Question 1. The key observation is that P consists of repeated occurrences of the same string of length 2^ℓ, namely, the string $P' = p_1 p_2 \dots p_{2^\ell}$. In other words $P = P'P' \dots P'$ where the number of copies of P' that are concatenated to form P is

$$n/2^\ell = 2^{q-\ell}$$

P occurs in T at the beginning of each of the above-mentioned $2^{q-\ell}$ copies of P', plus one occurrence that begins at position $n+1$ in T. Therefore the total number of occurrences of P in T is $2^{q-\ell}+1$.

Question 2. In what follows whenever we say "add an extra edge from v_i to v_j" it is always implied that no such edge is added if it already exists. The procedure for adding the extra edges is recursive.

If $n = 2$ there is no need to add any edge (this is the bottom of the recursion). If $n > 2$ then add an extra edge from every vertex in the left half of the chain to the middle of the chain (where the middle is taken to be the rightmost vertex of the left half), and from the middle of the chain to every vertex in the right half of the chain. Recurse on the left half of the chain, and on the right half of the chain. The total number $f(n)$ of newly added edges obeys the recurrence

- $f(n) = 2f(n/2) + ((n/2) - 2) + ((n/2) - 1) = 2f(n/2) + n - 3$
- $f(2) = 0$

whose solution is $O(n \log n)$. That there is always a path of length ≤ 2 from any v_i to any v_j, $i < j$, is proved by considering the recursion tree for the above-described recursive construction. Let w be the lowest (i.e., farthest from the root) node of the recursion tree whose chain (call it $\mathcal{C}$) contains both v_i and v_j. We distinguish two cases:

1. w is a leaf node in the recursion tree. In that case $j = i + 1$, and the claim trivially holds (because there is an original edge from v_i to v_{i+1}).
2. w is an internal node in the recursion tree. In that case the path of length 2 from v_i to v_j consists of the edge from v_i to the middle vertex in $\mathcal{C}$, followed by the edge from that middle vertex to v_j.

Question 3. In what follows whenever we say "add an extra edge from v_i to v_j" it is always implied that no such edge is added if it already exists. The procedure for adding the extra edges is recursive.

1. If $q = 0$ then $n = 2$ and there is no need to add any edge (this is the bottom of the recursion).

2. If $q > 0$ then consider every vertex v_k for which k is a multiple of $\sqrt{n}$ to be *special*; note that v_k is special if $k = 2^{(2^{q-1})}$. The special vertices partition the chain into $\sqrt{n}$ smaller chains of length $\sqrt{n}$ each; we call each such smaller chain a chunk, and we note that the rightmost vertex of each chunk is special.

   (a) Add an extra edge (directed left to right, of course) between every pair of special vertices; the number of such edges added is quadratic in $\sqrt{n}$, hence linear.

   (b) Add an extra edge from every vertex v_i to the special vertex at the end of the chunk to which v_i belongs; the number of such added edges is $n - 2\sqrt{n}$.

   (c) For every special vertex v_i other than v_n, add an edge from v_i to every vertex in the chunk immediately to the right of v_i; this adds $\sqrt{n}-2$ new edges that leave a special v_i, and since there are $\sqrt{n}$ special nodes the number of such added edges is again linear.

   (d) Recurse on each of the $\sqrt{n}$ chunks.

The total number $f(n)$ of extra edges added by the above recursive procedure obeys the recurrence

- $f(n) \leq \sqrt{n} f(\sqrt{n}) + cn$
- $f(2) = 0$

whose solution is $O(n \log\log n)$. That there is always a path of length ≤ 3 from any v_i to any v_j, $i < j$, is proved by considering the recursion tree for the above-described recursive construction. Let w be the lowest (i.e., farthest from the root) node of the recursion tree whose chain (call it $\mathcal{C}$) contains both v_i and v_j. We distinