup_{i \in [1,N]} (s_i, t_i) + W_{r_k}$, $(s_i, t_i) \in \mathbb{Z} \times \mathbb{N}$. $\mathcal{C} = \{ (a_i, b_i) + W_{dr_{k-1}} : i \in [1, n] \}$ is a k-*cover* of φ if

$$\varphi \upharpoonright (A \setminus \bigcup_{B \in \mathcal{C}} B)$$

is $(k-1)$-black. $\mathcal{C}$ is *minimal* if whenever $\mathcal{C}'$ is a k-cover, this implies $|\mathcal{C}| \leq |\mathcal{C}'|$.

In the proof of Lemma 3.2.2 we will use coverings of k-sparse errors which are understood similarly with cover elements now being space-time squares of size $3r_{k-1} \times 3r_{k-1}$. To distinguish between the two notions, we will call them k-*black-cover* and k-*sparse-cover*, respectively, when both are used at the same time.

Remark 3.3.10 We will use the term "well-separatedness" frequently in subsequent proofs. The meaning of the term will be two-fold depending on the context. First, if *well-separatedness* is used in the context of k-sparse errors, it will mean that we are choosing the constant c sufficiently large such that $3r_{k-1} \times 3r_{k-1}$ error windows are located far from each other in space-time in the Euclidean distance sense ($c \gg 3$). This implication holds since $r_k/r_{k-1} = ck$. Second, if the term is used in the context

of k-black regions and their minimal coverings, it will mean that c is chosen large enough relative to d ($c \gg d$) so that the elements of a minimal k-cover are remotely located from each other in space-time.

Proposition 3.3.11 *Let $\varphi : A \to S$ be k-black where $A \subset \mathbb{Z} \times \mathbb{N}$ is bounded. Let $\mathcal{C}$ be a minimal k-cover of A. Then $\forall (x, y) \in \mathbb{Z} \times \mathbb{N}$ and $\forall B, B' \in \mathcal{C}$ with $B \neq B'$,*

$$(x, y) + W_{r_k - 6dr_{k-1}} \cap B \neq \emptyset \quad \implies \quad (x, y) + W_{r_k - 6dr_{k-1}} \cap B' = \emptyset.$$

Proof. Suppose for some $(x, y) \in \mathbb{Z} \times \mathbb{N}$ and $B, B' \in \mathcal{C}$, $B \neq B'$,

$$(x, y) + W_{r_k - 6dr_{k-1}} \cap B \neq \emptyset \quad \wedge \quad (x, y) + W_{r_k - 6dr_{k-1}} \cap B' \neq \emptyset.$$

It follows that $\exists (x', y') \in \mathbb{Z} \times \mathbb{N}$ such that $B, B' \subseteq (x', y') + W_{r_k - 4dr_{k-1}}$. Let us consider its dilation $D = (x' - dr_{k-1}, y' - dr_{k-1}) + W_{r_k - 2dr_{k-1}}$. By assumption of $\varphi \upharpoonright A$ being k-black, there exists $(a, b) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\varphi \upharpoonright A \cap D \setminus (a, b) + W_{dr_{k-1}} \quad \text{is } (k-1)\text{-black.}$$

Let $G = (a, b) + W_{dr_{k-1}}$. We will show that $\mathcal{C} \cup \{G\} \setminus \{B, B'\}$ is a k-cover of $\varphi \upharpoonright A$ which contradicts the minimality assumption of $\mathcal{C}$.

We need to show that for every $(r_{k-1} - 2dr_{k-2})$-window H,

$$\varphi \upharpoonright H \cap A \setminus \left(\bigcup_{U \in \mathcal{C} \setminus \{B, B'\}} U \ \cup \ G \right)$$

is $(k-1)$-black. If H does not intersect B, B' then we are done since, by assumption, $\mathcal{C}$ is a k-cover of A. If $H \cap (B \cup B') \neq \emptyset$ then $H \subset D$. But we know that $\varphi \upharpoonright A \cap D \setminus G$ is $(k-1)$-black. Hence, $\varphi \upharpoonright H \cap A \setminus G$ is $(k-1)$-black. ∎

Thus Proposition 3.3.11 shows that elements of a minimal k-cover are well-separated in space-time. A similar result holds for minimal k-sparse-covers.

Proof of Lemma 3.2.2.   The proof goes by induction on k. Let $k = 0$. First, Lemma 3.2.4 is applied to the all-black space interval $[0, w_0)$ at time $t = 0$ to yield

the all-black space interval $[-h_0 + 14r_0, w_0 + h_0 - 14r_0)$ at $t = h_0$. Since $\mathcal{T}_0$ occupies $[-\beta_0 h_0, w_0 + \beta_0 h_0)$ at $t = h_0$ and $\beta_0 = 0.9$, $h_0 = 280r_0$, it follows that $[-h_0 + 14r_0, w_0 + h_0 - 14r_0)$ is strictly larger than $[-\beta_0 h_0, w_0 + \beta_0 h_0)$ by $14r_0$ on both sides. Hence Lemma 3.2.4 can be applied again to the space interval $[-h_0 + 14r_0, w_0 + h_0 - 14r_0)$ at $t = h_0$ to conclude that η^ξ is 0-black on the trapezoid $(-h_0 + 28r_0, w_0 + h_0 - 28r_0, h_0, h_0 + h, 1)$. Since

$$\mathcal{T}_0' \subset (-h_0 + 28r_0, w_0 + h_0 - 28r_0, h_0, h_0 + h, 1),$$

the basis is proven.

Assume the statement holds for $k \geq 0$. Let $\mathcal{C}_b$ be a minimum $(k+1)$-black-cover of $\mathcal{T}_{k+1}$ and let $\mathcal{C}_s$ be a minimum $(k+1)$-sparse-cover of U_{k+1}. Let $B_i = (s_i, t_i) + W_{3r_k}$, $i = 1, 2, \dots, N$ be an enumeration of $\mathcal{C}_s$ such that $i < j$ iff

$$t_i < t_j \qquad \text{or} \qquad t_i = t_j \,\wedge\, s_i < s_j.$$

To each B_i, we associate $C_i = (s_i - dr_k/2, t_i) + W_{dr_k}$. Let $\mathcal{C}^* = \{\, C_i : i \in [1, N] \,\}$. We will prove that

$$\mathcal{C}^* \text{ is a } (k+1)\text{-black-cover of } \mathcal{T}_{k+1}^* \tag{3.3.12}$$

where $\mathcal{T}_{k+1}^* = (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}') \cap \mathbb{Z} \times [h_{k+1} - r_{k+1}, \infty)$. The Lemma follows from the above statement by the next fact.

Claim I $(3.3.12) \implies \eta^\xi \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}')$ is $(k+1)$-black.

Proof: Let $D = (a, b) + W_{r_{k+1} - 2dr_k}$ be a test window of $(k+1)$-blackness. Let $H = D \cap (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}')$. If $H \subset \mathcal{T}_{k+1}$, then by assumption of $\eta^\xi \upharpoonright \mathcal{T}_{k+1}$ being $(k+1)$-black, there exists $A = (x, y) + W_{dr_k}$ such that $\eta^\xi \upharpoonright H \setminus A$ is k-black. If $H \subset \mathcal{T}_{k+1}^*$, then there exists $A \in \mathcal{C}^*$ such that $\eta^\xi \upharpoonright H \setminus A$ is k-black. This follows from the minimality of $\mathcal{C}_s$ which assures that elements in $\mathcal{C}_s$ are well-separated which, in turn, implies that $\forall B_i, B_j \in \mathcal{C}^*$, $i \neq j$, $B_i \cap D \neq \emptyset \implies B_j \cap D = \emptyset$. ▲

The proof of (3.3.12) goes by induction on the size of $\mathcal{C}_s$. Assume $n = |\mathcal{C}_s| = 0$. That is, U_{k+1} is k-sparse. Let

$$\mathcal{C}'_b = \{\, B \in \mathcal{C}_b : B \cap \mathcal{T}_{k+1} \cap \mathbb{Z} \times [0, h_k) \neq \emptyset \,\}.$$

If $|\mathcal{C}'_b| = 0$, then by the inductive assumption, η^ξ is a k-continuation at $(-\beta_{k+1} h_k, h_k)$ with width $w_{k+1} + 2\beta_{k+1} h_k$ and extension $h_{k+1} + h - h_k$. Hence, $\eta^\xi \restriction (-\beta_k h_k, w_{k+1} + \beta_k h_k, h_k, h_{k+1} + h, \beta_k)$ is k-black. Since

$$\mathcal{T}^*_{k+1} \subset (-\beta_k h_k, w_{k+1} + \beta_k h_k, h_k, h_{k+1} + h, \beta_k),$$

$\eta^\xi \restriction \mathcal{T}^*_{k+1}$ is k-black.

Assume $|\mathcal{C}'_b| > 0$. Let $(s^*, t^*) + W_{dr_k} \in \mathcal{C}'_b$ be an element such that t^* is maximal. Let $\mathcal{K} = \mathcal{T}_{k+1} \cap \mathbb{Z} \times [t^* + dr_k, t^* + dr_k + h_k)$.

Claim II $\eta^\xi \restriction \mathcal{K}$ is k-black.

Proof: Let $D = (x, y) + W_{r_k - 2dr_{k-1}}$ be any test window such that $H = D \cap \mathcal{K} \neq \emptyset$. Since $B \in \mathcal{C}'_b \implies B \cap H = \emptyset$, we only need consider $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$ such that $B \cap H \neq \emptyset$. Let $B = (a, b) + W_{dr_k}$ be such an element. Well-separatedness, $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$, and $\mathcal{C}_b$ being a $(k+1)$-black-cover imply that η^ξ is a k-continuation at (a', b'), $b' = b - r_k$, $a' = \max\{a - r_k, -\beta_{k+1} b'\}$, with width $\ell = \min\{dr_k + 2r_k, w_{k+1} + \beta_{k+1} b' - a'\}$ and extension $dr_k + h_k + r_k$. Hence,

$$\eta^\xi \restriction (a', a' + \ell, b', b' + dr_k + h_k + r_k, \beta_k) \quad \text{is } k\text{-black}.$$

Since $\beta_k > 1/2$ for all $k \geq 0$, $H \subset (a', a' + \ell, b', b' + dr_k + h_k + r_k, \beta_k)$ and $\eta^\xi \restriction H$ is k-black. $\blacktriangle$

Claim II implies that η^ξ is a k-continuation at $(-\beta_{k+1}(t^* + dr_k + h_k), t^* + dr_k + h_k)$ with width $w_{k+1} + 2\beta_{k+1}(t^* + dr_k + h_k)$ and extension $h_{k+1} + h - t^* - dr_k - h_k$. It follows that $\eta^\xi \restriction \mathcal{T}^*_{k+1}$ is k-black.

Assume (3.3.12) holds for $|\mathcal{C}_s| = n \geq 0$. Let $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r_k}$ be the last element in the enumeration of $\mathcal{C}_s$. We will consider the cases $t_{n+1} < h_{k+1} - r_{k+1} -$

$dr_k - h_k - 3r_k$ and $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, separately.

Claim III If $t_{n+1} < h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$ then (3.3.12) holds for $|\mathcal{C}_s| = n + 1$.

Proof: The assumption of the claim implies that

$$U_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r_{k+1} - dr_k - h_k, \infty) \quad \text{is } k\text{-sparse.}$$

Thus an argument analogous to the proof of Claim II can be applied to the smaller trapezoid $\mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r_{k+1} - dr_k - h_k, \infty)$ to conclude that $\eta^\xi \upharpoonright \mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r_{k+1}, h_{k+1})$ is k-black from which the claim follows. ▲

Let $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$. We will consider the three cases

$$s_{n+1} > w_{k+1} + \beta_{k+1} t_{n+1} + 5r_k,$$

$$w_{k+1} + \beta_{k+1} t_{n+1} - 3dr_k < s_{n+1} \leq w_{k+1} + \beta_{k+1} t_{n+1} + 5r_k,$$

$$-\beta_{k+1} t_{n+1} + 3dr_k \leq s_{n+1} \leq w_{k+1} + \beta_{k+1} t_{n+1} - 3dr_k,$$

separately. Note that without loss of generality (by symmetry) we may consider only the right boundary.

Case (i). Let $s_{n+1} > w_{k+1} + \beta_{k+1} t_{n+1} + 5r_k$. By the speed-of-light, $\eta^\xi \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathcal{L}$,

$$\mathcal{L} = \{ (s, t) : s \leq -2(t - t_{n+1}) + s_{n+1}, \ t \geq 0 \},$$

takes on the same values irrespective of whether the error event B_{n+1} occurred or not. Let $D = (x, y) + W_{r_k - 2dr_k - 1}$ be any test window such that

$$H = D \cap \left(\mathcal{T}^*_{k+1} \setminus \bigcup_{C \in \mathcal{C}^*} C \right) \neq \emptyset. \tag{3.3.13}$$

If $D \subseteq \mathcal{L}$ then by the inductive assumption on $|\mathcal{C}_s|$, $\eta^\xi \upharpoonright H$ is k-black. Assume $D \cap \mathcal{L}^c \neq \emptyset$. Let $\mathcal{K}$ be the trapezoid

$$(w_{k+1} + \beta_{k+1} t_{n+1} - 2(dr_k + 2h_k + 3r_k) - r_k, w_{k+1} + \beta_{k+1} t_{n+1}, t_{n+1} - h_k, t_{n+1}, \beta_k).$$

If $s_{n+1} \leq w_{k+1} + \beta_{k+1}t_{n+1} + 3(dr_k + 2h_k + 3r_k)$, then $\eta^\xi \restriction \mathcal{K}$ is k-black. This is a straightforward consequence of well-separatedness since if $t_{n+1} \geq h_{k+1}$, then by inductive assumption on $|\mathcal{C}_s|$,

$$\mathcal{C}^* \setminus \{C_{n+1}\} \quad \text{is a } (k+1)\text{-black-cover of} \quad \mathcal{T}_{k+1}^* \cap \mathbb{Z} \times [0, t_{n+1}).$$

If $t_{n+1} < h_{k+1}$ and for some $B = (a,b) + W_{dr_k} \in \mathcal{C}_b$, $B \cap \mathcal{K} \neq \emptyset$, then by well-separatedness η^ξ is a k-continuation at $(a - dr_k/2, b)$ with width $\min\{2dr_k, w_{k+1} + \beta_{k+1}b - a + dr_k/2\}$ and extension $2dr_k$ which implies that $\eta^\xi \restriction B' \cap \mathcal{T}_{k+1}$ is k-black where B' is the r_k-dilation of B. By Proposition 3.3.11, this contradicts the minimality of $\mathcal{C}_b$ and hence $\eta^\xi \restriction \mathcal{K}$ must be k-black.

Since η^ξ is a k-continuation at $(w_{k+1} + \beta_{k+1}t_{n+1} - 2(dr_k + 2h_k + 3r_k) - r_k, t_{n+1})$ with width $2(dr_k + 2h_k + 3r_k) + r_k$ and extension $h_{k+1} + h - t_{n+1}$, $\eta^\xi \restriction \mathcal{K} \cup \mathcal{K}'$ is k-black where $\mathcal{K}'$ is the trapezoid

$$(w_{k+1} + \beta_{k+1}t_{n+1} - 2(dr_k + 2h_k + 3r_k) - r_k, w_{k+1} + \beta_{k+1}t_{n+1}, t_{n+1}, h_{k+1} + h, \beta_k).$$

Let $\mathcal{M} = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, t_{n+1} + 3r_k + dr_k + 2h_k)$.

Claim IV $\eta^\xi \restriction \mathcal{M}$ is k-black.

Proof: If $s_{n+1} > w_{k+1} + \beta_{k+1}t_{n+1} + 3(dr_k + 2h_k + 3r_k)$ then $\mathcal{M} \subset \mathcal{L}$, and by the inductive assumption on $|\mathcal{C}_s|$, $\eta^\xi \restriction \mathcal{M}$ is k-black. Assume $s_{n+1} \leq w_{k+1} + \beta_{k+1}t_{n+1} + 3(dr_k + 2h_k + 3r_k)$. Let $U = (a,b) + W_{r_k - 2dr_{k-1}}$ be any test window such that $U \cap \mathcal{M} \neq \emptyset$. By the definition of $\mathcal{L}$ and $\mathcal{K}'$,

$$U \cap \mathcal{M} \subset \mathcal{L} \quad \text{or} \quad U \cap \mathcal{M} \subset \mathcal{K}'.$$

We already know that $\eta^\xi \restriction \mathcal{K}'$ is k-black. Assume $U \cap \mathcal{M} \subset \mathcal{L}$. Since $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, $\mathcal{M} \subset \mathcal{T}_{k+1}^*$. By the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-black-cover of $\mathcal{T}_{k+1}^* \cap \mathcal{L}$, and since C_{n+1} is the last element in the enumeration of $\mathcal{C}^*$, it follows that $\eta^\xi \restriction \mathcal{M} \cap \mathcal{L}$ is k-black. Since $U \cap \mathcal{M} \subset \mathcal{M} \cap \mathcal{L}$, $\eta^\xi \restriction U \cap \mathcal{M}$ is k-black. ▲

Let $\mathcal{M}' = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, \infty)$. An immediate consequence of Claim IV (by the inductive assumption on k) is that $\eta^{\xi} \restriction \mathcal{M}'$ is k-black. Since $D \cap \mathcal{L}^c \neq \emptyset$, by the definition of $\mathcal{L}$, $\mathcal{K}'$, and $\mathcal{M}'$,

$$H \subset \mathcal{K} \cup \mathcal{K}' \qquad \text{or} \qquad H \subset \mathcal{M}'.$$

Hence, (3.3.12) holds for $|\mathcal{C}_s| = n + 1$.

Case (ii). Let $-\beta_{k+1} t_{n+1} + 3dr_k \leq s_{n+1} \leq w_{k+1} + \beta_{k+1} t_{n+1} - 3dr_k$. If $t_{n+1} \geq h_{k+1}$, then by the inductive assumption on $|\mathcal{C}_s|$, η^{ξ} is a k-continuation at both $(s_{n+1} - 6r_k - 3dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 9r_k, t_{n+1} + 3r_k)$ with extension $h_{k+1} + h - t_{n+1} - 3r_k$ and widths $3dr_k$ and $\max\{ 3dr_k - 9r_k, w_{k+1} + \beta_{k+1}(t_{n+1} + 3r_k) - s_{n+1} - 9r_k \}$, respectively.

If $t_{n+1} < h_{k+1}$, η^{ξ} remains a k-continuation at both $(s_{n+1} - 6r_k - 3dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 9r_k, t_{n+1} + 3r_k)$ with the same parameters as before. Suppose this were not the case, i.e., for some $B = (a, b) + W_{dr_k} \in \mathcal{C}_b$,

$$B \cap (\mathcal{K}^L \cup \mathcal{K}^R) \neq \emptyset$$

where $\mathcal{K}^L$ and $\mathcal{K}^R$ are the contexts corresponding to the two k-continuations at $(s_{n+1} - 6r_k - 3dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 9r_k, t_{n+1} + 3r_k)$, respectively. Let (x, y) denote the upper-left corner of $\mathcal{K}^L$ and let w_L denote the width of $\mathcal{K}^L$ at $t = y$. If $b > y$, then η^{ξ} is a k-continuation at (x, y) with width w_L and extension $h_{k+1} + h - y$, and by the inductive assumption on k, $\eta^{\xi} \restriction \mathcal{K}^L$ is k-black. If $b \leq y$, then η^{ξ} is a k-continuation at (x, b) with width $w_L - \beta_k(y - b)$ and extension $h_{k+1} + h - b$, hence $\eta^{\xi} \restriction \mathcal{K}^L$ is k-black. A similar argument holds for $\mathcal{K}^R$.

Using Lemma 3.3.5 (error correction), $\eta^{\xi} \restriction \mathcal{K} \setminus C_{n+1}$ is k-black where

$$\mathcal{K} = (s_{n+1} - 6r_k - 3dr_k, w_{k+1} + \beta_{k+1} t_{n+1}, t_{n+1}, h_{k+1} + h, \beta_k).$$

By SOL, $\eta^{\xi} \restriction (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathcal{L}$ takes on the same values irrespective of whether the error event B_{n+1} occurred or not where

$$\mathcal{L} = \{ (s, t) : s \leq -2(t - t_{n+1}) + s_{n+1} \text{ or } s \geq 2(t - t_{n+1}) + s_{n+1} + 3r_k, \ t \geq t_{n+1} \}.$$

Let $\mathcal{M} = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, t_{n+1} + 3r_k + dr_k + 2h_k)$.

Claim V $\eta^\xi \upharpoonright \mathcal{M}$ is k-black.

Proof: Let $U = (a, b) + W_{r_k - 2dr_{k-1}}$ be any test window such that $U \cap \mathcal{M} \neq \emptyset$. By the definition of $\mathcal{L}$ and $\mathcal{K}$,

$$U \cap \mathcal{M} \subset \mathcal{L} \qquad \text{or} \qquad U \cap \mathcal{M} \subset \mathcal{K} \setminus C_{n+1}.$$

We have already established that $\eta^\xi \upharpoonright \mathcal{K} \setminus C_{n+1}$ is k-black. Let $U \cap \mathcal{M} \subset \mathcal{L}$. Since $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, $\mathcal{M} \subset \mathcal{T}^*_{k+1}$. By the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-black-cover of $\mathcal{T}^*_{k+1} \cap \mathcal{L}$, and since C_{n+1} is the last element in the enumeration of $\mathcal{C}^*$, it follows that $\eta^\xi \upharpoonright \mathcal{M} \cap \mathcal{L}$ is k-black. Since $U \cap \mathcal{M} \subset \mathcal{M} \cap \mathcal{L}$, $\eta^\xi \upharpoonright U \cap \mathcal{M}$ is k-black. $\blacktriangle$

Let $D = (x, y) + W_{r_k - 2dr_{k-1}}$ be any test window satisfying (3.3.13). If $D \subseteq \mathcal{L}$ we are done. If $D \cap \mathcal{L}^c \neq \emptyset$, then

$$H \subset \mathcal{K} \setminus C_{n+1} \qquad \text{or} \qquad H \subset \mathcal{M}'$$

where $\mathcal{M}' = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, \infty)$. Since Claim V implies that $\eta^\xi \upharpoonright \mathcal{M}'$ is k-black, (3.3.12) holds for $|\mathcal{C}_s| = n + 1$.

Case (iii). Let $w_{k+1} + \beta_{k+1} t_{n+1} - 3dr_k < s_{n+1} \leq w_{k+1} + \beta_{k+1} t_{n+1} + 5r_k$. It suffices to show that η^ξ is k-black on $\mathcal{K}$ where

$$
\begin{aligned}
\mathcal{K} = \{ (s, t) : \; &w_{k+1} + \beta_{k+1} t_{n+1} - 3dr_k + \beta_k (t - t_{n+1}) \; \leq \; s \; \leq \\
&w_{k+1} + \beta_{k+1} t_{n+1} + 3dr_k + \beta_k (t - t_{n+1}), \;\; t_{n+1} - 2dr_k \leq t \leq t_{n+1} \},
\end{aligned}
$$

since then the arguments of Case (ii) can again be applied from which (3.3.12) follows.

Consider the space-time point

$$p = (w_{k+1} + \beta_{k+1} (t_{n+1} + 3r_k - u), t_{n+1} + 3r_k - u)$$

where $u = r_{k+1}/3$. Let $p' = p + (-5dr_k, 0)$ be the space-translation of p by $-5dr_k$.

Claim VI η^ξ is a k-continuation at p' with width $5dr_k$ and extension $u - 3r_k$.

Proof: First, for any window $V = (s, t) + W_u$ such that $B_{n+1} \subset V$ (recall that B_{n+1} is the last element in the enumeration of $\mathcal{C}_s$), $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap V$ is k-sparse. This is a direct consequence of well-separatedness (i.e., choose c sufficiently large such that $r_{k+1}/3 < r_{k+1} - 6r_k$) and the definition of sparsity. Let $\mathcal{H}$ be the context of the (as yet to be determined) k-continuation in the claim. Let (a, b) be the upper-left corner point of $\mathcal{H}$, and let $w_{\mathcal{H}}$ denote the width of $\mathcal{H}$ at $t = b$. To prove the claim, it suffices to show that $\eta^\xi \restriction \mathcal{H}$ is k-black since by the definition of u and well-separatedness ($c \gg d$), the trapezoid

$$(a - 2r_k, a + w_{\mathcal{H}} + 2r_k, b, t_{n+1}, 1)$$

is k-sparse. Suppose for some $B = (a', b') + W_{dr_k} \in \mathcal{C}_b$, $B \cap \mathcal{H} \neq \emptyset$. If $b' > b$, then by $\mathcal{C}_b$ being a minimal $(k+1)$-cover, η^ξ is a k-continuation at (a, b) with width $w_{\mathcal{H}}$ and extension h_k which implies that $\eta^\xi \restriction \mathcal{H}$ is k-black. If $b' \leq b$, then η^ξ is a k-continuation at (a, b') with width $w_{k+1} + \beta_{k+1} b' - a$ and extension $dr_k + h_k$, and hence $\eta^\xi \restriction \mathcal{H}$ is k-black. ▲

Let $\mathcal{H}' = (a, a + w_{\mathcal{H}}, b, t_{n+1}, \beta_k)$. Claim VI implies that $\eta^\xi \restriction \mathcal{H}'$ is k-black. We will prove that $\mathcal{K} \subset \mathcal{H}'$ from which (3.3.12) follows. Since $p' = p + (-5dr_k, 0)$ and $1/2 < \beta_k < 1$, it suffices to show that

$$w_{k+1} + \beta_{k+1} t_{n+1} + 3dr_k \quad \leq \quad w_{k+1} + \beta_{k+1}(t_{n+1} + 3r_k - u) + \beta_k(u - 3r_k)$$

$$\Longleftrightarrow \qquad 3dr_k \quad \leq \quad (\beta_k - \beta_{k+1})(u - 3r_k). \tag{3.3.14}$$

Claim VII There exist $c, d > 0$ such that (3.3.14) holds.
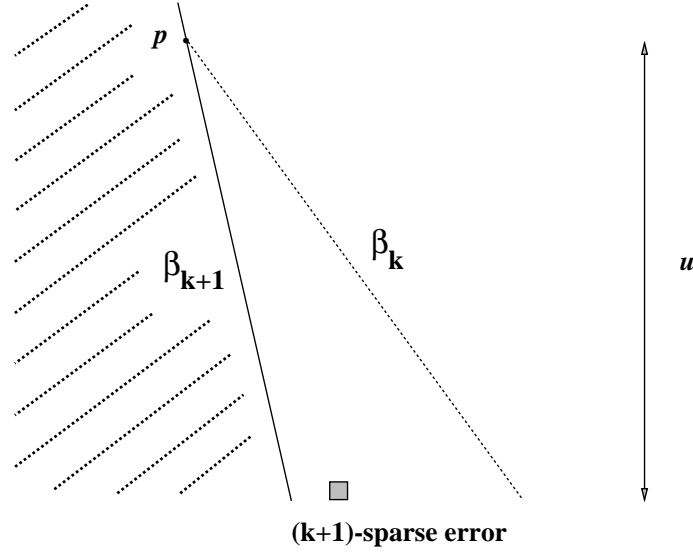
(k+1)-sparse error

Figure 3.3.3: Slope deflection in the proof of Lemma 3.2.2, case (iii).

Proof: By the inductive assumption on k, $\beta_k = 1/2 + 1/(k + 2.5)$, and (3.3.14) is equivalent to

$$3dr_k \leq \left(\frac{1}{2} + \frac{1}{k + 2.5} - \frac{1}{2} - \frac{1}{k + 3.5} \right) (r_{k+1}/3 - 3r_k)$$

$$\Longleftrightarrow \qquad 9dr_k \leq \frac{4}{(2k + 5)(2k + 7)} \left(c(k + 1)^2 r_k - 9r_k \right)$$

$$\Longleftrightarrow \qquad 9d \leq \frac{4}{(2k + 5)(2k + 7)} \left(c(k + 1)^2 - 9 \right) \qquad (3.3.15)$$

where we have used $u = r_{k+1}/3$ and $r_{k+1} = c(k + 1)^2 r_k$. Upon rearrangement, (3.3.15) holds iff

$$(4c - 36d)k^2 + (8c - 216d)k + 4c - 315d - 36 \geq 0.$$

This is satisfied, for all $k \geq 0$, if $c \geq 79d + 9$. ▲

It is easily checked that all the claims in the proof using the well-separatedness property with respect to $c \gg d$ are satisfied if $c = 100d$. ∎

Figure 3.3.3 depicts the occurrence of a $(k + 1)$-sparse error near the right boundary of the trapezoid $\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}$ (case (iii) in the proof of Lemma 3.2.2) and the "slope

deflection" it induces $(\beta_{k+1} < \beta_k)$.

3.4   Spreading of agreement

In this subsection, we will prove that if the coupled processes η_0, η_1 are k-black on a trapezoid R_k, then they concur on the slightly smaller trapezoid $R_k(b_k) \subset R_k$ (Lemma 2.2.5). Thus the problem of showing the spreading of agreement is reduced to the problem of showing the spreading of blackishness (Lemma 2.2.9). First, we state some simple facts.

Fact 3.4.1   *If η_0, η_1 concur on a space interval $[x, x + a) \times T$, then they continue to concur on the space-time triangle*

$$\{\, (s, t) : x + 2(t - T) \leq s \leq x + a - 2(t - T),\, t \geq T \,\}.$$

Proof.   A direct consequence of SOL since by the definition of basic coupling, if $\eta_0(s, t) = \eta_1(s, t)$ and η_0, η_1 concur in the rest of their neighborhood $\{\, (s - 2, t), (s - 1, t), (s + 1, t), (s + 2, t) \,\}$, then $\eta_0(s, t + 1) = \eta_1(s, t + 1)$.   ∎

Fact 3.4.2   *Let $\varphi, \varphi' : \mathbb{Z} \times \mathbb{N} \to S$ be space-time configurations such that $\varphi \leq \varphi'$. Let $A \subseteq \mathbb{Z} \times \mathbb{N}$. Then*

$$\varphi \restriction A \text{ is } k\text{-black} \quad \Longrightarrow \quad \varphi' \restriction A \text{ is } k\text{-black}.$$

Proof.   The basis ($k = 0$) holds trivially. Assume the statement holds for $k > 0$. Let W be a $(r_{k+1} - 2dr_k)$-window such that $A \cap W \neq \emptyset$. Let $\varphi \restriction A$ be $(k + 1)$-black. This implies that there exists dr_k-window B such that $\varphi \restriction (A \cap W \setminus B)$ is k-black. By inductive assumption, $\varphi' \restriction (A \cap W \setminus B)$ is k-black as well. Since this holds for an arbitrary W and we are able to identify an exclusion window B for φ' via φ, $\varphi' \restriction A$ is $(k + 1)$-black.   ∎

Fact 3.4.3 *Let $\varphi, \varphi' : \mathbb{Z} \times \mathbb{N} \to S$ be space-time configurations such that $\varphi \leq \varphi'$. Then for all $A \subseteq \mathbb{Z} \times \mathbb{N}$,*

$$\mathcal{C} \text{ is a } k\text{-cover of } \varphi \restriction A \implies \mathcal{C} \text{ is a } k\text{-cover of } \varphi' \restriction A.$$

Proof. Assume $k = 1$. (Note that a k-cover is only meaningful for $k \geq 1$ although for $k = 0$ we may define the 0-cover to be the empty set.) Let $\mathcal{C}$ be a 1-cover of $\varphi \restriction A$; i.e., φ is black on $A \setminus \bigcup_{B \in \mathcal{C}} B$. If $\mathcal{C}$ were not a 1-cover of $\varphi' \restriction A$, this would lead to a contradiction since $\varphi \leq \varphi'$.

Assume the statement holds for $k > 0$. Let $\mathcal{C}$ be a $(k+1)$-cover of $\varphi \restriction A$. That is, φ is k-black on $A \setminus \bigcup_{B \in \mathcal{C}} B$. By Fact 3.4.2, this implies φ' is k-black on $A \setminus \bigcup_{B \in \mathcal{C}} B$. If $\mathcal{C}$ were not a $(k+1)$-cover of $\varphi' \restriction A$, it would mean that $\varphi' \restriction A \setminus \bigcup_{B \in \mathcal{C}} B$ is not k-black which is a contradiction. ∎

Proof of Lemma 2.2.5. Let $k = 0$. Since η_0, η_1 are 0-black on trapezoid $R_0 = (y, z, u, v, q)$, i.e., $\eta_0(s, t) = \eta_1(s, t) = 11$ for $(s, t) \in R_0$, the basis follows directly.

Assume the statement holds for $k > 0$. Let $\mathcal{C}$ be a minimum $(k+1)$-cover of $\eta_0 \restriction R_{k+1}$. Monotonicity allows us to work with a single cover since by Fact 3.4.3, $\mathcal{C}$ is also a $(k+1)$-cover of $\eta_1 \restriction R_{k+1}$. If $\mathcal{C} = \emptyset$, then $\eta_0(s, t) = \eta_1(s, t)$ on $R_{k+1}(b_k)$ and the lemma is trivially true since $R_{k+1}(b_{k+1}) \subset R_{k+1}(b_k)$.

Assume $\mathcal{C} \neq \emptyset$. Let us consider the trapezoid $R'_{k+1} = (y + 5dr_k, z - 5dr_k, u + 2dr_k, v, q)$.

Claim If for all $B \in \mathcal{C}$, $B \subset R'_{k+1}$, then η_0, η_1 concur on R'_{k+1}.

Proof: Since elements in $\mathcal{C}$ are well-separated, using Fact 3.4.1, it is easily checked by induction that for all $B \in \mathcal{C}$,

$$\eta_0(s, t) = \eta_1(s, t), \qquad (s, t) \in B$$

from which the claim follows. ▲

Let us call

$$y + 5dr_k \leq s \leq z - 5dr_k \tag{3.4.4}$$

the *space-separation* condition and let

$$t \geq u + 2dr_k \tag{3.4.5}$$

be the *time-separation* condition. The additional dr_k in the definition of R'_{k+1} stems from an application of the inductive hypothesis which by itself yields the trapezoid $R_{k+1}(b_k)$. We remind the relationship

$$R'_{k+1} \subset R_{k+1}(b_k) \subset R_{k+1}.$$

The previous claim fails to hold if for some $B \in \mathcal{C}$ the space-and/or time-separation conditions are violated.

First, consider the time-separation case. Let $B = (x, w) + W_{dr_k} \in \mathcal{C}$ be an element with maximum time coordinate w such that (3.4.5) is violated, i.e., $w < u + 2dr_k$. Ignoring the effect of space-separation on x for the moment, we obtain the time sufficiency condition

$$2dr_k + dr_k + dr_k < dr_{k+1} = b_{k+1} \tag{3.4.6}$$

where a further dr_k is needed to account for the size of B, and the final dr_k represents a subsequent application of the inductive hypothesis. The inequality holds since $r_{k+1} = c(k + 1)^2 r_k$ and $c \gg d$.

Second, let us consider the space-separation case. W.l.o.g. (by symmetry), we will consider the left boundary of the trapezoid R_{k+1}. Let $B = (x, w) + W_{dr_k} \in \mathcal{C}$ be an element with maximum space coordinate x violating (3.4.4). That is,

$$x < y + q(t - w) + 5dr_k.$$

A space sufficiency condition is obtained by adding dr_k to account of the size of B followed by a further application of the inductive hypothesis:

$$5dr_k + dr_k + dr_k < dr_{k+1} = b_{k+1}. \tag{3.4.7}$$

Thus, combining conditions (3.4.6) and (3.4.7), we can see that $R_{k+1}(b_{k+1})$ satisfies both, and using the previous claim it follows that η_0 and η_1 concur on $R_{k+1}(b_{k+1}) \subset R'_{k+1}$. $\blacksquare$

Chapter 4

Relaxation time

4.1 Relaxation lower bound

4.1.1 Supersparsity

To show that the all-00 initial configuration remains "whitish" for a long time, we will use a scheme analogous to the proof of mixing. We need the new notion "k-supersparse" which is a more restrictive counterpart of the sparsity definition used earlier. Let us define a new window size r'_k,

$$r'_k = \varepsilon^{-1/4} c^k.$$

Definition 4.1.1 (supersparsity)   The error process is 0-*supersparse* on a space-time set A if it is empty. It is k-*supersparse*, $k \geq 1$, if for all squares B of the form $(s, t) + W_{r'_k}$ either $A \cap B$ is $(k-1)$-supersparse, or there exists (s', t') such that the following two conditions hold:

(i) $(A \cap B) \setminus (s', t') + W_{3r'_{k-1}}$ is $(k-1)$-supersparse,

(ii) there are no errors on $(A \cap B \cap (s' - dr'_{k-1}/2, t') + W_{dr'_{k-1}}) \setminus (s', t') + W_{3r'_{k-1}}$.

Lemma 4.1.2 (supersparsity)   *There exists $0 < \varepsilon_0 < 1/2$, $v > c > 1$ such that for all $0 < \varepsilon < \varepsilon_0$, $0 \leq k \leq (\log 1/\varepsilon)/8 \log v$, $\beta > 0$ the following holds. Let η denote*

an ε-perturbation of K, $K_{\varepsilon,\beta}$. Let $A = \bigcup_{i \in [1,N]} (a_i, b_i) + W_{r'_k}$, $a_i, b_i \in \mathbb{Z}$, where $\eta \upharpoonright A$ is defined. Let q_k be the probability that η is not k-supersparse on A. Then

$$q_k \; < \; N \varepsilon^{(k+1.2)/4} v^{k(k-1)}.$$

Proof. The proof goes by induction on k. Let $k = 0$. For any window $W^i = (a_i, b_i) + W_{r'_0}$, consider the probability that η is not 0-supersparse on W^i. Since $r'_0 = \varepsilon^{-1/4}$,

$$\Pr(\eta \text{ is not } 0\text{-supersparse on } W^i) \; \leq \; 2(\varepsilon^{-1/4})^2 \varepsilon \; = \; 2\varepsilon^{1/2} \; < \; \varepsilon^{0.3}$$

for ε sufficiently small. Hence $q_0 < N\varepsilon^{0.3}$ which equals the probability bound with $k = 0$.

Assume the relation holds for $k > 0$. For $i, j \in \{0, 1\}$, define partition $\mathcal{P}_{i,j}$ of A as follows:

$$\mathcal{P}_{i,j} = \big\{ A \, \cap \, r'_{k+1}(2s + i, 2t + j) + W_{2r'_{k+1}} \; : \; s, t \in \mathbb{Z} \big\}.$$

Each $(a_l, b_l) + W_{r'_{k+1}}$ is intersected by at most four elements of $\mathcal{P}_{i,j}$, hence $|\mathcal{P}| \leq 16N$ where $\mathcal{P} = \bigcup_{i,j} \mathcal{P}_{i,j}$. Suppose there exists $B = (a, b) + W_{r'_{k+1}}$ such that η is not $(k+1)$-supersparse on $A \cap B$, $A \cap B \neq \emptyset$. Since $A \cap B \subset V$ for some $V \in \mathcal{P}$, η is not $(k+1)$-supersparse on V. Thus,

$$q_{k+1} \; \leq \; 16Nq' \tag{4.1.3}$$

where $q' = \Pr(\eta \text{ is not } (k+1)\text{-supersparse on } V)$.

Partition V as before but with r'_k in place of r'_{k+1}. Denote the four partitions by $\mathcal{R}_{i,j}$, $i, j \in \{0, 1\}$, and let $\mathcal{R} = \bigcup_{i,j} \mathcal{R}_{i,j}$. Since $r'_{k+1}/r'_k = c$, $|\mathcal{R}_{i,j}| \leq (c+1)^2$. Consider the events $\mathcal{E}, \mathcal{E}'$ where $\mathcal{E}$ is the event that there exist $U, U' \in \mathcal{R}$, $U \cap U' = \emptyset$, such that η is not k-supersparse on U and U', and $\mathcal{E}'$ denotes the event that V is not k-supersparse and there exists $r'_k(2s+i, 2t+j)+W_{3r'_k}$ such that $V \backslash r'_k(2s+i, 2t+j)+W_{3r'_k}$ is k-supersparse and

$$\Big(V \cap (r'_k(2s+i) - dr'_{k-1}/2, r'_k(2t+j)) + W_{dr'_k} \Big) \, \backslash \, r'_k(2s+i, 2t+j) + W_{3r'_k}$$

is not error-free.

Claim $\quad q' < \Pr(\mathcal{E}) + \Pr(\mathcal{E}')$.

Proof: We will prove the contrapositive: $\neg\mathcal{E} \wedge \neg\mathcal{E}' \implies \eta$ is $(k+1)$-supersparse on V. Suppose $\neg\mathcal{E}$. That is, if $U, U' \in \mathcal{R}$ and η is not k-supersparse on U and U', then $U \cap U' \neq \emptyset$. In particular, if we fix U, then all elements of $\mathcal{R}$ on which η is not k-supersparse must intersect with U as well as with each other. By definition of $\mathcal{R}_{i,j}$, $\forall U, U' \in \mathcal{R}$ with $U \cap U' \neq \emptyset$, there exists $r_k'(2s+i, 2t+j)$ such that $U, U' \subset r_k'(2s+i, 2t+j) + W_{3r_k'}$. $r_k'(2s+i, 2t+j) + W_{3r_k'}$ covers all elements of $\mathcal{R}$ on which η is not k-supersparse, and by containment, any r_k'-window in V on which η is not k-supersparse. Hence η is k-supersparse on $V \setminus r_k'(2s+i, 2t+j) + W_{3r_k'}$. The possibility that

$$\left(V \cap (r_k'(2s+i) - dr_{k-1}'/2, r_k'(2t+j)) + W_{dr_k'} \right) \setminus r_k'(2s+i, 2t+j) + W_{3r_k'}$$

is not error-free is excluded by $\neg\mathcal{E}'$. $\quad\blacktriangle$

By independence, $\Pr(\eta$ is not k-supersparse on U and $U') \leq q''^2$ where $U, U' \in \mathcal{R}$, $U \cap U' = \emptyset$, and $q'' = \Pr(\eta$ is not k-supersparse on $A \cap (s,t) + W_{2r_k'})$. Since the total number of disjoint pairs in $\mathcal{R}$ is strictly less than $|\mathcal{R}|^2$,

$$\Pr(\mathcal{E}) \quad < \quad 16(c+1)^4 q''^2.$$

To bound $\Pr(\mathcal{E}')$, notice that

$$\mathcal{E}' \quad \subseteq \quad \bigvee_{s,t,i,j} \left(V \cap r_k'(2s+i, 2t+j) + W_{3r_k'} \text{ is not } k\text{-supersparse} \bigwedge \right.$$

$$\left. (r_k'(2s+i) - dr_{k-1}'/2, r_k'(2t+j)) + W_{dr_k'} \setminus r_k'(2s+i, 2t+j) + W_{3r_k'} \text{ is not error-free} \right).$$

Thus,

$$\Pr(\mathcal{E}') \quad \leq \quad 9(c+1)^2 q_k 2(dr_k')^2 \varepsilon \quad = \quad 18 d^2 (c+1)^2 q_k \varepsilon^{1/2} c^{2k},$$

and we have

$$q' \; < \; 16(c+1)^4 q''^2 \; + \; 18d^2(c+1)^2 q_k \varepsilon^{1/2} c^{2k}.$$

Using (4.1.3) and the inductive hypothesis on q'',

$$
\begin{aligned}
q_{k+1} \;\; \leq \;\; & 16Nq' \;\; < \;\; 16N \left(16(c+1)^4 4^2 q_k^2 \; + \; 18d^2(c+1)^2 q_k \varepsilon^{1/2} c^{2k} \right) \\
< \;\; & N \left(16^3(c+1)^4 \varepsilon^{(k+1.2)/2} v^{2k(k-1)} \; + \; 288d^2(c+1)^2 \varepsilon^{(k+1.2)/4} v^{k(k-1)} \varepsilon^{1/2} c^{2k} \right) \\
= \;\; & N \varepsilon^{(k+2.2)/4} v^{(k+1)k} \left(16^3(c+1)^4 \varepsilon^{(k+0.2)/4} v^{k(k-3)} \; + \; 288d^2(c+1)^2 (c/v)^{2k} \varepsilon^{1/4} \right).
\end{aligned}
$$

Clearly, $288d^2(c+1)^2(c/v)^{2k}\varepsilon^{1/4} < 1/2$ for $v > c$ and ε sufficiently small.

To achieve $16^3(c+1)^4 \varepsilon^{(k+0.2)/4} v^{k(k-3)} < 1/2$, first

$$
\begin{aligned}
16^3(c+1)^4 \varepsilon^{(k+0.2)/4} v^{k(k-3)} \;\; = \;\; & 16^3(c+1)^4 \exp((k+0.2)(\log \varepsilon)/4 + k(k-3)\log v) \\
\leq \;\; & 16^3(c+1)^4 \exp(k^2 \log v - (k/4)\log 1/\varepsilon - 0.05 \log 1/\varepsilon).
\end{aligned}
$$

Let $F(k) = k^2 \log v - (k/4)\log 1/\varepsilon - 0.05 \log 1/\varepsilon$ denote the exponent. $F'(k) = 2k \log v - (1/4)\log 1/\varepsilon$, and the parabola attains its minimum at $F'(k) = 0$, i.e.,

$$k_0 = (1/8 \log v) \log 1/\varepsilon.$$

Since $F'(0) = -(1/4)\log 1/\varepsilon < 0$ and $F(0) = -0.05 \log 1/\varepsilon < 0$, $F(k)$ is monotonically decreasing for $0 \leq k \leq (1/8 \log v) \log 1/\varepsilon$. Hence,

$$
\begin{aligned}
16^3(c+1)^4 \exp(k^2 \log v - (k/4)\log 1/\varepsilon - 0.05 \log 1/\varepsilon) \;\; \leq \;\; & 16^3(c+1)^4 e^{-0.05 \log 1/\varepsilon} \\
= \;\; & 16^3(c+1)^4 \varepsilon^{0.05} \\
< \;\; & 1/2
\end{aligned}
$$

for sufficiently small ε where we have substituted $k = 0$. ∎

We remark that by a similar argument q_k is monotonically decreasing for $0 \leq k \leq (\log 1/\varepsilon)/8 \log v$.

Let $k_0 = (\log 1/\varepsilon)/8 \log v$. We will consider a sequence of space-time rectangles S'_k, $k = 0, 1, \ldots, k_0$, where S'_k ($k < k_0$) has size $17r'_k \times 4r'_k$, and S'_{k_0} is of the size

$$17 \cdot 2^{c_2 \log^2 1/\varepsilon} \times 4 \cdot 2^{c_2 \log^2 1/\varepsilon}$$

where $c_2 > 0$ is a positive constant. We will view S'_k as a union of two subrectangles $S^U_k, S^B_k \subset S'_k$ each of size $17r'_k \times 3r'_k$ if $k < k_0$ and

$$17 \cdot 2^{c_2 \log^2 1/\varepsilon} \times 3 \cdot 2^{c_2 \log^2 1/\varepsilon}$$

if $k = k_0$. Note that given S'_k, S^U_k and S^B_k are uniquely determined.

Lemma 4.1.4 *There exist $\delta > 0$, $c_2 > 0$ such that for all sufficiently small ε (depending on δ)*

$$\Pr\left(\bigwedge_{k=0}^{k_0} (S^U_k \text{ is } k\text{-supersparse} \wedge S^B_k \text{ is } (k-1)\text{-supersparse}) \right) > \delta$$

where (-1)-supersparse is interpreted to mean 0-supersparse.

Proof. Let us upper-bound the probability of the complement event: for some $k \leq k_0$, S^U_k is not k-supersparse or S^B_k is not $(k-1)$-supersparse. First, since q_k is monotonically decreasing for $0 \leq k \leq (\log 1/\varepsilon)/8 \log v$, we have

$$\Pr(S^U_k \text{ is not } k\text{-supersparse}) < 51\varepsilon^{0.3},$$

$$\Pr(S^B_k \text{ is not } (k-1)\text{-supersparse}) < 51c^2\varepsilon^{0.3},$$

where we have used $r'_{k+1} = cr'_k$ in the second bound. Second, it is easy to check that

$$\Pr(S^U_k \text{ is not } k\text{-supersparse}) \leq \Pr(S^U_k \text{ is not } (k-1)\text{-supersparse}).$$

With these two facts in hand,

$\Pr(\exists k \leq k_0 : S_k^U$ is not k-supersparse $\lor$ S_k^B is not $(k-1)$-supersparse$)$

$$\leq \sum_{k=0}^{k_0} \Big(\Pr(S_k^U \text{ is not } k\text{-supersparse}) + \Pr(S_k^B \text{ is not } (k-1)\text{-supersparse}) \Big)$$

$$\leq 2\Pr(S_{k_0}^B \text{ is not } (k_0-1)\text{-supersparse}) + 2\sum_{k=0}^{k_0-1} \Pr(S_k^B \text{ is not } (k-1)\text{-supersparse})$$

$$< 51 \cdot 2^{2c_2 \log^2 1/\varepsilon} \varepsilon^{((k_0-1)+1.2)/4} v^{(k_0-1)(k_0-2)} + \varepsilon^{0.3} 51 c^2 k_0$$

$$< 51 \exp(2c_2 \log 2 \log^2 1/\varepsilon + k_0^2 \log v - (k_0/4) \log 1/\varepsilon - 0.05 \log 1/\varepsilon)$$

$$+ \varepsilon^{0.3} 51 c^2 (\log 1/\varepsilon)/8 \log v$$

$$= 51 \exp((2c_2 \log 2 - 1/64 \log v) \log^2 1/\varepsilon - 0.05 \log 1/\varepsilon) + \varepsilon^{0.3}(\log 1/\varepsilon) 51 c^2/8 \log v$$

Clearly, for $c_2 < 1/(128 \log 2 \log v)$ and ε sufficiently small (depending on δ),

$$\Pr(\exists k \leq k_0 : S_k^U \text{ is not } k\text{-supersparse} \lor S_k^B \text{ is not } (k-1)\text{-supersparse}) \leq 1 - \delta$$

which completes the proof. ∎

4.1.2 Shrinking region of consolidation

For each $\delta > 0$, site x, time $t \leq Relax(n, \delta, K_{\varepsilon,\beta})$, and $k = 0, \ldots, k_0$, we define a sequence $R_k' \subset S_k'$ of trapezoids

$$R_k' = (x_k, x_k + 17\ell_k, y_k, y_k + 4\ell_k, -2)$$

extending into the past such that

(i) $\ell_{k_0} = 2^{c_2 \log^2 1/\varepsilon}$, $\ell_k = r_k'$, $k \in [0, k_0)$;

(ii) $y_{k_0} = -3\ell_{k_0}$, $y_k = y_{k+1} + 4\ell_{k+1} - 3\ell_k$, $k \in [0, k_0)$;

(iii) $x_{k_0} = x - \ell_{k_0}/2$, $x_k = x_{k+1} + 8\ell_{k+1} + (\ell_{k+1} - 17\ell_k)/2$, $k \in [0, k_0)$;

(iv) $R_k' \cap \mathbb{Z} \times [y_k, y_k + 3\ell_k) \subset S_k^U$, $R_k' \cap \mathbb{Z} \times [y_k + \ell_k, y_k + 4\ell_k) \subset S_k^B$.

Condition (ii) allows for a $3\ell_k$ overlap between R'_{k+1} and R'_k, and condition (iii) implies that $(x,t) \in R'_0 \cap \mathbb{Z} \times [y_0 + 3\ell_0, \infty)$. The lower bound on the relaxation time is implied by the next lemma.

Lemma 4.1.5 *In η_0, we have $\Pr(\bigwedge_{k=0}^{k_0} \eta_0 \upharpoonright R'_k$ is k-white) $> \delta$. The same holds with η_1 and k-black.*

Thus an immediate corollary of Lemma 4.1.5 is that $\eta_0(x,t) = 00$ with probability at least δ if $t = O(2^{c_2 \log^2 1/\varepsilon})$. Note that R'_{k_0} is the essential trapezoid for achieving the relaxation time lower-bound, the remaining trapezoids being there to pinpoint a cell with state 00 as the stringency of the supersparsity condition is reduced. Lemma 4.1.5 will be implied by Lemma 4.1.4 and Lemma 4.1.6 which shows the existence of a shrinking region of increasing "white-consolidation" in space-time.

Lemma 4.1.6 (consolidation) *For all $k = 0, \dots, k_0 - 1$, if $\eta_0 \upharpoonright R'_{k+1}$ is $(k+1)$-white, S_{k+1}^U is $(k+1)$-supersparse, S_{k+1}^B is k-supersparse, and S_k^U is k-supersparse, then $\eta_0 \upharpoonright R'_k$ is k-white. The same holds with k-black.*

Before proving Lemma 4.1.6, we will establish a couple of useful facts. The next result states that a k-white (or k-black) configuration "quickly" returns to all-white (all-black) in the absence of errors.

Proposition 4.1.7 (attraction) *Let $k \geq 1$. Let $\eta : \mathbb{Z} \times \mathbb{N} \to S$ be a deterministic orbit such that $\eta(\cdot, 0)$ is k-white (k-black). Then $\eta(\cdot, \kappa r'_{k-1})$ is 0-white (0-black) where κ, $4d < \kappa < c$, is a fixed constant.*

Proof. The proof goes by induction on k. Let $k = 1$. Let $\mathcal{C}$ be a minimum 1-cover of $\eta(\cdot, 0)$. Note that if $\mathcal{C} = \emptyset$ then the basis is trivially true. By Theorem 1.1.4, for every $B \in \mathcal{C}$ there exists $s_B \in \mathbb{Z}$ such that

$$\eta \upharpoonright \left((\mathbb{Z} \times \mathbb{N}) \setminus \bigcup_{B \in \mathcal{C}} (s_B, 0) + W_{3dr'_{k-1}} \right)$$

is 0-white. Since $\kappa > 4d$ the basis is proven.

Assume the statement holds for $k \geq 1$. Let $\mathcal{C}$ be a minimum $(k+1)$-cover of $\eta(\cdot, 0)$. Let $B = (s_B, 0) + W_{dr'_k}$ be an element of $\mathcal{C}$. Let us consider η restricted on the space interval

$$I = (s_B + dr'_k, s_B + r'_{k+1} - dr'_k].$$

Clearly, $\eta \upharpoonright (I \times [0,0])$ is k-white.

By the inductive assumption and speed-of-light,

$$\eta \upharpoonright I' \times [\kappa r'_{k-1}, \kappa r'_{k-1}) \quad \text{is 0-white}$$

where $I' = (s_B + dr'_k + 2\kappa r'_{k-1}, s_B + r'_{k+1} - dr'_k - 2\kappa r'_{k-1}]$. Using well-separatedness and symmetry, we can apply Theorem 1.1.4 to conclude that $\eta(\cdot, \kappa r'_{k-1} + 3(dr'_k + 4\kappa r'_{k-1}))$ is 0-white. Since

$$\kappa r'_{k-1} + 3(dr'_k + 4\kappa r'_{k-1}) < 4dr'_k < \kappa r'_k,$$

the proposition follows. ∎

Lemma 4.1.8 *Let $k \geq 0$ and let η be an orbit of $K_{\varepsilon, \beta}$. Let $E = (x, y) + W_{3r'_k}$ and let $U = (x - 50r'_k, y) + W_{100r'_k}$. Let*

$$\mathcal{M} = \{(s, t) : x - dr'_k/2 + 2(t - y) \leq s < x + dr'_k/2 - 2(t - y), \ t \geq y\}.$$

If $\mathcal{M} \setminus E$ is 0-supersparse and $\eta \upharpoonright (\mathcal{M} \setminus E) \cap \mathbb{Z} \times [y, y]$ is k-white, then

$$\eta \upharpoonright (\mathcal{M} \cap \mathbb{Z} \times [y + 3r'_k, \infty)) \setminus U \quad \text{is 0-white.}$$

Proof. By the speed-of-light, $\eta \upharpoonright \mathcal{M}$ is not affected by what values η takes on on $\mathcal{M}^c \cap \mathbb{Z} \times [y, \infty)$. Let

$$\mathcal{K} = \{(s, t) : x - 2(t - y) \leq s < x + 3r'_k + 2(t - y), \ t \geq y\}.$$
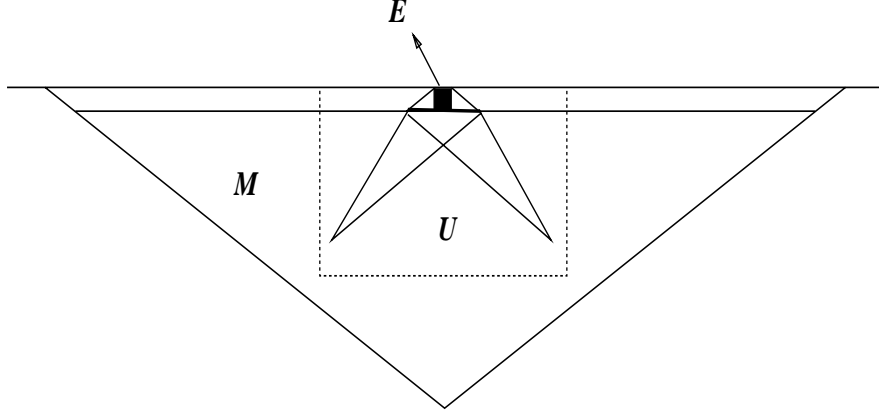
Figure 4.1.1: Error correction under k-supersparse errors.

By Proposition 4.1.7 and noting that $\kappa r'_{k-1} < 3r'_k$ (κ is the time variable in the proposition),

$$\eta \restriction (\mathcal{M} \setminus \mathcal{K}) \cap \mathbb{Z} \times [y + 3r'_k, y + 3r'_k] \quad \text{is 0-white.}$$

Theorem 1.1.4 (deterministic eroder property) implies that the error island

$$\mathcal{K} \cap \mathbb{Z} \times [y + 3r'_k, y + 3r'_k]$$

which has length $15r'_k$ is corrected within a space-time rectangle of size at most $75r'_k \times 75r'_k$. Hence the error correction process, inclusive the $(k+1)$-supersparse error E, is covered by U. For the previous arguments to hold, we must choose d sufficiently large such that $U \subset \mathcal{M}$. It is easily checked that this is the case if $d = 1000$. ∎

Figure 4.1.1 depicts the error correction process subject to a $(k+1)$-supersparse error described in the proof of Lemma 4.1.8.

Remark 4.1.9 Let (a, b) be the intersection point of the left boundaries of $\mathcal{M}$ and $\mathcal{K}$. If $d = 1000$, then $b = y + 125r'_k$ and it follows that $\eta \restriction \mathcal{M} \cap \mathbb{Z} \times [y + 100r'_k, y + 125r'_k)$ is 0-white. We will use this property in the proof of Lemma 4.1.10.

Lemma 4.1.10   *Let* $\mathcal{T}_k = (0, w_k, 0, h_k, -2)$, $\mathcal{T}'_k = (2h_k, w_k - 2h_k, h_k, h_k + h, -2)$ *where* $w_k \geq 17r'_k$, $h_k = 3r'_k$, *and* $h \geq 0$. *Let* η *be an orbit of* $K_{\varepsilon, \beta}$. *Then,*

$$\eta \restriction \mathcal{T}_k \text{ is } k\text{-white} \ \wedge \ \mathcal{T}_k \cup \mathcal{T}'_k \text{ is } k\text{-supersparse} \implies \eta \restriction (\mathcal{T}_k \cup \mathcal{T}'_k) \text{ is } k\text{-white.}$$

The same holds true if k*-white is replaced by* k*-black.*

We will call η a *k-continuation (with respect to supersparsity)* at $(2h_k, h_k)$ with *width* $w_k - 4h_k$ and *extension* h. This lemma is the main technical tool in the proof of Lemma 4.1.6, and its structure follows the inductive proof of the Expansion Lemma. However, it is much simpler due to absence of boundary effects facilitated by the two sides of the trapezoids $\mathcal{T}_k$, $\mathcal{T}'_k$ shrinking with the speed-of-light.

Proof of Lemma 4.1.10.   The proof goes by induction on k. Let $k = 0$. Since $\eta \restriction \mathcal{T}_0$ is all-white and no errors occur in $\mathcal{T}_0 \cup \mathcal{T}'_0$, the basis follows trivially from SOL.

Assume the statement holds for $k \geq 0$. Let $\mathcal{C}_b$, $\mathcal{C}'_b$, $\mathcal{C}_s$, $\mathcal{C}^*$, and $\mathcal{T}^*_k$ be defined as in the proof of the Expansion Lemma (Lemma 3.2.2) where $\mathcal{C}_b$ is now a minimum $(k+1)$-white-cover of $\mathcal{T}_{k+1}$. Let $B_i = (s_i, t_i) + W_{3r'_k}$, $i = 1, 2, \dots, n$, be the corresponding enumeration of $\mathcal{C}_s$. It suffices to prove

$$\mathcal{C}^* \text{ is a } (k+1)\text{-white-cover of } \mathcal{T}^*_{k+1} \tag{4.1.11}$$

since (4.1.11) implies that $\eta \restriction (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})$ is $(k+1)$-white. (See Claim I in the proof of the Expansion Lemma with k-black in place of k-white.)

The proof of (4.1.11) goes by induction on the size of $\mathcal{C}_s$. Assume $n = |\mathcal{C}_s| = 0$. If $|\mathcal{C}'_b| = 0$, then η is a k-continuation at $(2h_k, h_k)$ with width $w_{k+1} - 4h_k$ and extension $h_{k+1} + h - h_k$. By the inductive assumption on k,

$$\eta \restriction (2h_k, w_{k+1} - 2h_k, h_k, h_{k+1} + h, -2)$$

is k-white from which (4.1.11) follows.

Assume $|\mathcal{C}'_b| > 0$. Let $(s^*, t^*) + W_{dr'_k} \in \mathcal{C}'_b$ be an element such that t^* is maximal. Let $\mathcal{K} = \mathcal{T}_{k+1} \cap \mathbb{Z} \times [t^* + dr'_k, t^* + dr'_k + h_k)$.

Claim I $\eta \upharpoonright \mathcal{K}$ is k-white.

Proof: Let $D = (x,y) + W_{r'_k - 2dr'_{k-1}}$ be any test window such that $H = D \cap \mathcal{K} \neq \emptyset$. Since $B \in \mathcal{C}'_b \implies B \cap H = \emptyset$, we only need consider $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$ such that $B \cap H \neq \emptyset$. Let $B = (a,b) + W_{dr'_k}$ be such an element. Well-separatedness, $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$, and $\mathcal{C}_b$ being a $(k+1)$-white-cover imply that η is a k-continuation at (a',b), $a' = \max\{a - 3r'_k - 2dr'_k, 2b\}$, with width $\ell = \min\{5dr'_k + 6r'_k, w_{k+1} - 2b - a'\}$ and extension $dr'_k + h_k$. Hence,

$$\eta \upharpoonright (a' - 2r'_k, a' + \ell + 2r'_k, b - r'_k, b + r'_k + dr'_k + h_k, -2)$$

is k-white. Since $H \subset (a' - 2r'_k, a' + \ell + 2r'_k, b - r'_k, b - r'_k + dr'_k + h_k, -2)$, it follows that $\eta \upharpoonright H$ is k-white. ▲

Claim I implies that η is a k-continuation at $(2(t^* + dr'_k + h_k), t^* + dr'_k + h_k)$ with width $w_{k+1} - 4(t^* + dr'_k + h_k)$ and extension $h_{k+1} + h - t^* - dr'_k - h_k$. Hence, $\eta \upharpoonright \mathcal{T}^*_{k+1}$ is k-white.

Assume (4.1.11) holds for $|\mathcal{C}_s| = n \geq 0$. Let $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r'_k}$ be the last element in the enumeration of $\mathcal{C}_s$. If $t_{n+1} < h_{k+1} - r'_{k+1} - dr'_k - h_k - 3r'_k$ then

$$(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1} - dr'_k - h_k, \infty) \quad \text{is } k\text{-supersparse},$$

and hence an argument analogous to the proof of Claim I can be applied to the smaller trapezoid $\mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1} - dr'_k - h_k, \infty)$ to conclude that

$$\eta \upharpoonright \mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1}, h_{k+1})$$

is k-white from which (4.1.11) follows.

Let $t_{n+1} \geq h_{k+1} - r'_{k+1} - dr'_k - h_k - 3r'_k$. If $B_{n+1} \subset (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c$, then by SOL and the inductive assumption on $|\mathcal{C}_s|$, (4.1.11) holds for $|\mathcal{C}_s| = n + 1$. Let $B_{n+1} \cap (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \neq \emptyset$. Note that since $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})$ is independent of

$$\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty),$$

we may assume any values for $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty)$, in particular, all-white, without affecting the analysis. By the same reason, we may view $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty)$ as being 0-supersparse. Let

$$\mathcal{M} = \{\, (s,t) : s_{n+1} - dr'_k/2 + 2(t - t_{n+1}) \le s < s_{n+1} + dr'_k/2 - 2(t - t_{n+1}),\ t \ge t_{n+1} \,\}.$$

Claim II $\eta \upharpoonright (\mathcal{M} \setminus B_{n+1}) \cap \mathbb{Z} \times [t_{n+1}, t_{n+1}]$ is k-white.

Proof: If $t_{n+1} \ge h_{k+1}$, then by the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-white-cover of $\eta \upharpoonright \mathcal{T}^*_{k+1} \cap \mathbb{Z} \times (-\infty, t_{n+1} - 1]$, and by well-separatedness and SOL, the claim follows. Let $t < h_{k+1}$ and assume for some $B = (a,b) + W_{dr'_k} \in \mathcal{C}_b$,

$$B \cap (\mathcal{M} \setminus B_{n+1}) \cap \mathbb{Z} \times [t_{n+1}, t_{n+1}] \ne \emptyset.$$

Well-separatedness implies that η is a k-continuation at $(a - 2dr'_k - 2r'_k, b)$ with width $5dr'_k + 4r'_k$ and extension $t_{n+1} - b$ from which the claim follows. ▲

Using Lemma 4.1.8, an immediate consequence of Claim II is that

$$\eta \upharpoonright (\mathcal{M} \cap \mathbb{Z} \times [t_{n+1} + 3r'_k, \infty)) \setminus U \quad \text{is 0-white} \tag{4.1.12}$$

where $U = (s_{n+1} - 50r'_k, t_{n+1}) + W_{100r'_k}$. Let

$$\mathcal{K} = \{\, (s,t) : s_{n+1} - 2(t - t_{n+1}) \le s < s_{n+1} + 3r'_k + 2(t - t_{n+1}),\ t \ge t_{n+1} \,\}.$$

Let $v = 2dr'_k + 2h_k + 2(dr'_k - 100r'_k - h_k) + 2h_k$.

Claim III η is a k-continuation at $(s_{n+1} - v, t_{n+1} + 100r'_k + h_k)$ with width $2v + 3r'_k$ and extension $dr'_k - 100r'_k$.

Proof: We need to show that η is k-white on the trapezoid

$$\mathcal{A} = (s_{n+1} - v - 2h_k, s_{n+1} + v + 3r'_k + 2h_k, t_{n+1} + 100r'_k, t_{n+1} + 100r'_k + h_k, -2).$$

Let $D = (a,b) + W_{r'_k - 2dr'_{k-1}}$ be a test window such that $H = D \cap \mathcal{A} \ne \emptyset$. By the definition of v, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{A}$,

$$H \subset \mathcal{K}^c \quad \text{or} \quad H \subset \mathcal{M}.$$

If $H \subset \mathcal{K}^c$, then by well-separatedness and the inductive assumption on $|\mathcal{C}_s|$, $\eta \restriction H$ is k-white. If $H \subset \mathcal{M}$, then by (4.1.12) and Remark 4.1.9, $\eta \restriction H$ is 0-white. ▲

Claim IV $\mathcal{C}^*$ is a $(k+1)$-white cover of $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [0, t_{n+1} + dr'_k + h_k)$.

Proof: Let $D = (a,b) + W_{r'_k - 2dr'_{k-1}}$ be a test window such that

$$H = D \cap \left((\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [0, t_{n+1} + dr'_k + h_k) \setminus \bigcup_{C \in \mathcal{C}^*} C \right) \neq \emptyset.$$

Let $\mathcal{A}'$ be the same trapezoid as $\mathcal{A}$ in the proof of Claim III except that its second time parameter is changed from $t_{n+1} + 100r'_k + h_k$ to $t_{n+1} + dr'_k + h_k$. By the definition of $\mathcal{K}$ and $\mathcal{A}'$,

$$H \subset \mathcal{K}^c \quad \text{or} \quad H \subset \mathcal{A}'.$$

In either case, $\eta \restriction H$ is k-white which proves the claim. ▲

Since $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r'_k}$ is the last element in the enumeration of $\mathcal{C}_s$, Claim IV implies that η is a k-continuation at $(2(t_{n+1} + dr'_k + h_k), t_{n+1} + dr'_k + h_k)$ with width $w_{k+1} - 4(t_{n+1} + dr'_k + h_k)$ and extension $h_{k+1} + h - (t_{n+1} + dr'_k + h_k)$. It follows that $\mathcal{C}^*$ is a $(k+1)$-white-cover of $\mathcal{T}^*_{k+1}$. ∎

Proof of Lemma 4.1.6. First, S^B_{k+1} being k-supersparse implies that $R'_{k+1} \cap \mathbb{Z} \times [y_{k+1} + r_{k+1}, \infty)$ is k-supersparse. Since the last element of $\mathcal{C}_s$, $B_n = (s_n, t_n) + W_{3r'_k}$, has $t_n < y_{k+1} + r_{k+1}$, it is easy to deduce from the proof of Lemma 4.1.10 that $\eta \restriction R'_{k+1} \cap \mathbb{Z} \times [y_{k+1} + 2r_{k+1}, \infty)$ is k-white. Since R'_k is k-supersparse and R'_{k+1}, R'_k overlap by $3r'_k$, it follows by Lemma 4.1.10 that $\eta \restriction R'_k$ is k-white. ∎

Note that for $k = k_0$, ε needs to be chosen sufficiently small such that $2^{c_2 \log^2 1/\varepsilon} \geq r'_{k_0}$. This has the effect of making the width w_{k_0} and extension h in Lemma 4.1.6 large when applying the lemma to R'_{k_0}, S'_{k_0}. The height of the context, h_{k_0}, may be kept at r'_{k_0}.

4.2 Relaxation upper bound

The upper bound will follow from a lower bound to $\Pr(\mathcal{E}_1(c_0\varepsilon^{-1/2}, T))$ as a function of T. For this, we will use the fact that in the absence of bad errors, an "approximately" well-placed good error can increase the size of the island by a constant factor. Therefore an island of size $c_0\varepsilon^{-1/2}$ arises from $O(\log(1/\varepsilon))$ well-placed good errors, and the probability for this to happen will be of the order of $\varepsilon^{c_2\log(1/\varepsilon)} = 2^{-c_2\log^2(1/\varepsilon)}$ for some constant $c_2 > 0$. If we wait until $2^{c_2\log^2(1/\varepsilon)}$ we will see such an event with constant probability, i.e., $\mathcal{E}_1(c_0\varepsilon^{-1/2}, 2^{c_2\log^2(1/\varepsilon)})$ has constant probability.

Let us define the window size

$$r_k = c_1 c^k$$

where $c_1 > c > 1$ are fixed constants. Consider the initial configuration ξ where $\xi(s) = 11$ for $s \leq 0$ and $\xi(s) = 00$ for $s > 0$. Let φ be a sample of $K_{\varepsilon,\beta}$ with initial configuration ξ such that at the sequence of space-time points $(T_i, T_i), i = 0, 1, 2, \ldots$ an error event setting $\varphi(T_i, T_i) = 11$ occurs but otherwise φ is error-free. By Proposition 3.3.1, a left-moving black cone arises at these space-time points. Fixing (T_0, T_0), the sequence (T_i, T_i) is inductively defined as follows: For $i \geq 1$, (T_i, T_i) is the unique point such that the cones starting at (T_{i-1}, T_{i-1}) and (T_i, T_i) meet for the first time at site 0. A simple calculation shows that

$$T_i = \left(\frac{4}{3}\right)^i T_0.$$

The maximum width of the cone starting at (T_i, T_i) is given by $w_i = T_i/2$.

To amplify the probability of φ occurring for $i = O(\log(1/\varepsilon))$, we define a growing area A_i in which at least one good error is required to occur. We will define a new sequence of space-time points (T_i', T_i') lying adjacent to the diagonal $s = t$ as follows. Let $T_0' > 0$ be given. Let $A_0 = (T_0', T_0' - \alpha T_0') + W_{\alpha T_0' \times \alpha T_0'}$ where $0 < \alpha < 1$ is a constant. Let $a_0^u = (T_0', T_0' - \alpha T_0')$ be the upper-left corner of A_0 and let $a_0^l = (T_0' + \alpha T_0', T_0')$ be the lower-right corner point. Define a cone C_0 induced by the two lines passing

through a_0^u, a_0^l with slope -1 and -2, respectively:

$$C_0 = \{\, (s,t) : -2(t - T_0' - \alpha T_0') + T_0' \leq s \leq -(t - T_0') + T_0' - \alpha T_0',\ s \geq 0 \,\}.$$

For $i \geq 1$, (T_i', T_i') is defined to be the unique space-time point such that C_{i-1} and C_i meet for the first time at site 0 where $A_i = (T_i', T_i' - \alpha T_i') + W_{\alpha T_i' \times \alpha T_i'}$. A pictorial depiction is shown in Figure 4.2.1.

Lemma 4.2.1   *Let T_i', $i = 0, 1, 2, \ldots$ be defined as above. Then*

$$T_i' = \left(\frac{4 - 2\alpha}{3 + 2\alpha} \right)^i T_0'.$$

Moreover, $w_i' = (1 + 3\alpha)T_i'/2$.

Proof.   Assume we are given (T_i', T_i'). To find (T_{i+1}', T_{i+1}'), first note that the line passing through a_i^u with slope -1 intersects the space axis $s = 0$ at $(0, 2T_i' - \alpha T_i')$. By definition of (T_{i+1}', T_{i+1}'), a_{i+1}^l must lie on the line passing through $(0, 2T_i' - \alpha T_i')$ with slope -2,

$$s = -2(t - (2 - \alpha)T_i'). \tag{4.2.2}$$

Since $a_{i+1}^l = (T_{i+1}', T_{i+1}' + \alpha T_{i+1}')$, substituting a_{i+1}^l into equation (4.2.2) yields

$$T_{i+1}' = \frac{4 - 4\alpha}{3 + 2\alpha} T_i'.$$

The width w_i' is obtained by computing the intersection point of the lines passing through a_{i+1}^u and a_{i+1}^l with slopes -1 and -2, respectively, then halving the space coordinate. Simple calculation yields

$$((1 + 3\alpha)T_{i+1}', (1 - 2\alpha)T_{i+1}'),$$

from which $w_{i+1}' = (1 + 3\alpha)T_{i+1}'/2$ follows immediately. ∎

Thus, $\alpha < 1/4$ ensures that a constant factor expansion is attained at successive iterations. At $\alpha = 0$, we have $T_i' = T_i$ and $w_i' = w_i$. In the following, we will set

$$c = \frac{4 - 2\alpha}{3 + 2\alpha}.$$

Hence, $T_i' = c_1 c^i$ where $T_0' = c_1$.

We will use a sequence S_i of space-time rectangles where

$$S_i = (0, r_i) + W_{4r_i}.$$

Let A_i^+ be the space-time square of size $\alpha r_i \times \alpha r_i$:

$$A_i^+ = (r_i - \alpha r_i, r_i) + W_{\alpha r_i}.$$

Let A_i^- be the symmetric reflection of A_i^+ around the space axis $s = 0$. That is, $A_i^- = (-r_i, r_i) + W_{\alpha r_i}$. Given the error processes $E_{x,t,j}, B_{x,t,j}$, we will call S_i black-spotted if

(i) There are no bad errors in S_i.

(ii) There is at least one good error in each A_i^+, A_i^-, respectively.

Lemma 4.2.3 *Let $k \leq (\log 1/2c_1^2\beta + \log 1/\varepsilon)/2\log c$. There exists $c_1 > 0$ such that*

$$\Pr\left(\bigwedge_{i=0}^{k} S_i \text{ is black-spotted}\right) > e^{-a\log^2 1/\varepsilon}$$

where $a = (1 + \log 1/2c_1^2\beta)/2\log c$.

Proof. First, $\Pr(S_i \text{ contains no bad errors}) \geq 1 - \varepsilon\beta c_1^2 c^{2i}$ and

$$\Pr(A_i^+, A_i^- \text{ each contain at least one good error}) \geq \varepsilon(1 - \beta)\alpha^2 c_1^2 c^{2i}.$$

Since

$$\Pr(S_i \text{ is black-spotted}) \geq \Pr(S_i \text{ contains no bad errors}) \cdot$$

$$\Pr(A_i^+, A_i^- \text{ each contain at least one good error}),$$

we have

$$\Pr\Big(\bigwedge_{i=0}^{k} S_i \text{ is black-spotted}\Big) \;\geq\; \prod_{i=0}^{k}(1 - \varepsilon\beta c_1^2 c^{2i})(1-\beta)^k \varepsilon^k \alpha^{2k} c_1^{2k} c^{\sum_i 2i}$$

$$\geq\; \Big(\frac{\alpha^2 c_1^2}{2}\Big)^k \prod_{i=0}^{k}(1 - \varepsilon\beta c_1^2 c^{2i})\varepsilon^k c^{k^2+k}$$

$$\geq\; \Big(\frac{\alpha^2 c_1^2}{4}\Big)^k \varepsilon^k c^{k^2+k}$$

where the last inequality follows from the restriction on k. For sufficiently large c_1, $(\alpha^2 c_1^2/4)^k = 1$, and we have

$$\Pr\Big(\bigwedge_{i=0}^{k} S_i \text{ is black-spotted}\Big) \;\geq\; \varepsilon^k \;\geq\; e^{-a \log^2 1/\varepsilon}$$

where $a = (1 + \log 1/2c_1^2\beta)/2\log c$. $\blacksquare$

Theorem 4.2.4 *There are constants $c_2, \delta > 0$ such that*

$$\Pr(\mathcal{E}_1(c_0\varepsilon^{-1/2}, 2^{c_2 \log^2 1/\varepsilon})) > \delta.$$

Proof. Lemma 4.2.1 gives us the constant expansion factor $c = (4 - 2\alpha)/(3 + 2\alpha) > 1$ for $\alpha < 1/4$, and Lemma 4.2.3 restricts the number of iterations to $k \leq (\log 1/2c_1^2\beta + \log 1/\varepsilon)/2\log c$. Starting with an initial island of size $T_0' = c_1$, after k iterations

$$w_k' \;\geq\; \frac{c_1}{2} c^k \;=\; \frac{c_1}{2} c^{(\log 1/2c_1^2\beta + \log 1/\varepsilon)/2\log c}$$

$$=\; \frac{c_1}{2} e^{(\log 1/2c_1^2\beta + \log 1/\varepsilon)/2} \;=\; \frac{c_1}{2} \frac{1}{\sqrt{2c_1^2\beta}} \varepsilon^{-1/2}$$

$$=\; \frac{1}{2\sqrt{2\beta}} \varepsilon^{-1/2} \;\geq\; \frac{\varepsilon^{-1/2}}{2}$$

for $\beta \leq 1/2$.

Since we want $2w_k' \geq c_0\varepsilon^{-1/2}$, we need c_0 copies of such events occurring adjacent to each other at the same time. Let us denote this event by $\mathcal{A}$. Then

$$\Pr(\mathcal{A}) \;>\; \varepsilon^{c_0 c_1} e^{-c_0 a \log^2 1/\varepsilon} \;=\; e^{c_0 c_1 \log \varepsilon} e^{-c_0 a \log^2 1/\varepsilon} \;\geq\; e^{-c_0(a+c_1) \log^2 1/\varepsilon}$$

where $a = (1+\log 1/2c_1^2\beta)/2\log c$ and we have used Lemma 4.2.3. Thus $c_2 = c_0(a+c_1)$ which completes the proof. ∎
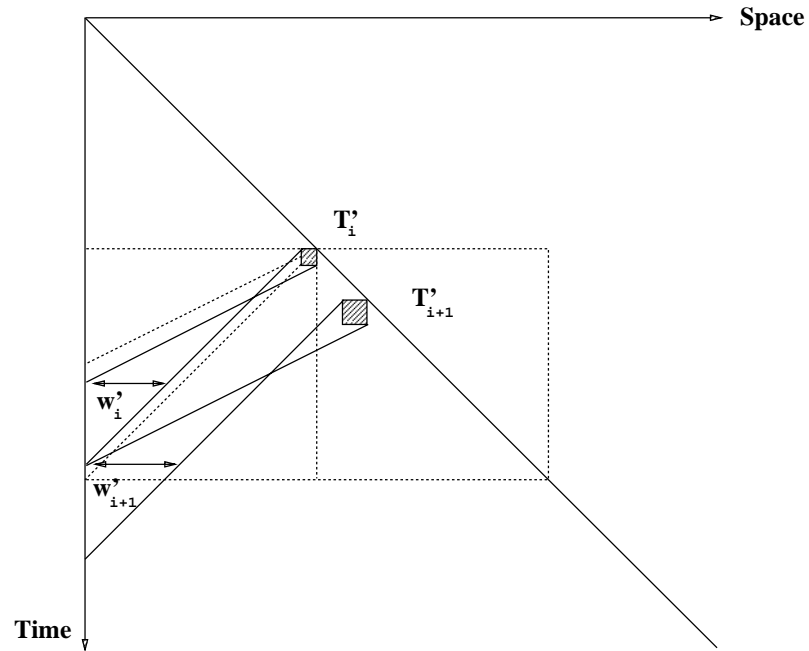

Figure 4.2.1: Constant factor growth of black island due to approximately placed single error.

Chapter 5

GKL rule

5.1 Proof structure of mixing

The proof structure is similar to two-line voting except that the nonmonotonic nature of the GKL rule forces us to consider the behavior of all trajectories η^ξ, $\xi \in S^{\mathbb{Z}}$, rather than just the two $\eta^{\zeta_{00}}$, $\eta^{\zeta_{11}}$ afforded by the "sandwiching property" in the monotonic case. (Note, we are using $S = \{0, 1\}$ as the state set instead of $\{-1, 1\}$.) One further difference is the presence of two different interfaces, one behaving similarly to the interface of two-line voting, and the other resembling a simple random walk.

Let us define a system of trajectories of the probabilistic cellular automaton $L_{\varepsilon,\beta}$ joined by a common probability space using basic coupling analogous to two-line voting. First, for all x, t we independently toss a coin $E_{x,t}$ with the property

$$\Pr(E_{x,t} = u) = \begin{cases} 1 - \varepsilon & \text{if } u = 0, \\ \varepsilon & \text{if } u = 1. \end{cases}$$

This is followed by a coin $B_{x,t}$ which will be 1 with probability $1 - \beta$ and 0 with probability β. Now, for each $\eta = \eta^\xi$, $\xi \in S^{\mathbb{Z}}$, we proceed as follows. Suppose that $\eta(\cdot, t)$ is defined up to $t > 0$. We will define it for $t+1$. If $E_{x,t+1} = 1$ then the definition obeys the deterministic transition given by (1.1.1). Otherwise, $\eta(x, t + 1) = B_{x,t+1}$.

Proposition 5.1.1 *If there is a $\delta_1 > 0$ such that $\forall n \in \mathbb{N}$, $\exists t_0 > 0$, $\forall t > t_0$, $\forall \xi, \xi' \in S^{\mathbb{Z}}$*

$$\Pr(\eta^{\xi}(x,t) = \eta^{\xi'}(x,t), \ -n \le x \le n) > \delta_1, \qquad (5.1.2)$$

then $L_{\varepsilon,\beta}$ is mixing.

Proof. It is sufficient to show that the supposition implies

$$\lim_{t \to \infty} \Pr(\eta^{\xi}(x,t) = \eta^{\xi'}(x,t), \ -n \le x \le n) = 1.$$

Consider a sequence of trapezoids $C_i = (-w_i, w_i, u_i, u_i + t_0(w_i), 3)$, $i = 1, \dots, m$,

$$\{ (x,t) : u_i - t_0(w_i) \le t \le u_i, \ -w_i + 3t < x < w_i - 3t \}$$

going backward in time such that

$$w_1 = n,$$

$$w_{i+1} = w_i + 3t_0(w_i), \qquad 0 < i \le m-1,$$

$$u_1 > 0,$$

$$u_{i+1} = u_i - t_0(w_i), \qquad 0 < i \le m-1,$$

where $t_0(\cdot)$ is the time lower bound function. (See Figure 5.1.1.) Clearly, for any $m > 0$, we can choose $u_1 > 0$ sufficiently large such that

$$C_i \subseteq \mathbb{Z} \times [t_0(w_m), \infty), \quad i = 1, \dots, m.$$

Let $\mathcal{A}_i$ denote the event

$$\eta^{\xi}(x, u_i) = \eta^{\xi'}(x, u_i), \quad -w_i \le x \le w_i.$$

Using the assumption, we have $\Pr(\neg \mathcal{A}_i \mid \mathcal{B}_i(\zeta, \zeta')) \le 1 - \delta_1$, for all $\zeta, \zeta' \in S^{\mathbb{Z}}$, where $\mathcal{B}_i(\zeta, \zeta')$ is the event

$$\eta^{\xi}(\cdot, u_i - t_0(w_i)) = \zeta \quad \wedge \quad \eta^{\xi'}(\cdot, u_i - t_0(w_i)) = \zeta'.$$
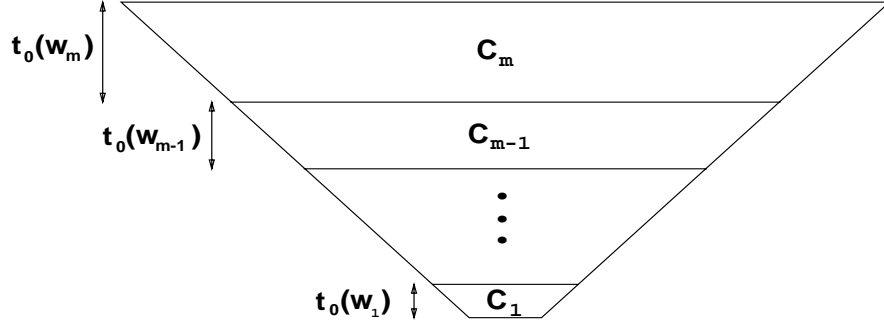
Figure 5.1.1: Amplification trapezoids.

Hence,

$$\Pr\left(\bigwedge_{i=1}^{m} \neg \mathcal{A}_i \;\Big|\; \mathcal{B}_i(\zeta, \zeta')\right) \;\leq\; (1 - \delta_1)^m \;\leq\; e^{-\delta_1 m}.$$

However, by SOL, if $\mathcal{A}_i$ holds for some $1 < i \leq m$, then $\mathcal{A}_0$ must hold as well. Thus,

$$\Pr(\mathcal{A}_0) \;>\; 1 - e^{-\delta_1 m},$$

and $\Pr(\mathcal{A}_0) \to 1$ as $m \to \infty$ $(t \to \infty)$. ■

We will prove (5.1.2) in the following way. Let $\mathcal{E}_0(n, s, t)$ be the event

$$E_{x,t} = 1 \;\wedge\; B_{x,t} = 1, \qquad x \in [s - n, s + n],$$

and let $\mathcal{E}_1(n, s, t) = \bigcup_{t' \leq t} \mathcal{E}_0(n, s, t)$. That is, $\mathcal{E}_i(n, s, t)$, $i = 1, 2$, correspond to the $\mathcal{E}_i$'s of two-line voting translated in space by $s \in \mathbb{Z}$.

Lemma 5.1.3 *There are $c_0, \delta_2 > 0$ such that $\forall \xi, \xi' \in S^{\mathbb{Z}}$, $\forall s \in \mathbb{Z}$, $\forall t \geq 0$, $\forall \varepsilon > 0$,*

$$\Pr(\eta^{\xi}(x, t) = \eta^{\xi'}(x, t), \; s + t/8 \leq x \leq s + t/2 \mid \mathcal{E}_1(c_0 \varepsilon^{-1/2}, s, t)) > \delta_2.$$

Notice that the trapezoid of agreement is skewed in the positive direction with a net expansion factor of $3/8$.

Proposition 5.1.4 *Lemma 5.1.3 $\implies$ (5.1.2).*

Proof. Since $\mathcal{E}_1(c_0\varepsilon^{-1/2}, s, T)$ has constant probability lower bound for $T \geq \varepsilon^{-c_0\varepsilon^{-1/2}}$,

$$\delta_1 = \delta_2 \Pr(\mathcal{E}_1(c_0\varepsilon^{-1/2}, s, T)).$$

For any n, let

$$t_0 = T + \frac{16}{3}n.$$

Since the skewed trapezoids in Lemma 5.1.3 expand linearly with a net speed of $3/8$, the width of agreement at time t_0 is at least $2n$. Clearly, for each $t > t_0$, we can choose $s \in \mathbb{Z}$ such that

$$[-n, n] \times t \subset B$$

where B is a skewed trapezoid emanating from $(s, 0) + [0, c_0\varepsilon^{-1/2}) \times T$ (due to $\mathcal{E}_1(c_0\varepsilon^{-1/2}, s, T)$) on which all η^ξ, $\xi \in S^\mathbb{Z}$, concur. ∎

Lemma 5.1.3 will be implied by the following two lemmas, both of which depend on a sequence $R_i = (y_i, z_i, u_i, v_i, q_i, p_i)$, $i = 0, 1, \ldots$ of *skewed trapezoids*

$$\{\, (x, t) : u_i \leq t \leq v_i, \ y_i + q_i(t - u_i) < x < z_i + p_i(t - u_i) \,\},$$

$0 < q_i < p_i$, forward-linked in space-time. Let $(b_i)_{i \in \mathbb{N}}$ be an increasing sequence to be defined later. We will say that (R_i, b_i) is *forward-linked* to (R_{i+1}, b_{i+1}) if

$$\mathbb{Z} \times [\infty, u_{i+1} + b_{i+1}) \cap R_{i+1} \subset R_i(b_{i+1}/2).$$

Figure 5.1.2 depicts three forward-linked skewed trapezoids where the shaded region is the region of agreement. Analogously to two-line voting, we will define a sequence S_k, $k \in \mathbb{N}$, of space-time rectangles such that $R_k \subset S_k$.

Lemma 5.1.5 (agreement) *For all $\xi, \xi' \in S^\mathbb{Z}$, if $\eta^\xi, \eta^{\xi'}$ are k-black on skewed trapezoid R_k then $\forall (x, t) \in R_k(b_k)$,*

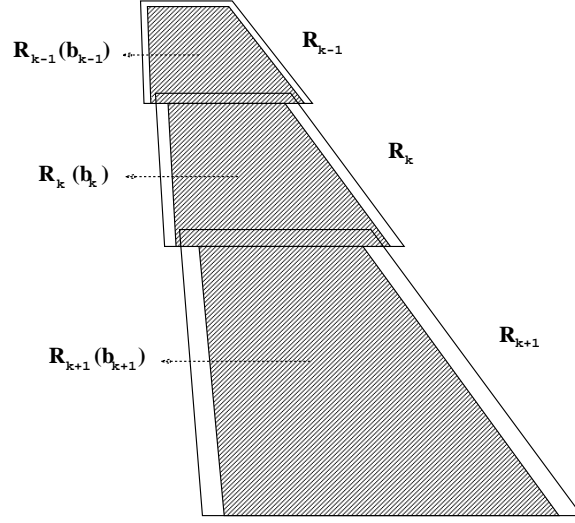$$\eta^\xi(x, t) = \eta^{\xi'}(x, t).$$

Figure 5.1.2: Forward-linked skewed trapezoids R_{k-1}, R_k, R_{k+1}. The enclosed, shaded trapezoids represent regions of agreement $R_{k-1}(b_{k-1})$, $R_k(b_k)$, $R_{k+1}(b_{k+1})$.

Lemma 5.1.6   *There are $c_0, \delta_2 > 0$ such that for all $\varepsilon > 0$, $\xi \in S^{\mathbb{Z}}$,*

$$\Pr\Big(\bigwedge_{k=0}^{\infty} \eta^{\xi} \restriction R_k \text{ is } k\text{-black} \;\Big|\; \mathcal{E}_0\big(c_0 \varepsilon^{-1/2}, 0\big) \Big) > \delta_2.$$

The following two lemmas will finish the proof.

Lemma 5.1.7   *There exists $\beta < 1/2$ such that for all $\varepsilon > 0$,*

$$\Pr\Big(\bigwedge_{k=0}^{\infty} S_k \text{ is } k\text{-sparse with black-bias } \beta \Big) > \delta_2.$$

Lemma 5.1.8 (inheritance)

(a) *There is a c_0 such that for all $\xi \in S^{\mathbb{Z}}$ if $\mathcal{E}_0(c_0 \varepsilon^{-1/2}, 0)$ holds and S_0 is 0-sparse then $\eta^{\xi} \restriction R_0$ is 0-black.*

(b) *For all $\xi \in S^{\mathbb{Z}}$, $k \in \mathbb{N}$, if $\eta^{\xi} \restriction R_k$ is k-black and S_{k+1} is $(k+1)$-sparse then $\eta^{\xi} \restriction R_{k+1}$ is $(k+1)$-black.*

5.2 Sparsity and blackishness

Given the error processes $E_{x,t}$, $B_{x,t}$, we will say a *good error* has occured at (x,t) if

$$E_{x,t} = 1 \ \wedge \ B_{x,t} = 1,$$

and there is a *bad error* at (x,t) if $E_{x,t} = 1 \wedge B_{x,t} = 0$. Sparsity is defined analogously to two-line voting. The only difference is that there is one less index. The window size is given by

$$r_k \ = \ c_1 \varepsilon^{-1/2} c^k (k!)^2$$

for $k = 0, 1, 2, \ldots$. The constants, including c_0, d, will be slightly different from the two-line voting case; however, otherwise, there are no essential differences.

The sparsity lemma has the same form as before (Lemma 3.1.2). It is stated again for $L_{\varepsilon,\beta}$.

Lemma 5.2.1 (sparsity) $\forall c > 1$, $\exists C_1 > 0$, $0 < \beta_0 < 1/2$ *such that* $\forall 0 < \beta < \beta_0$, $0 < \varepsilon \leq 1$, $c_1 > C_1$, $k \in \mathbb{N}$, *the following holds. Let* η *denote an* ε*-perturbation of* L, $L_{\varepsilon,\beta}$. *Let* $A = \bigcup_{i \in [1,N]} (a_i, b_i) + W_{r_k}$, $a_i, b_i \in \mathbb{Z}$, *where* $\eta \upharpoonright A$ *is defined. Let* q_k *be the probability that* η *is not* k*-sparse on* A. *Then*

$$q_k \ < \ N\gamma^{2^{k-1}+(k+1)/2}$$

where $0 < \gamma < 1$ *is a constant depending only on* c.

Remark 5.2.2 The proof is exactly the same as for $K_{\varepsilon,\beta}$ except for a missing constant 2 in the basis case. Specifically, the upper bounding calculation of $\Pr(\eta$ is not 0-sparse on $W^i)$ is replaced by

$$\Pr(\eta \text{ is not 0-sparse on } W^i) \ \leq \ (c_1 \varepsilon^{-1/2})^2 \varepsilon \beta + (1 - \varepsilon(1 - \beta))^{(c_1 \varepsilon^{-1/2})^2}$$
$$\leq \ c_1^2 \beta + e^{-c_1^2(1-\beta)} \ < \ \gamma,$$

which again holds for c_1 sufficiently large and β sufficiently small.

Let us define the sequences R_k, b_k, and S_k for $k \in \mathbb{N}$. First, $b_k = dr_k$ where $d = 150$ is a constant used in the definition of k-black in Section 3.2. The skewed trapezoid $R_k = (y_k, z_k, u_k, v_k, q_k, p_k)$ is given by

$$z_k - y_k = (50 + 2d)r_k,$$

$$v_k - u_k = 8(80 + 3d)r_{k+1}/3,$$

$$q_k = 1/8 - 1/(k+8),$$

$$p_k = 1/2 + 1/(k+2.5).$$

Based on where R_0 is located in space-time, R_k is placed such that it is forward-linked to R_{k-1} for $k \geq 1$. Given R_k, S_k is the space-time rectangle of size

$$56(80 + 3d)r_{k+1}/3 \times 8(80 + 3d)r_{k+1}/3$$

such that $R_k \subset S_k$ is centered within S_k at the top ($t = u_k$). Since $r_{k+1}/r_k = c(k+1)^2$, S_k can be expressed as the disjoint union of $448(80 + 3d)^2 c^2 (k+1)^4/9$ space-time windows W_{r_k}.

Proof of Lemma 5.1.7.   We will show that Lemma 3.1.2 $\implies$ Lemma 5.1.7. To lower-bound $\Pr(\bigwedge_{k=0}^{\infty} S_k$ is k-sparse with bias $\beta)$, let us upper-bound the probability of its complement event $\Pr(\exists k : S_k$ is not k-sparse with bias $\beta)$.

$$
\begin{aligned}
\Pr(\exists k : S_k \text{ is not } k\text{-sparse with bias } \beta) &\leq \sum_{k=0}^{\infty} \Pr(S_k \text{ is not } k\text{-sparse with bias } \beta) \\
&< \sum_{k=0}^{\infty} N_0(k+1)^4 \gamma^{2^{k-1}+k/2+1/2} \\
&= \sum_{k=0}^{\infty} N_0 \gamma^{2^{k-1}+k/2+1/2-\alpha \log(k+1)} \\
&< \frac{N_0 \gamma}{1 - \gamma} < 1 - \delta_2
\end{aligned}
$$

where $\alpha = -4/\log \gamma$, $N_0 = 448(80 + 3d)^2 c^2/9$, and we have used Lemma 3.1.2 with $N = N_0(k+1)^4$ for each $k \geq 0$. For γ sufficiently small, $\alpha < 1/2$, and the last two inequalities hold. ∎

5.3 Spreading of blackishness

The definition of k-black is the same as in two-line voting. The next lemma is the main technical lemma in the mixing proof of the GKL rule.

Lemma 5.3.1 (expansion) *Let $\beta_k^L = 1/8 - 1/(k+8)$, $\beta_k^R = 1/2 + 1/(k+2.5)$. Let $\mathcal{T}_k = (0, w_k, 0, h_k, \beta_k^L, \beta_k^R)$ be a skewed trapezoid where $w_0 \geq 50r_0$, $h_0 = 3r_0$, and $w_k \geq r_k$, $h_k = 3r_k$ for $k > 0$. Let*

$$\mathcal{T}_k' = (\beta_k^L h_k, w_k + \beta_k^R h_k, h_k, h_k + h, \beta_k^L, \beta_k^R)$$

where $h \geq 0$. Let $U_k = (-2r_k, w_k + 2r_k, 0, h_k + h, 0, 1)$. Then, $\forall \xi \in S^{\mathbb{Z}}$,

$$\eta^\xi \upharpoonright \mathcal{T}_k \text{ is } k\text{-black} \;\; \wedge \;\; U_k \text{ is } k\text{-sparse} \;\; \implies \;\; \eta^\xi \upharpoonright (\mathcal{T}_k \cup \mathcal{T}_k') \text{ is } k\text{-black}.$$

We will say that η^ξ is a *k-continuation* at $(\beta_k^L h_k, h_k)$ with *width* $w_k + (\beta_k^R - \beta_k^L)h_k$ and *extension* h. Notice that this completely determines $\mathcal{T}_k$, $\mathcal{T}_k'$, and U_k. We will call $\mathcal{T}_k$ the *context* of the k-continuation.

Lemma 5.3.2 (bootstrap) *Let φ be a sample path of $L_{\varepsilon,\beta}$ such that $\varphi(s, 0) = 11$ for $s \in [0, \ell)$ where $\ell \geq 50r_0$. Let $(-2r_0, \ell + 2r_0, 0, h, 0, 1)$ be 0-sparse. Then $\varphi \upharpoonright \mathcal{K}$ is 0-black where*

$$\mathcal{K} = \big\{ (s, t) : 0 \leq s < \ell - 14r_0 + t, \; 0 \leq t < h \big\}.$$

Proof. The proof is exactly analogous to the two-line voting case. The only two differences are, one, that the left-moving black cones now travel with speed 3 at the front instead of 2 which only increases the volume of the backward cone T' in the proof of Lemma 3.2.4; and, two, that the left boundary remains stationary. In fact, it expands slightly toward the left, a property which we are not going to exploit. ∎

Proof of Lemma 5.1.8. Lemma 5.3.2 $\implies$ Lemma 5.1.8 (a). Consider a sample path φ of $L_{\varepsilon,\beta}$ with $\mathcal{E}_0(c_0 \varepsilon^{-1/2}, 0)$ where $c_0 = (80 + 2d)c_1$. Let $\mathcal{K}(\ell)$ denote the trapezoid

in Lemma 5.3.1 with width ℓ, height $h = 8(80 + 3d)r_1/3$, centered at 0. Since $R_0 = (y_0, z_0, u_0, v_0, 0, 0.9)$ where

$$z_0 - y_0 = (50 + 2d)c_1\varepsilon^{-1/2},$$

$$v_0 - u_0 = 8(80 + 3d)r_1/3,$$

$R_0 \subset \mathcal{K}(c_0\varepsilon^{-1/2})$. S_0 is 0-sparse and its width $56(80 + 3d)r_1/3$ satisfies the width requirement in the supposition of Lemma 5.3.2:

$$(50 + 2d)c_1\varepsilon^{-1/2} + 32(80 + 3d)r_1/3 < 56(80 + 3d)r_1/3.$$

Hence, $\varphi \restriction \mathcal{K}(c_0\varepsilon^{-1/2})$ is 0-black which by containment implies $\varphi \restriction R_0$ is 0-black.

Lemma 5.3.1 $\implies$ Lemma 5.1.8 (b). The proof goes by induction on k. Since R_k being k-black trivially implies it is $(k+1)$-black, the implication holds if (R_k, b_k) being forward-linked to (R_{k+1}, b_{k+1}) satisfies the supposition of Lemma 5.3.1. Let $k = 0$. Part (a) has shown that $\varphi \restriction R_0$ is 0-black. By the definition of forward-linkedness, it is easily checked that φ is a 1-continuation at $(y_1 - \alpha_1, u_1)$ with parameters $(z_1 - y_1 + 2\alpha_1, 3r_k, v_1 - u_1)$. Hence, $\varphi \restriction R_1$ is 1-black. Since $v_1 - u_1 = 8(80 + 3d)r_2/3$ and the net expansion factor is at least $3/8$, i.e.,

$$\beta_k^R - \beta_k^L = \left(\frac{1}{2} + \frac{1}{k + 2.5}\right) - \left(\frac{1}{8} - \frac{1}{k + 8}\right) > \frac{3}{8},$$

the width of R_1 at time v_1 is at least $(80 + 3d)r_2$. The previous argument applies to any $k > 0$ which carries the induction step. ∎

Lemma 5.3.3 (error-correction) *Let φ be a k-continuation at both $(0,0)$ and $(a, 0)$ with extension h and widths ℓ, ℓ', respectively. Let $a > \ell$ and $h = a - \ell + dr_k$. Let*

$$\mathcal{L} = \{ (s, t) : \beta_k^L t < s < \beta_k^R t + a + \ell', \ 0 \leq t < h \},$$

$$B = (\ell - r_k, 0) + W_{8(a-\ell)/3 + 2r_k}.$$

Then $\varphi \restriction \mathcal{L} \setminus B$ is k-black.

Fact 5.3.4 *For each $k \geq 0$, Lemma 5.3.1 $\Longrightarrow$ Lemma 5.3.3.*

Proof. Let

$$\mathcal{K} \;=\; \{\, (s,t) : \beta_k^L t < s < \beta_k^R t + \ell,\; 0 \leq t < h \,\},$$

$$\mathcal{K}' \;=\; \{\, (s,t) : \beta_k^L t + a < s < \beta_k^R t + a + \ell,\; 0 \leq t < h \,\}.$$

By Lemma 5.3.1, φ is k-black on $\mathcal{K}$ and $\mathcal{K}'$.

Let (x,y) be the intersection point of the two lines given by

$$s \;=\; t/2 + \ell,$$

$$s \;=\; t/8 + a.$$

Calculation of the intersection point yields

$$(x,y) \;=\; ((4a - \ell)/3,\; 8(a - \ell)/3).$$

Since

$$\{\, (s,t) : t/2 + \ell < s < t/8 + a \,\} \;\subset\; B$$

for $t \geq 0$, and

$$(s,t) + W_{r_k} \subset \mathcal{L} \setminus B \quad \Longrightarrow \quad (s,t) + W_{r_k} \subset \mathcal{K} \;\lor\; (s,t) + W_{r_k} \subset \mathcal{K}',$$

noting $\beta_k^L < 1/8$, $\beta_k^R < 1/2$, it follows that $\varphi \restriction \mathcal{L}$ is k-black. The last implication holds since the size of B was chosen $2r_k$ larger than necessary to cover the correction process. The width requirement on ℓ, ℓ' is independent of the size of the error island $a - \ell$ since the widths of K, K' both expand in time. ∎

Figure 5.3.1 is a depiction (not drawn to scale) of the error-correction process facilitated by Lemma 5.3.3. We will be interested in the case when $a - \ell = 21 r_k$ which accounts for the worst-case spreading effect of a $3r_k \times 3r_k$ error window under the speed-of-light 3. The exclusion window that covers the error effect is contained in
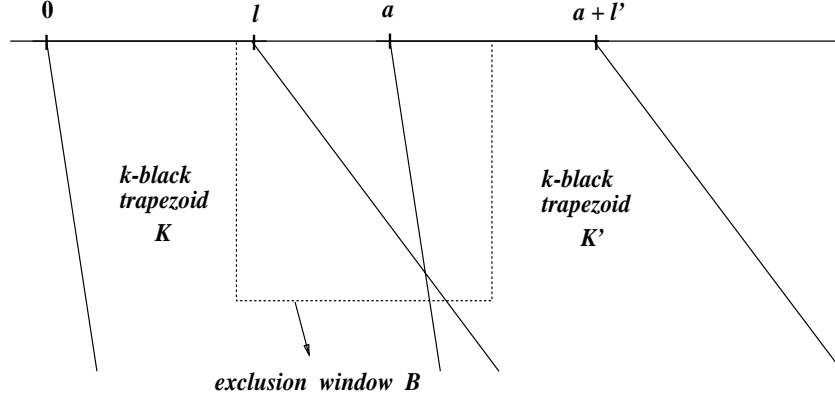
Figure 5.3.1: Error-correction Lemma in action: error region sandwiched between the k-black trapezoids K, K' is eaten up; $\varphi \upharpoonright (K \cup K') \setminus B$ is k-black.

$(\ell - 16r_k, 0) + W_{60r_k}$ since

$$8(a - \ell)/3 + 2r_k = 56r_k + 2r_k < 60r_k$$

where we have substituted $21r_k$ for $a - \ell$. The additional $2r_k$ in the window size is needed to get the inclusion implication at the end of the proof. Hence, $d = 100$ suffices to cover the error correction process including the k-sparse error.

Again, Lemma 5.3.3, although implied by Lemma 5.3.1 for each level k, is itself used in the proof of the latter. Hence, Lemma 5.3.3 it is proven conjointly with Lemma 5.3.1 in the induction.

Proposition 5.3.5 *Let φ be a trajectory of L with initial condition $\varphi(0,0) = \varphi(-1,0) = \varphi(1,0) = 1$, and for all other s, $\varphi(s,0) \neq \varphi(s+1,0)$ and $\varphi(s,0) \neq \varphi(s-1,0)$. Let $\theta^l, \theta^r : \mathbb{N} \to \mathbb{Z}$ denote the endpoint processes of the maximum interval $[\theta^l(t), \theta^r(t)]$ such that*

$$\varphi(s,t) = \begin{cases} 1 & \text{if } s \in [\theta^l(t), \theta^r(t)], \\ 0 & \text{otherwise.} \end{cases}$$

Then $\theta^l(t) = -3t - 1$ and $\theta^r(t) = -t + 1$.

Proof. Let $t = 0$. Clearly, $\theta^l(0) = -1$ and $\theta^r(0) = 1$. Assume the relations hold for $t > 0$. Consider the space-time points $(-3t - 4, t + 1)$, $(-3t - 3, t + 1)$, $(-3t - 2, t + 1)$, and $(-t + 1, t + 1)$. By the action of L, $\varphi(-3t - 4, t + 1) = \varphi(-3t - 3, t + 1) = \varphi(-3t - 2, t + 1) = 1$, and $\varphi(-t + 1, t + 1) = 0$. Moreover, for all $-3t - 1 \leq s < -t + 1$ the sites remain unchanged whereas for all other sites their values toggle to the opposite value. Hence, $\theta^l(t + 1) \leq -3(t + 1) - 1$ and $\theta^r(t + 1) \geq -(t + 1) + 1$. ∎

Proof of Lemma 5.3.1. The proof is analogous to two-line voting and contains no essential differences. The constants change slightly due to the increased speed-of-light 3, and we need a separate slope deflection calculation for the left boundary. We will repeat the induction proof, leaving out parts which are obviously the same and concentrating on the differences which themselves are relatively minor.

The proof goes by induction on k. Assume $k = 0$. First, Lemma 5.3.2 is applied to the all-black space interval $[0, w_0)$ at time $t = 0$ to yield the all-black space interval $[0, w_0 + h_0 - 14r_0)$ at $t = h_0$. Since $\mathcal{T}_0$ occupies $[\beta_0^L h_0, w_0 + \beta_0^R h_0)$ at $t = h_0$ where $\beta_0^L = 0$, $\beta_0^R = 0.9$, and $h_0 = 280r_0$, it follows that $[0, w_0 + h_0 - 14r_0)$ is strictly larger than $[\beta_0^L h_0, w_0 + \beta_0^R h_0)$ by $14r_0$ on the right side. Hence Lemma 5.3.2 can be applied again to the space interval $[0, w_0 + h_0 - 14r_0)$ at $t = h_0$ to conclude that η^ξ is 0-black on the trapezoid $(0, w_0 + h_0 - 28r_0, h_0, h_0 + h, 0, 1)$. Since

$$\mathcal{T}_0' \subset (0, w_0 + h_0 - 28r_0, h_0, h_0 + h, 0, 1),$$

the basis is proven.

Assume the statement holds for $k \geq 0$. Let $\mathcal{C}_s$, $\mathcal{C}_b$, $\mathcal{C}_b'$, $\mathcal{C}^*$, and $\mathcal{T}_{k+1}^*$ be defined exactly as in the two-line voting case. We will prove that

$$\mathcal{C}^* \text{ is a } (k + 1)\text{-black-cover of } \mathcal{T}_{k+1}^* \tag{5.3.6}$$

which implies the Expansion Lemma by the following implication:

$$(5.3.6) \implies \eta^\xi \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}').$$

The implication is stated as Claim I in the proof of two-line voting and its proof is exactly the same.

The proof of (5.3.6) goes by induction on the size of $\mathcal{C}_s$. Assume $n = |\mathcal{C}_s| = 0$. That is, U_{k+1} is k-sparse. If $|\mathcal{C}_b'| = 0$, then by the inductive assumption, η^ξ is a k-continuation at $(\beta_{k+1}^L h_k, h_k)$ with width $w_{k+1} + (\beta_{k+1}^R - \beta_{k+1}^L)h_k$ and extension $h_{k+1} + h - h_k$. Hence, $\eta^\xi \upharpoonright (\beta_{k+1}^L h_k, w_{k+1} + (\beta_{k+1}^R - \beta_{k+1}^L)h_k, h_k, h_{k+1} + h, \beta_k^L, \beta_k^R)$ is k-black. Since

$$\mathcal{T}_{k+1}^* \subset (\beta_{k+1}^L h_k, w_{k+1} + (\beta_{k+1}^R - \beta_{k+1}^L)h_k, h_k, h_{k+1} + h, \beta_k^L, \beta_k^R),$$

$\eta^\xi \upharpoonright \mathcal{T}_{k+1}^*$ is k-black.

Assume $|\mathcal{C}_b'| > 0$. Let $(s^*, t^*) + W_{dr_k} \in \mathcal{C}_b'$ be an element such that t^* is maximal. Let $\mathcal{K} = \mathcal{T}_{k+1} \cap \mathbb{Z} \times [t^* + dr_k, t^* + dr_k + h_k)$.

Claim II $\eta^\xi \upharpoonright \mathcal{K}$ is k-black.

Proof: Let $D = (x, y) + W_{r_k - 2dr_{k-1}}$ be any test window such that $H = D \cap \mathcal{K} \neq \emptyset$. Since $B \in \mathcal{C}_b' \implies B \cap H = \emptyset$, we only need consider $B \in \mathcal{C}_b \backslash \mathcal{C}_b'$ such that $B \cap H \neq \emptyset$. Let $B = (a, b) + W_{dr_k}$ be such an element. Well-separatedness, $B \in \mathcal{C}_b \backslash \mathcal{C}_b'$, and $\mathcal{C}_b$ being a $(k+1)$-black-cover imply that η^ξ is a k-continuation at (a', b'), $b' = b - r_k$, $a' = \max\{a - 3r_k - dr_k/8, \beta_{k+1}^L b'\}$, with width $\ell = \min\{dr_k + 4r_k + dr_k/8, w_{k+1} + \beta_{k+1}^R b' - a'\}$ and extension $dr_k + h_k + r_k$. Hence,

$$\eta^\xi \upharpoonright (a', a' + \ell, b', b' + dr_k + h_k + r_k, \beta_k^L, \beta_k^R)$$

is k-black. Since $H \subset (a', a' + \ell, b', b' + dr_k + h_k + r_k, \beta_k^L, \beta_k^R)$, $\eta^\xi \upharpoonright H$ is k-black.   ▲

Claim II implies that η^ξ is a k-continuation at $(\beta_{k+1}^L(t^* + dr_k + h_k), t^* + dr_k + h_k)$ with width $w_{k+1} + (\beta_{k+1}^R - \beta_{k+1}^L)(t^* + dr_k + h_k)$ and extension $h_{k+1} + h - t^* - dr_k - h_k$. It follows that $\eta^\xi \upharpoonright \mathcal{T}_{k+1}^*$ is k-black.

Assume (5.3.6) holds for $|\mathcal{C}_s| = n \geq 0$. Let $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r_k}$ be the last element in the enumeration of $\mathcal{C}_s$. If $t_{n+1} < h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, then by Claim III in the proof of the Expansion Lemma of two-line voting, (5.3.6) holds for

$|\mathcal{C}_s| = n+1$. Let $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$. We will consider the five cases

$$s_{n+1} > w_{k+1} + \beta_{k+1}^R t_{n+1} + 5r_k, \tag{5.3.7a}$$

$$w_{k+1} + \beta_{k+1}^R t_{n+1} - 5dr_k < s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} + 5r_k, \tag{5.3.7b}$$

$$\beta_{k+1}^L t_{n+1} + 5dr_k \leq s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} - 5dr_k, \tag{5.3.7c}$$

$$\beta_{k+1}^L t_{n+1} - 5r_k \leq s_{n+1} < \beta_{k+1}^L t_{n+1} + 5dr_k,, \tag{5.3.7d}$$

$$s_{n+1} < \beta_{k+1}^L t_{n+1} - 3r_k. \tag{5.3.7e}$$

Case (5.3.7e) is analogous to case (5.3.7a) and will not be considered separately. Cases (5.3.7a), (5.3.7b), and (5.3.7c) are proven analogously to their corresponding counterparts in two-line voting except for the change in speed-of-light from 2 to 3.

Case (i). Let $s_{n+1} > w_{k+1} + \beta_{k+1}^R t_{n+1} + 5r_k$. By the speed-of-light, $\eta^\xi \restriction (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}') \cap \mathcal{L}$,

$$\mathcal{L} = \{\, (s,t): \ s \leq -3(t - t_{n+1}) + s_{n+1}, \ t \geq 0 \,\},$$

takes on the same values irrespective of whether the error event B_{n+1} occurred or not. Let $D = (x,y) + W_{r_k - 2dr_{k-1}}$ be any test window such that

$$H = D \cap \left(\mathcal{T}_{k+1}^* \setminus \bigcup_{C \in \mathcal{C}^*} C\right) \neq \emptyset. \tag{5.3.8}$$

If $D \subset \mathcal{L}$ then by the inductive assumption on $|\mathcal{C}_s|$, $\eta^\xi \restriction H$ is k-black. Assume $D \cap \mathcal{L}^c \neq \emptyset$. Let $\mathcal{K}$ be the trapezoid

$$(w_{k+1} + \beta_{k+1}^R t_{n+1} - 4(dr_k + 2h_k + 3r_k) - r_k, w_{k+1} + \beta_{k+1}^R t_{n+1}, t_{n+1} - h_k, t_{n+1}, \beta_k^L, \beta_k^R).$$

If $s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} + 4(dr_k + 2h_k + 3r_k)$, then $\eta^\xi \restriction \mathcal{K}$ is k-black. This is a straightforward consequence of well-separatedness since if $t_{n+1} \geq h_{k+1}$, then by the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-black-cover of $\mathcal{T}_{k+1}^* \cap \mathbb{Z} \times [0, t_{n+1})$. If $t_{n+1} < h_{k+1}$ and for some $B = (a,b) + W_{dr_k} \in \mathcal{C}_b$, $B \cap \mathcal{K} \neq \emptyset$, then by well-separatedness η^ξ is a k-continuation at $(a - dr_k/2, b)$ with width $\min\{2dr_k, w_{k+1} +$

$\beta_{k+1}^R b - a + dr_k/2$ } and extension $2dr_k$ which implies that $\eta^\xi \upharpoonright B' \cap \mathcal{T}_{k+1}$ is k-black where B' is the r_k-dilation of B. By Proposition 3.3.11, this contradicts the minimality of $\mathcal{C}_b$ and hence $\eta^\xi \upharpoonright \mathcal{K}$ must be k-black. Since η^ξ is a k-continuation at $(w_{k+1} + \beta_{k+1}^R t_{n+1} - 4(dr_k + 2h_k + 3r_k) - r_k, t_{n+1})$ with width $4(dr_k + 2h_k + 3r_k) + r_k$ and extension $h_{k+1} + h - t_{n+1}$, $\eta^\xi \upharpoonright \mathcal{K} \cup \mathcal{K}'$ is k-black where $\mathcal{K}'$ is the trapezoid

$$\left(w_{k+1} + \beta_{k+1}^R t_{n+1} - 4(dr_k + 2h_k + 3r_k) - r_k, w_{k+1} + \beta_{k+1}^R t_{n+1}, t_{n+1}, h_{k+1} + h, \beta_k^L, \beta_k^R\right).$$

Let $\mathcal{M} = (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}') \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, t_{n+1} + 3r_k + dr_k + 2h_k)$.

Claim IV $\eta^\xi \upharpoonright \mathcal{M}$ is k-black.

Proof: If $s_{n+1} > w_{k+1} + \beta_{k+1}^R t_{n+1} + 4(dr_k + 2h_k + 3r_k)$ then $\mathcal{M} \subset \mathcal{L}$, and by the inductive assumption (on $|\mathcal{C}_s|$), $\eta^\xi \upharpoonright \mathcal{M}$ is k-black. Assume $s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} + 4(dr_k + 2h_k + 3r_k)$. Let $U = (a, b) + W_{r_k - 2dr_{k-1}}$ be any test window such that $U \cap \mathcal{M} \neq \emptyset$. By the definition of $\mathcal{L}$ and $\mathcal{K}'$,

$$U \cap \mathcal{M} \subset \mathcal{L} \quad \text{or} \quad U \cap \mathcal{M} \subset \mathcal{K}'.$$

We already know that $\eta^\xi \upharpoonright \mathcal{K}'$ is k-black. Assume $U \cap \mathcal{M} \subset \mathcal{L}$. Since $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, $\mathcal{M} \subset \mathcal{T}_{k+1}^*$. By the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-black-cover of $\mathcal{T}_{k+1}^* \cap \mathcal{L}$, and since C_{n+1} is the last element in the enumeration of $\mathcal{C}^*$, it follows that $\eta^\xi \upharpoonright \mathcal{M} \cap \mathcal{L}$ is k-black. Since $U \cap \mathcal{M} \subset \mathcal{M} \cap \mathcal{L}$, $\eta^\xi \upharpoonright U \cap \mathcal{M}$ is k-black. $\blacktriangle$

Let $\mathcal{M}' = (\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}') \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, \infty)$. An immediate consequence of Claim IV (by the inductive assumption on k) is that $\eta^\xi \upharpoonright \mathcal{M}'$ is k-black. Since $D \cap \mathcal{L}^c \neq \emptyset$, by the definition of $\mathcal{L}$, $\mathcal{K}'$, and $\mathcal{M}'$,

$$H \subset \mathcal{K} \cup \mathcal{K}' \quad \text{or} \quad H \subset \mathcal{M}'.$$

Hence, (5.3.6) holds for $|\mathcal{C}_s| = n + 1$.

Case (ii). Let $\beta_{k+1}^L t_{n+1} + 5dr_k \leq s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} - 5dr_k$. If $t_{n+1} \geq h_{k+1}$, then by the inductive assumption on $|\mathcal{C}_s|$, η^ξ is a k-continuation at both $(s_{n+1} - 9r_k - $

$4dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 12r_k, t_{n+1} + 3r_k)$ with extension $h_{k+1} + h - t_{n+1} - 3r_k$ and widths $4dr_k + r_k$ and $\max\{ 5dr_k - 12r_k, w_{k+1} + \beta_{k+1}^R(t_{n+1} + 3r_k) - s_{n+1} - 12r_k \}$, respectively.

If $t_{n+1} < h_{k+1}$, η^ξ remains a k-continuation at both $(s_{n+1} - 9r_k - 4dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 12r_k, t_{n+1} + 3r_k)$ with the same parameters as before. Suppose this were not the case, i.e., for some $B = (a, b) + W_{dr_k} \in \mathcal{C}_b$,

$$B \cap (\mathcal{K}^L \cup \mathcal{K}^R) \neq \emptyset$$

where $\mathcal{K}^L$ and $\mathcal{K}^R$ are the contexts corresponding to the two k-continuations at $(s_{n+1} - 9r_k - 4dr_k, t_{n+1} + 3r_k)$ and $(s_{n+1} + 12r_k, t_{n+1} + 3r_k)$, respectively. Let (x, y) denote the upper-left corner of $\mathcal{K}^L$ and let w_L denote the width of $\mathcal{K}^L$ at $t = y$. If $b > y$, then η^ξ is a k-continuation at (x, y) with width w_L and extension $h_{k+1} + h - y$, and by the inductive assumption on k, $\eta^\xi \restriction \mathcal{K}^L$ is k-black. If $b \leq y$, then η^ξ is a k-continuation at (x, b) with width $w_L - \beta_k^R(y - b)$ and extension $h_{k+1} + h - b$, hence $\eta^\xi \restriction \mathcal{K}^L$ is k-black. A similar argument holds for $\mathcal{K}^R$.

Using Lemma 5.3.3 (error correction), $\eta^\xi \restriction \mathcal{K} \setminus C_{n+1}$ is k-black where

$$\mathcal{K} = (s_{n+1} - 9r_k - 4dr_k, w_{k+1} + \beta_{k+1}^R t_{n+1}, t_{n+1}, h_{k+1} + h, \beta_k^L, \beta_k^R).$$

By SOL, $\eta^\xi \restriction (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathcal{L}$ takes on the same values irrespective of whether the error event B_{n+1} occurred or not where

$$\mathcal{L} = \{ (s, t) : s \leq -3(t - t_{n+1}) + s_{n+1} \text{ or } s \geq 3(t - t_{n+1}) + s_{n+1} + 3r_k, \ t \geq t_{n+1} \}.$$

Let $\mathcal{M} = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, t_{n+1} + 3r_k + dr_k + 2h_k)$.

Claim V $\eta^\xi \restriction \mathcal{M}$ is k-black.

Proof: Let $U = (a, b) + W_{r_k - 2dr_{k-1}}$ be any test window such that $U \cap \mathcal{M} \neq \emptyset$. By the definition of $\mathcal{L}$ and $\mathcal{K}$,

$$U \cap \mathcal{M} \subset \mathcal{L} \qquad \text{or} \qquad U \cap \mathcal{M} \subset \mathcal{K} \setminus C_{n+1}.$$

We have already established that $\eta^\xi \restriction \mathcal{K} \setminus C_{n+1}$ is k-black. Let $U \cap \mathcal{M} \subset \mathcal{L}$. Since $t_{n+1} \geq h_{k+1} - r_{k+1} - dr_k - h_k - 3r_k$, $\mathcal{M} \subset \mathcal{T}_{k+1}^*$. By the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-black-cover of $\mathcal{T}_{k+1}^* \cap \mathcal{L}$, and since C_{n+1} is the last element in the enumeration of $\mathcal{C}^*$, it follows that $\eta^\xi \restriction \mathcal{M} \cap \mathcal{L}$ is k-black. Since $U \cap \mathcal{M} \subset \mathcal{M} \cap \mathcal{L}$, $\eta^\xi \restriction U \cap \mathcal{M}$ is k-black. ▲

Let $D = (x,y) + W_{r_k - 2dr_{k-1}}$ be any test window satisfying (5.3.8). If $D \subseteq \mathcal{L}$ we are done. If $D \cap \mathcal{L}^c \neq \emptyset$, then

$$H \subset \mathcal{K} \setminus C_{n+1} \qquad \text{or} \qquad H \subset \mathcal{M}'$$

where $\mathcal{M}' = (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [t_{n+1} + 3r_k + dr_k + h_k, \infty)$. Since Claim V implies that $\eta^\xi \restriction \mathcal{M}'$ is k-black, (5.3.6) holds for $|\mathcal{C}_s| = n+1$.

Case (iii). Let $w_{k+1} + \beta_{k+1}^R t_{n+1} - 5dr_k < s_{n+1} \leq w_{k+1} + \beta_{k+1}^R t_{n+1} + 3r_k$. It suffices to show that η^ξ is k-black on $\mathcal{K}$ where

$$
\begin{aligned}
\mathcal{K} \;=\; \{\, (s,t) : \; & w_{k+1} + \beta_{k+1}^R t_{n+1} - 5dr_k + \beta_k^R(t - t_{n+1}) \;\leq\; s \;\leq\; \\
& w_{k+1} + \beta_{k+1}^R t_{n+1} + 5dr_k + \beta_k^R(t - t_{n+1}), \;\; t_{n+1} - 2dr_k \leq t \leq t_{n+1} \,\},
\end{aligned}
$$

since then the arguments of Case (ii) can again be applied from which (5.3.6) follows.

Consider the space-time point

$$p = (w_{k+1} + \beta_{k+1}^R(t_{n+1} + 3r_k - u), t_{n+1} + 3r_k - u)$$

where $u = r_{k+1}/3$. Let $p' = p + (-5dr_k, 0)$ be the space-translation of p by $-5dr_k$.

Claim VI η^ξ is a k-continuation at p' with width $5dr_k$ and extension $u - 3r_k$.

Proof: First, for any window $V = (s,t) + W_u$ such that $B_{n+1} \subset V$ (recall that B_{n+1} is the last element in the enumeration of $\mathcal{C}_s$), $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap V$ is k-sparse. This is a direct consequence of well-separatedness (i.e., choose c sufficiently large such that $r_{k+1}/3 < r_{k+1} - 6r_k$) and the definition of sparsity. Let $\mathcal{H}$ be the context of the (as yet to be determined) k-continuation in the claim. Let (a,b) be the upper-left corner point of $\mathcal{H}$, and let $w_\mathcal{H}$ denote the width of $\mathcal{H}$ at $t = b$. To prove

the claim, it suffices to show that $\eta^\xi \upharpoonright \mathcal{H}$ is k-black since by the definition of u and well-separatedness $(c \gg d)$, the trapezoid

$$(a - 2r_k, a + w_\mathcal{H} + 2r_k, b, t_{n+1}, 0, 1)$$

is k-sparse. Suppose for some $B = (a', b') + W_{dr_k} \in \mathcal{C}_b$, $B \cap \mathcal{H} \neq \emptyset$. If $b' > b$, then by $\mathcal{C}_b$ being a minimal $(k + 1)$-cover, η^ξ is a k-continuation at (a, b) with width $w_\mathcal{H}$ and extension h_k which implies that $\eta^\xi \upharpoonright \mathcal{H}$ is k-black. If $b' \leq b$, then η^ξ is a k-continuation at (a, b') with width $w_{k+1} + \beta_{k+1}^R b' - a$ and extension $dr_k + h_k$, and hence $\eta^\xi \upharpoonright \mathcal{H}$ is k-black. ▲

Let $\mathcal{H}' = (a, a + w_\mathcal{H}, b, t_{n+1}, \beta_k^L, \beta_k^R)$. Claim VI implies that $\eta^\xi \upharpoonright \mathcal{H}'$ is k-black. We will prove that $\mathcal{K} \subset \mathcal{H}'$ from which (5.3.6) follows. Since $p' = p + (-5dr_k, 0)$, $\beta_k^L < \beta_{k+1}^L$, and $\beta_{k+1}^R < \beta_k^R < 1$, it suffices to show that

$$w_{k+1} + \beta_{k+1}^R t_{n+1} + 5dr_k \quad \leq \quad w_{k+1} + \beta_{k+1}^R(t_{n+1} + 3r_k - u) + \beta_k^R(u - 3r_k)$$

$$\Longleftrightarrow \qquad 5dr_k \quad \leq \quad (\beta_k^R - \beta_{k+1}^R)(u - 3r_k). \tag{5.3.9}$$

Claim VII There exist $c, d > 0$ such that (5.3.9) holds.

Proof: By the inductive assumption on k, $\beta_k^R = 1/2 + 1/(k + 2.5)$, and (5.3.9) is equivalent to

$$5dr_k \quad \leq \quad \left(\frac{1}{2} + \frac{1}{k + 2.5} - \frac{1}{2} - \frac{1}{k + 3.5}\right)(r_{k+1}/3 - 3r_k)$$

$$\Longleftrightarrow \qquad 15dr_k \quad \leq \quad \frac{4}{(2k + 5)(2k + 7)}(c(k + 1)^2 r_k - 9r_k)$$

$$\Longleftrightarrow \qquad 15d \quad \leq \quad \frac{4}{(2k + 5)(2k + 7)}(c(k + 1)^2 - 9) \tag{5.3.10}$$

where we have used $u = r_{k+1}/3$ and $r_{k+1} = c(k + 1)^2 r_k$. Upon rearrangement, (5.3.10) holds iff

$$(4c - 60d)k^2 + (8c - 360d)k + 4c - 525d - 36 \quad \geq \quad 0.$$

This is satisfied, for all $k \geq 0$, if $c \geq 132d + 9$. ▲

Case (iv). Let $\beta_{k+1}^L t_{n+1} - 5r_k \leq s_{n+1} < \beta_{k+1}^L t_{n+1} + 5dr_k$. It suffices to show that η^ξ is k-black on $\mathcal{K}$ where

$$\mathcal{K} \;=\; \{\, (s,t): \beta_{k+1}^L t_{n+1} - 5dr_k + \beta_k^L(t - t_{n+1}) \;\leq\; s \;\leq$$

$$\beta_{k+1}^L t_{n+1} + 5dr_k + \beta_k^L(t - t_{n+1}), \;\; t_{n+1} - 2dr_k \leq t \leq t_{n+1} \,\},$$

since then the arguments of Case (ii) can again be applied from which (5.3.6) follows.

Consider the space-time point

$$p = (w_{k+1} + \beta_{k+1}^L(t_{n+1} + 3r_k - u), t_{n+1} + 3r_k - u)$$

where $u = r_{k+1}/3$.

Claim VIII η^ξ is a k-continuation at p with width $5dr_k$ and extension $u - 3r_k$.

Proof: First, for any window $V = (s,t) + W_u$ such that $B_{n+1} \subset V$ (recall that B_{n+1} is the last element in the enumeration of $\mathcal{C}_s$), $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap V$ is k-sparse. This is a direct consequence of well-separatedness (i.e., choose c sufficiently large such that $r_{k+1}/3 < r_{k+1} - 6r_k$) and the definition of sparsity. Let $\mathcal{H}$ be the context of the (as yet to be determined) k-continuation in the claim. Let (a,b) be the upper-left corner point of $\mathcal{H}$, and let $w_{\mathcal{H}}$ denote the width of $\mathcal{H}$ at $t = b$. To prove the claim, it suffices to show that $\eta^\xi \restriction \mathcal{H}$ is k-black since by the definition of u and well-separatedness $(c \gg d)$, the trapezoid

$$(a - 2r_k, a + w_{\mathcal{H}} + 2r_k, b, t_{n+1}, 0, 1)$$

is k-sparse. Suppose for some $B = (a', b') + W_{dr_k} \in \mathcal{C}_b$, $B \cap \mathcal{H} \neq \emptyset$. If $b' > b$, then by $\mathcal{C}_b$ being a minimal $(k+1)$-cover, η^ξ is a k-continuation at (a,b) with width $w_{\mathcal{H}}$ and extension h_k which implies that $\eta^\xi \restriction \mathcal{H}$ is k-black. If $b' \leq b$, then η^ξ is a k-continuation at $(a - \beta_k^L(b - b'), b')$ with width $w_{\mathcal{H}}$ and extension $dr_k + h_k$, and hence $\eta^\xi \restriction \mathcal{H}$ is k-black. $\blacktriangle$

Let $\mathcal{H}' = (a, a + w_{\mathcal{H}}, b, t_{n+1}, \beta_k^L, \beta_k^R)$. Claim VIII implies that $\eta^\xi \restriction \mathcal{H}'$ is k-black. We will prove that $\mathcal{K} \subset \mathcal{H}'$ from which (5.3.6) follows. Since $\beta_k^L < \beta_{k+1}^L$, using Claim VIII,

it is sufficient to show that

$$\beta_{k+1}^L(t_{n+1} + 3r_k - u) + \beta_k^L(u - 3r_k) \leq \beta_{k+1}^L t_{n+1} - 5dr_k$$

$$\Longleftrightarrow \qquad -(\beta_{k+1}^L - \beta_k^L)(u - 3r_k) \leq -5dr_k \tag{5.3.11}$$

Claim IX There exist $c, d > 0$ such that (5.3.11) holds.

Proof: By the inductive assumption on k, $\beta_k^L = 1/8 - 1/(k+8)$, and (5.3.11) is equivalent to

$$5dr_k \leq \left(\frac{1}{8} - \frac{1}{k+9} - \frac{1}{8} + \frac{1}{k+8}\right)(r_{k+1}/3 - 3r_k)$$

$$\Longleftrightarrow \qquad 15dr_k \leq \frac{1}{(k+9)(k+8)}\left(c(k+1)^2 r_k - 9r_k\right)$$

$$\Longleftrightarrow \qquad 15d \leq \frac{1}{(k+9)(k+8)}\left(c(k+1)^2 - 9\right) \tag{5.3.12}$$

where we have used $u = r_{k+1}/3$ and $r_{k+1} = c(k+1)^2 r_k$. Upon rearrangement, (5.3.12) holds iff

$$(c - 15d)k^2 + (2c - 255d)k + c - 1080d - 9 \geq 0.$$

This is satisfied, for all $k \geq 0$, if $c \geq 1080d + 9$. ▲

It is easily checked that all the claims in the proof using the well-separatedness property with respect to $c \gg d$ are satisfied if $c = 1500d$. ∎

5.4 Spreading of agreement

Fact 5.4.1 *For all $\xi, \xi' \in S^{\mathbb{Z}}$, if $\eta^\xi, \eta^{\xi'}$ concur on a space interval $[x, x + a) \times T$, then they continue to concur on the space-time triangle*

$$\{(s, t) : x + 3(t - T) \leq s \leq x + a - 3(t - T), t \geq T\}.$$

Proof. Immediate consequence of SOL and basic coupling. ∎

Proof of Lemma 5.1.5. Assume $k = 0$. Since $\eta^\xi, \eta^{\xi'}$ are 0-black on R_0, i.e., $\eta^\xi(s,t) = \eta^{\xi'}(s,t) = 1$ for $(s,t) \in R_0$, the basis follows directly.

Assume the statement holds for $k > 0$. The inductive step is similar to the two-line voting case except that Fact 3.4.3 cannot be used since the GKL rule is nonmonotonic. Hence, we need to work with two coverings instead of just one. Let $\mathcal{C}, \mathcal{C}'$ be minimum $(k+1)$-covers of $\eta^\xi \restriction R_{k+1}$ and $\eta^{\xi'} \restriction R_{k+1}$, respectively. If $\mathcal{C} = \mathcal{C}' = \emptyset$, then by the inductive assumption,

$$\eta^\xi(s,t) = \eta^{\xi'}(s,t), \qquad \forall (s,t) \in R_{k+1}(b_k).$$

Hence they also concur on $R_{k+1}(b_{k+1})$. Assume $\mathcal{C} \neq \emptyset \ \vee \ \mathcal{C}' \neq \emptyset$. Let $\mathcal{T} = \mathcal{C} \cup \mathcal{C}'$.

Claim If for every $10dr_k$-window W with $W \cap R_{k+1} \neq \emptyset$ at most one element from $\mathcal{T}$ has nonempty intersection with W, then η^ξ and $\eta^{\xi'}$ concur on $R_{k+1}(b_{k+1})$.

Proof: We can apply the same argument as in the proof of Lemma 2.2.5 which only depends on the well-separatedness of elements belonging to a covering. The fact that $\mathcal{T}$ is a union of $\mathcal{C}, \mathcal{C}'$ and the skewed nature of the trapezoid R_{k+1} does not affect the argument. Both η^ξ and $\eta^{\xi'}$ are k-black on

$$R_{k+1} \setminus \bigcup_{B \in \mathcal{C} \cup \mathcal{C}'} B$$

which allows us to carry out the induction if well-separatedness of elements in $\mathcal{T}$ is ensured. The size of the window W, $10dr_k$, can be easily checked to be sufficient to apply Fact 5.4.1 which reflects the speed-of-light 3 of the GKL rule. Changes in constants are absorbed by $c \gg d$. ▲

More generally, let us consider a special covering (by intersection) $\mathcal{U}$ of $\mathcal{T}$ by $10dr_k$-windows W given by

$$\forall B \in \mathcal{T}, \ \exists W \in \mathcal{U} : \ W \cap B \neq \emptyset.$$

It is an immediate consequence of well-separatedness that at most two elements from $\mathcal{T}$, one from each $\mathcal{C}$ and $\mathcal{C}'$, can have nonempty intersection with a $10dr_k$-window W.

Let $\mathcal{U}$ be also minimal. Let us partition $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ where

$$\mathcal{U}_2 = \{\, W \in \mathcal{U} : \exists B, B' \in \mathcal{T}, B \neq B', W \cap B \neq \emptyset \,\wedge\, W \cap B' \neq \emptyset \,\}.$$

For $W \in \mathcal{U}_1$, we can use the size of W itself as the separation condition to apply Fact 5.4.1 to the element $B \in \mathcal{T}$ with $W \cap B \neq \emptyset$. Let $\mathcal{T}_1$ denote the set of all such $B \in \mathcal{T}$ corresponding to some $W \in \mathcal{U}_1$. For $W \in \mathcal{U}_2$, we can choose c sufficiently large, so that Fact 5.4.1 can be applied to $2W$, the dilution of W by factor 2 such that both its intersecting elements of $\mathcal{T}$ are contained in $2W$. Since both η^ξ and $\eta^{\xi'}$ are k-black on

$$R_{k+1} \setminus \bigcup_{\substack{B \in \mathcal{T}_1 \\ W \in \mathcal{U}_2}} (B \,\cup\, 2W),$$

this allows us to repeat the argument of Lemma 2.2.5 as in the previous claim. ∎

5.5   Relaxation time

5.5.1   Relaxation lower bound

The proof structure of the relaxation time lower bound is the same as in two-line voting. The main difference arises from the change in speed of light which changes the dimensions of the space-time rectangles and trapezoids, and the statement of deterministic eroder property for the GKL rule which is used in Proposition 5.5.5. The definition of supersparsity is exactly as before.

Lemma 5.5.1 (supersparsity)   *There exists $0 < \varepsilon_0 < 1/2$, $v > c > 1$ such that for all $0 < \varepsilon < \varepsilon_0$, $0 \leq k \leq (\log 1/\varepsilon)/8 \log v$, $\beta > 0$ the following holds. Let η denote an ε-perturbation of L, $L_{\varepsilon,\beta}$. Let $A = \bigcup_{i \in [1,N]} (a_i, b_i) + W_{r'_k}$, $a_i, b_i \in \mathbb{Z}$, where $\eta \restriction A$ is defined. Let q_k be the probability that η is not k-supersparse on A. Then*

$$q_k \;<\; N \varepsilon^{(k+1.2)/4} v^{k(k-1)}.$$

The statement and proof of the Supersparsity Lemma are exactly as in two-line voting. The only difference in the proof is the absence of the constant 2 in the upper bounding of $\Pr(\eta$ is not 0-supersparse on $W^i)$ in the basis.

Let $k_0 = (\log 1/\varepsilon)/8 \log v$. We will consider a sequence of space-time rectangles S'_k, $k = 0, 1, \ldots, k_0$, where S'_k $(k < k_0)$ has size $25r'_k \times 4r'_k$, and S'_{k_0} is of the size

$$25 \cdot 2^{c_2 \log^2 1/\varepsilon} \times 4 \cdot 2^{c_2 \log^2 1/\varepsilon}$$

where $c_2 > 0$ is a positive constant. We will view S'_k as a union of two subrectangles $S^U_k, S^B_k \subset S'_k$ each of size $25r'_k \times 3r'_k$ if $k < k_0$ and

$$25 \cdot 2^{c_2 \log^2 1/\varepsilon} \times 3 \cdot 2^{c_2 \log^2 1/\varepsilon}$$

if $k = k_0$. Note that given S'_k, S^U_k and S^B_k are uniquely determined.

Lemma 5.5.2 *There exist $\delta > 0$, $c_2 > 0$ such that for all sufficiently small ε (depending on δ)*

$$\Pr\left(\bigwedge_{k=0}^{k_0} (S^U_k \text{ is } k\text{-supersparse} \ \wedge \ S^B_k \text{ is } (k-1)\text{-supersparse}) \right) > \delta$$

where (-1)-supersparse is interpreted to mean 0-supersparse.

Proof. Let us upper-bound the probability of the complement event: for some $k \leq k_0$, S^U_k is not k-supersparse or S^B_k is not $(k-1)$-supersparse. First, since q_k is monotonically decreasing for $0 \leq k \leq (\log 1/\varepsilon)/8 \log v$, we have

$$\Pr(S^U_k \text{ is not } k\text{-supersparse}) < 75\varepsilon^{0.3},$$

$$\Pr(S^B_k \text{ is not } (k-1)\text{-supersparse}) < 75c^2\varepsilon^{0.3},$$

where we have used $r'_{k+1} = cr'_k$ in the second bound. Second, it is easy to check that

$$\Pr(S^U_k \text{ is not } k\text{-supersparse}) \leq \Pr(S^U_k \text{ is not } (k-1)\text{-supersparse}).$$

With these two facts in hand,

$$\Pr(\exists k \le k_0 : S_k^U \text{ is not } k\text{-supersparse} \ \lor \ S_k^B \text{ is not } (k-1)\text{-supersparse})$$

$$\le \sum_{k=0}^{k_0} \Big(\Pr(S_k^U \text{ is not } k\text{-supersparse}) + \Pr(S_k^B \text{ is not } (k-1)\text{-supersparse}) \Big)$$

$$\le \ 2\Pr(S_{k_0}^B \text{ is not } (k_0-1)\text{-supersparse}) + 2\sum_{k=0}^{k_0-1} \Pr(S_k^B \text{ is not } (k-1)\text{-supersparse})$$

$$< \ 75 \cdot 2^{2c_2 \log^2 1/\varepsilon} \varepsilon^{((k_0-1)+1.2)/4} v^{(k_0-1)(k_0-2)} \ + \ \varepsilon^{0.3} 75 c^2 k_0$$

$$< \ 75 \exp(2c_2 \log 2 \log^2 1/\varepsilon \ + \ k_0^2 \log v \ - \ (k_0/4) \log 1/\varepsilon \ - \ 0.05 \log 1/\varepsilon)$$

$$+ \ \varepsilon^{0.3} 75 c^2 (\log 1/\varepsilon)/8 \log v$$

$$= \ 75 \exp((2c_2 \log 2 - 1/64 \log v) \log^2 1/\varepsilon - 0.05 \log 1/\varepsilon) \ + \ \varepsilon^{0.3} (\log 1/\varepsilon) 75 c^2/8 \log v$$

Clearly, for $c_2 < 1/(128 \log 2 \log v)$ and ε sufficiently small (depending on δ),

$$\Pr(\exists k \le k_0 : S_k^U \text{ is not } k\text{-supersparse} \ \lor \ S_k^B \text{ is not } (k-1)\text{-supersparse}) \ \le \ 1 - \delta$$

which completes the proof. ∎

5.5.2 Shrinking region of consolidation

For each $\delta > 0$, site x, time $t \le Relax(n, \delta, K_{\varepsilon,\beta})$, and $k = 0, \ldots, k_0$, we define a sequence $R_k' \subset S_k'$ of trapezoids

$$R_k' = (x_k, x_k + 25\ell_k, y_k, y_k + 4\ell_k, -3)$$

extending into the past such that

(i) $\ell_{k_0} = 2^{c_2 \log^2 1/\varepsilon}$, $\ell_k = r_k'$, $k \in [0, k_0)$;

(ii) $y_{k_0} = -3\ell_{k_0}$, $y_k = y_{k+1} + 4\ell_{k+1} - 3\ell_k$, $k \in [0, k_0)$;

(iii) $x_{k_0} = x - \ell_{k_0}/2$, $x_k = x_{k+1} + 12\ell_{k+1} + (\ell_{k+1} - 25\ell_k)/2$, $k \in [0, k_0)$;

(iv) $R_k' \cap \mathbb{Z} \times [y_k, y_k + 3\ell_k) \subset S_k^U$, $R_k' \cap \mathbb{Z} \times [y_k + \ell_k, y_k + 4\ell_k) \subset S_k^B$.

Condition (ii) allows for a $3\ell_k$ overlap between R'_{k+1} and R'_k, and condition (iii) implies that $(x,t) \in R'_0 \cap \mathbb{Z} \times [y_0 + 3\ell_0, \infty)$. The lower bound on the relaxation time is implied by the next lemma.

Lemma 5.5.3 *In η_0, we have $\Pr(\bigwedge_{k=0}^{k_0} \eta_0 \restriction R'_k$ is k-white$) > \delta$. The same holds with η_1 and k-black.*

Thus an immediate corollary of Lemma 5.5.3 is that $\eta_0(x,t) = 00$ with probability at least δ if $t = O(2^{c_2 \log^2 1/\varepsilon})$. Lemma 5.5.3 will be implied by Lemma 5.5.2 and Lemma 5.5.4 which shows the existence of a shrinking region of increasing "white-consolidation" in space-time.

Lemma 5.5.4 (consolidation) *For all $k = 0, \ldots, k_0 - 1$, if $\eta_0 \restriction R'_{k+1}$ is $(k+1)$-white, S^U_{k+1} is $(k+1)$-supersparse, S^B_{k+1} is k-supersparse, and S^U_k is k-supersparse, then $\eta_0 \restriction R'_k$ is k-white. The same holds with k-black.*

Before proving Lemma 5.5.4, we will establish a couple of useful facts. The next result states that a k-white (or k-black) configuration "quickly" returns to all-white (all-black) in the absence of errors.

Proposition 5.5.5 (attraction) *Let $k \geq 1$. Let $\eta : \mathbb{Z} \times \mathbb{N} \to S$ be a deterministic orbit such that $\eta(\cdot, 0)$ is k-white (k-black). Then $\eta(\cdot, \kappa r'_{k-1})$ is 0-white (0-black) where κ, $4d < \kappa < c$, is a fixed constant.*

Proof. The proof goes by induction on k. Let $k = 1$. Let $\mathcal{C}$ be a minimum 1-cover of $\eta(\cdot, 0)$. Note that if $\mathcal{C} = \emptyset$ then the basis is trivially true. By Theorem 1.1.2, for every $B \in \mathcal{C}$ there exists $s_B \in \mathbb{Z}$ such that

$$\eta \restriction \left((\mathbb{Z} \times \mathbb{N}) \setminus \bigcup_{B \in \mathcal{C}} (s_B, 0) + W_{3dr'_{k-1}} \right)$$

is 0-white. Since $\kappa > 4d$ the basis is proven.

Assume the statement holds for $k \geq 1$. Let $\mathcal{C}$ be a minimum $(k + 1)$-cover of $\eta(\cdot, 0)$. Let $B = (s_B, 0) + W_{dr'_k}$ be an element of $\mathcal{C}$. Let us consider η restricted on the space interval

$$I = (s_B + dr'_k, s_B + r'_{k+1} - dr'_k].$$

Clearly, $\eta \upharpoonright (I \times [0, 0])$ is k-white.

By the inductive assumption and speed-of-light,

$$\eta \upharpoonright I' \times [\kappa r'_{k-1}, \kappa r'_{k-1}) \quad \text{is 0-white}$$

where $I' = (s_B + dr'_k + 3\kappa r'_{k-1}, s_B + r'_{k+1} - dr'_k - 3\kappa r'_{k-1}]$. Using well-separatedness and symmetry, we can apply Theorem 1.1.2 to conclude that $\eta(\cdot, \kappa r'_{k-1} + 3(dr'_k + 6\kappa r'_{k-1}))$ is 0-white. Since

$$\kappa r'_{k-1} + 3(dr'_k + 6\kappa r'_{k-1}) < 4dr'_k < \kappa r'_k,$$

the proposition follows. ∎

Lemma 5.5.6 *Let $k \geq 0$ and let η be an orbit of $L_{\varepsilon, \beta}$. Let $E = (x, y) + W_{3r'_k}$ and let $U = (x - 50r'_k, y) + W_{100r'_k}$. Let*

$$\mathcal{M} = \{ (s, t) : x - dr'_k/2 + 3(t - y) \leq s < x + dr'_k/2 - 3(t - y), \ t \geq y \}.$$

If $\mathcal{M} \setminus E$ is 0-supersparse and $\eta \upharpoonright (\mathcal{M} \setminus E) \cap \mathbb{Z} \times [y, y]$ is k-white, then

$$\eta \upharpoonright (\mathcal{M} \cap \mathbb{Z} \times [y + 3r'_k, \infty)) \setminus U \quad \text{is 0-white.}$$

Proof. By the speed-of-light, $\eta \upharpoonright \mathcal{M}$ is not affected by what values η takes on on $\mathcal{M}^c \cap \mathbb{Z} \times [y, \infty)$. Let

$$\mathcal{K} = \{ (s, t) : x - 3(t - y) \leq s < x + 3r'_k + 3(t - y), \ t \geq y \}.$$
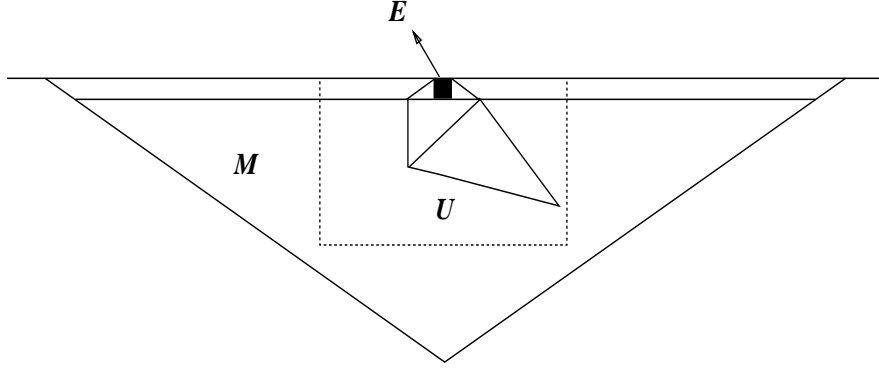
Figure 5.5.1: Error correction under k-supersparse errors.

By Proposition 5.5.5 and noting that $\kappa r'_{k-1} < 3r'_k$ (κ is the time variable in the proposition),

$$\eta \upharpoonright (\mathcal{M} \setminus \mathcal{K}) \cap \mathbb{Z} \times [y + 3r'_k, y + 3r'_k] \quad \text{is 0-white.}$$

Theorem 1.1.2 (deterministic eroder property) implies that the error island

$$\mathcal{K} \cap \mathbb{Z} \times [y + 3r'_k, y + 3r'_k]$$

which has length $21r'_k$ is corrected within a space-time rectangle of size at most $70r'_k \times 70r'_k$. Hence the error correction process, inclusive the $(k+1)$-supersparse error E, is covered by U. For the previous arguments to hold, we must choose d sufficiently large such that $U \subset \mathcal{M}$. It is easily checked that this is the case if $d = 1600$. ∎

Figure 5.5.1 depicts the error correction process subject to a $(k+1)$-supersparse error described in the proof of Lemma 5.5.6.

Remark 5.5.7   Let (a, b) be the intersection point of the left boundaries of $\mathcal{M}$ and $\mathcal{K}$. If $d = 1600$, then $b = y + 125r'_k$ and it follows that $\eta \upharpoonright \mathcal{M} \cap \mathbb{Z} \times [y + 100r'_k, y + 125r'_k)$ is 0-white. We will use this property in the proof of Lemma 5.5.8.

Lemma 5.5.8   *Let* $\mathcal{T}_k = (0, w_k, 0, h_k, -3)$, $\mathcal{T}'_k = (3h_k, w_k - 3h_k, h_k, h_k + h, -3)$ *where* $w_k \geq 25r'_k$, $h_k = 3r'_k$, *and* $h \geq 0$. *Let* η *be an orbit of* $L_{\varepsilon, \beta}$. *Then,*

$$\eta \upharpoonright \mathcal{T}_k \text{ is } k\text{-white} \;\wedge\; \mathcal{T}_k \cup \mathcal{T}'_k \text{ is } k\text{-supersparse} \quad\Longrightarrow\quad \eta \upharpoonright (\mathcal{T}_k \cup \mathcal{T}'_k) \text{ is } k\text{-white.}$$

The same holds true if k-*white is replaced by* k-*black.*

We will call η a *k-continuation (with respect to supersparsity)* at $(3h_k, h_k)$ with *width* $w_k - 6h_k$ and *extension* h. This lemma is the main technical tool in the proof of Lemma 5.5.4, and its structure follows the inductive proof of the Expansion Lemma. However, it is much simpler due to absence of boundary effects facilitated by the two sides of the trapezoids $\mathcal{T}_k$, $\mathcal{T}'_k$ shrinking with the speed-of-light.

Proof of Lemma 5.5.8.   The proof goes by induction on k. Let $k = 0$. Since $\eta \upharpoonright \mathcal{T}_0$ is all-white and no errors occur in $\mathcal{T}_0 \cup \mathcal{T}'_0$, the basis follows trivially from SOL.

Assume the statement holds for $k \geq 0$. Let $\mathcal{C}_b$, $\mathcal{C}'_b$, $\mathcal{C}_s$, $\mathcal{C}^*$, and $\mathcal{T}^*_k$ be defined as in the proof of the Expansion Lemma (Lemma 3.2.2) where $\mathcal{C}_b$ is now a minimum $(k+1)$-white-cover of $\mathcal{T}_{k+1}$. Let $B_i = (s_i, t_i) + W_{3r'_k}$, $i = 1, 2, \ldots, n$, be the corresponding enumeration of $\mathcal{C}_s$. It suffices to prove

$$\mathcal{C}^* \text{ is a } (k+1)\text{-white-cover of } \mathcal{T}^*_{k+1} \tag{5.5.9}$$

since (5.5.9) implies that $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})$ is $(k+1)$-white. (See Claim I in the proof of the Expansion Lemma with k-black in place of k-white.)

The proof of (5.5.9) goes by induction on the size of $\mathcal{C}_s$. Assume $n = |\mathcal{C}_s| = 0$. If $|\mathcal{C}'_b| = 0$, then η is a k-continuation at $(3h_k, h_k)$ with width $w_{k+1} - 6h_k$ and extension $h_{k+1} + h - h_k$. By the inductive assumption on k, $\eta \upharpoonright (3h_k, w_{k+1} - 3h_k, h_k, h_{k+1} + h, -3)$ is k-white from which (5.5.9) follows.

Assume $|\mathcal{C}'_b| > 0$. Let $(s^*, t^*) + W_{dr'_k} \in \mathcal{C}'_b$ be an element such that t^* is maximal. Let $\mathcal{K} = \mathcal{T}_{k+1} \cap \mathbb{Z} \times [t^* + dr'_k, t^* + dr'_k + h_k)$.

Claim I $\eta \upharpoonright \mathcal{K}$ is k-white.

Proof: Let $D = (x, y) + W_{r'_k - 2dr'_{k-1}}$ be any test window such that $H = D \cap \mathcal{K} \neq \emptyset$. Since $B \in \mathcal{C}'_b \implies B \cap H = \emptyset$, we only need consider $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$ such that $B \cap H \neq \emptyset$. Let $B = (a, b) + W_{dr'_k}$ be such an element. Well-separatedness, $B \in \mathcal{C}_b \setminus \mathcal{C}'_b$, and $\mathcal{C}_b$ being a $(k+1)$-white-cover imply that η is a k-continuation at (a', b), $a' = \max\{a - 7r'_k - 3dr'_k, 3b\}$, with width $\ell = \min\{7dr'_k + 14r'_k, w_{k+1} - 3b - a'\}$ and extension $dr'_k + h_k$. Hence,

$$\eta \upharpoonright (a' - 3r'_k, a' + \ell + 3r'_k, b - r'_k, b + r'_k + dr'_k + h_k, -3)$$

is k-white. Since $H \subset (a' - 2r'_k, a' + \ell + 2r'_k, b - r'_k, b - r'_k + dr'_k + h_k, -2)$, it follows that $\eta \upharpoonright H$ is k-white. $\blacktriangle$

Claim I implies that η is a k-continuation at $(3(t^* + dr'_k + h_k), t^* + dr'_k + h_k)$ with width $w_{k+1} - 6(t^* + dr'_k + h_k)$ and extension $h_{k+1} + h - t^* - dr'_k - h_k$. Hence, $\eta \upharpoonright \mathcal{T}^*_{k+1}$ is k-white.

Assume (5.5.9) holds for $|\mathcal{C}_s| = n \geq 0$. Let $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r'_k}$ be the last element in the enumeration of $\mathcal{C}_s$. If $t_{n+1} < h_{k+1} - r'_{k+1} - dr'_k - h_k - 3r'_k$ then

$$(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1} - dr'_k - h_k, \infty) \quad \text{is } k\text{-supersparse,}$$

and hence an argument analogous to the proof of Claim I can be applied to the smaller trapezoid $\mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1} - dr'_k - h_k, \infty)$ to conclude that $\eta \upharpoonright \mathcal{T}_{k+1} \cap \mathbb{Z} \times [h_{k+1} - r'_{k+1}, h_{k+1})$ is k-white from which (5.5.9) follows.

Let $t_{n+1} \geq h_{k+1} - r'_{k+1} - dr'_k - h_k - 3r'_k$. If $B_{n+1} \subset (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c$, then by SOL and the inductive assumption on $|\mathcal{C}_s|$, (5.5.9) holds for $|\mathcal{C}_s| = n + 1$. Let $B_{n+1} \cap (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \neq \emptyset$. Note that since $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})$ is independent of $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty)$, we may assume any values for $\eta \upharpoonright (\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty)$, in particular, all-white, without affecting the analysis. By the same reason, we may view $(\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1})^c \cap \mathbb{Z} \times [0, \infty)$ as being 0-supersparse. Let

$$\mathcal{M} = \{ (s, t) : s_{n+1} - dr'_k/2 + 3(t - t_{n+1}) \leq s < s_{n+1} + dr'_k/2 - 3(t - t_{n+1}), \ t \geq t_{n+1} \}.$$

Claim II $\eta \upharpoonright (\mathcal{M} \setminus B_{n+1}) \cap \mathbb{Z} \times [t_{n+1}, t_{n+1}]$ is k-white.

Proof: If $t_{n+1} \geq h_{k+1}$, then by the inductive assumption on $|\mathcal{C}_s|$, $\mathcal{C}^* \setminus \{C_{n+1}\}$ is a $(k+1)$-white-cover of $\eta \upharpoonright \mathcal{T}_{k+1}^* \cap \mathbb{Z} \times (-\infty, t_{n+1} - 1]$, and by well-separatedness and SOL, the claim follows. Let $t < h_{k+1}$ and assume for some $B = (a, b) + W_{dr_k'} \in \mathcal{C}_b$,

$$B \cap (\mathcal{M} \setminus B_{n+1}) \cap \mathbb{Z} \times [t_{n+1}, t_{n+1}] \neq \emptyset.$$

Well-separatedness implies that η is a k-continuation at $(a - 3dr_k' - 3r_k', b)$ with width $7dr_k' + 6r_k'$ and extension $t_{n+1} - b$ from which the claim follows. $\blacktriangle$

Using Lemma 5.5.6, an immediate consequence of Claim II is that

$$\eta \upharpoonright (\mathcal{M} \cap \mathbb{Z} \times [t_{n+1} + 3r_k', \infty)) \setminus U \quad \text{is 0-white} \tag{5.5.10}$$

where $U = (s_{n+1} - 50r_k', t_{n+1}) + W_{100r_k'}$. Let

$$\mathcal{K} = \{ (s, t) : s_{n+1} - 3(t - t_{n+1}) \leq s < s_{n+1} + 3r_k' + 3(t - t_{n+1}), \ t \geq t_{n+1} \}.$$

Let $v = 3dr_k' + 3h_k + 3(dr_k' - 100r_k' - h_k) + 3h_k$.

Claim III η is a k-continuation at $(s_{n+1} - v, t_{n+1} + 100r_k' + h_k)$ with width $2v + 3r_k'$ and extension $dr_k' - 100r_k'$.

Proof: We need to show that η is k-white on the trapezoid

$$\mathcal{A} = (s_{n+1} - v - 3h_k, s_{n+1} + v + 3r_k' + 3h_k, t_{n+1} + 100r_k', t_{n+1} + 100r_k' + h_k, -3).$$

Let $D = (a, b) + W_{r_k' - 2dr_{k-1}'}$ be a test window such that $H = D \cap \mathcal{A} \neq \emptyset$. By the definition of v, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{A}$,

$$H \subset \mathcal{K}^c \quad \text{or} \quad H \subset \mathcal{M}.$$

If $H \subset \mathcal{K}^c$, then by well-separatedness and the inductive assumption on $|\mathcal{C}_s|$, $\eta \upharpoonright H$ is k-white. If $H \subset \mathcal{M}$, then by (5.5.10) and Remark 5.5.7, $\eta \upharpoonright H$ is 0-white. $\blacktriangle$

Claim IV $\mathcal{C}^*$ is a $(k+1)$-white cover of $(\mathcal{T}_{k+1} \cup \mathcal{T}_{k+1}') \cap \mathbb{Z} \times [0, t_{n+1} + dr_k' + h_k)$.

Proof: Let $D = (a, b) + W_{r'_k - 2dr'_{k-1}}$ be a test window such that

$$H = D \cap \left((\mathcal{T}_{k+1} \cup \mathcal{T}'_{k+1}) \cap \mathbb{Z} \times [0, t_{n+1} + dr'_k + h_k) \setminus \bigcup_{C \in \mathcal{C}^*} C \right) \neq \emptyset.$$

Let $\mathcal{A}'$ be the same trapezoid as $\mathcal{A}$ in the proof of Claim III except that its second time parameter is changed from $t_{n+1} + 100r'_k + h_k$ to $t_{n+1} + dr'_k + h_k$. By the definition of $\mathcal{K}$ and $\mathcal{A}'$,

$$H \subset \mathcal{K}^c \quad \text{or} \quad H \subset \mathcal{A}'.$$

In either case, $\eta \restriction H$ is k-white which proves the claim. ▲

Since $B_{n+1} = (s_{n+1}, t_{n+1}) + W_{3r'_k}$ is the last element in the enumeration of $\mathcal{C}_s$, Claim IV implies that η is a k-continuation at $(3(t_{n+1} + dr'_k + h_k), t_{n+1} + dr'_k + h_k)$ with width $w_{k+1} - 6(t_{n+1} + dr'_k + h_k)$ and extension $h_{k+1} + h - (t_{n+1} + dr'_k + h_k)$. It follows that $\mathcal{C}^*$ is a $(k+1)$-white-cover of $\mathcal{T}^*_{k+1}$. ■

Proof of Lemma 5.5.4. First, S^B_{k+1} being k-supersparse implies that $R'_{k+1} \cap \mathbb{Z} \times [y_{k+1} + r_{k+1}, \infty)$ is k-supersparse. Since the last element of $\mathcal{C}_s$, $B_n = (s_n, t_n) + W_{3r'_k}$, has $t_n < y_{k+1} + r_{k+1}$, it is easy to deduce from the proof of Lemma 5.5.8 that $\eta \restriction R'_{k+1} \cap \mathbb{Z} \times [y_{k+1} + 2r_{k+1}, \infty)$ is k-white. Since R'_k is k-supersparse and R'_{k+1}, R'_k overlap by $3r'_k$, it follows by Lemma 5.5.8 that $\eta \restriction R'_k$ is k-white. ■

5.5.3 Relaxation upper bound

The results and proofs in the relaxation upper bound are exactly analogous to two-line voting except that a black cone now extends with speed 3 (instead of 2) and one of the boundaries remains stationary. The first difference only helps to reduce the number of good errors needed to form a black island of a certain size. However, since the left boundary is stationary, we are only able to grow an island of size at least $c_0 \varepsilon^{-1/2}/2$ in the same time (see Lemma 4.2.4), and we need twice as many copies as before which changes the constant c_2 in Lemma 4.2.4 by a factor of 2.

Chapter 6

Conclusion and future work

This thesis has shown that two simple, one-dimensional cellular automata possessing the eroder property—GKL (soldiers rule) and two-line voting—are not able to conserve information in the long run when subject to strongly biased, independent noise. The mixing property was shown to hold for any positive error probability $\varepsilon > 0$ when the bias β is a sufficiently small (or "large") constant. The strong bias assumption is a weak point and leaves room for further improvement since it does not preclude the possibility that for $\beta \approx 1/2$ the processes $K_{\varepsilon,\beta}$, $L_{\varepsilon,\beta}$ become nonergodic. We believe, however, that $K_{\varepsilon,\beta}$, $L_{\varepsilon,\beta}$ are mixing for all $0 \le \beta \le 1$. The bias assumption allowed us to show that $K_{\varepsilon,\beta}$ and $L_{\varepsilon,\beta}$ are mixing for all $\varepsilon > 0$ which is of independent interest given that most results in probabilistic cellular automata and interacting particle systems require that ε be sufficiently small, thus leaving a gap in the error probability.

We have also shown that the finite-time, information-conservation quality of a mixing system as represented by the relaxation time has a tight, slightly superpolynomial bound as a function of $1/\varepsilon$. The lower bound was shown to be independent of the bias assumption, and so was the expression of the upper bound but not its proof: after showing that black (or white) islands of a certain size arise with constant probability in time $2^{c \log^2 1/\varepsilon}$, we need to make use of the mixing property to predict the probable fate of those islands. Both the lower bound and upper bound results required that the error probability be sufficiently small.

With respect to future work, one immediate task is to narrow the bias gap and, as a special case, prove that $K_{\varepsilon,\beta}$ and $L_{\varepsilon,\beta}$ remain mixing when $\beta = 1/2$. As Larry Gray has pointed out, it may be the case that the most difficult situation arises when β is very close to $1/2$ but not equal to $1/2$ since, then, neither the strength in bias nor the symmetry available with $\beta = 1/2$ may be easily exploitable.

A second avenue of exploration lies in the detailed characterization of the interface arising in one-dimensional rules possessing the eroder property such as GKL and two-line voting. The width of the interface seems to be a function of the error probability ε, and the existence of unstable subconfigurations—light gray and dark gray regions in two-line voting, and regions of alternating black and white states in GKL—seems to give rise to complex interactions including phenomena resembling branching processes. This may render the analysis of such systems more difficult than, say, local majority voting. Even between two-line voting and GKL the interface dynamics is slightly different: white and black islands cannot "interpenetrate" each other in the GKL rule whereas in two-line voting they can. This may make the GKL rule a little easier to handle than two-line voting.

In this thesis, a notion of boundary was achieved by identifying lines in space-time on one side of which a sample path was assured to be "well-behaved," i.e., k-black or k-white. It is not clear whether the k-black property will remain useful in the context of symmetric errors since k-black subconfigurations may not be sufficiently persistent in the presence of unbiased sparse errors. Indeed, if they were, it would imply that $K_{\varepsilon,1/2}$ and $L_{\varepsilon,1/2}$ are nonergodic which goes against the conjecture held by many in this field. However, k-blackishness may still be useful if a weaker form of persistence could be fruitfully exploited. We note that supersparsity was introduced to yield more manageable error patterns which are oblivious to the sign of errors but which embody a stronger form "self-similar infrequency." The probability estimates for supersparse errors was sufficient to carry out the relaxation time lower-bound argument for k up to $O(\log 1/\varepsilon)$ while still making use of the k-black (white) space-

time configuration property. For higher values of k, supersparsity is a too strong requirement to yield effective probabilities. Our upper-bound result suggests that supersparsity may already be sufficiently weak.

A third, more practical research goal lies in the application of sparsity/renormalization techniques to the design and analysis of large-scale distributed systems such as the Internet. Fault-tolerance issues are ubiquitous in such environments, ranging from reliable distributed service provision including access to alternate name servers and document depositories, to the graceful, responsive provision of these services when multiple components are subject to failures including temporary congestion. For scaling arguments to yield effective estimates, the size of the underlying system needs to be sufficiently large so that constants arising in sparsity analysis can be absorbed. The rapid growth of computer networks promises to provide a domain where these techniques may be feasibly applied.

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Curriculum Vitae

Kihong Park received his B.A. in February 1988 from Seoul National University, Korea, in the School of Management. After staying on as a graduate student for one semester at SNU, he continued his study at the Computer Science Department at the University of South Carolina, U.S.A., where he earned his M.S. degree in May 1990. He went on to serve his military duties in the Korean Army, entering the 3rd Military Academy and completing his service as 2nd Lieutenant in May 1991. He resumed his studies in the Computer Science Department at Boston University, U.S.A., completing his thesis in July 1996 and receiving his Ph.D. degree in January 1997. Kihong Park was supported by numerous awards and grants during his graduate studies, from a scholarship at Seoul National University, to a teaching assistantship and two research assistantships (NSF, NIH) at the University of South Carolina, to a Presidential University Fellowship, teaching fellowship, and research assistantship (NSF) at Boston University. Other awards include short-term support from Carnegie Mellon University and the Santa Fe Institute for research visits. As of August 1996, Kihong Park is an Assistant Professor of Computer Science in the Department of Computer Sciences at Purdue University.

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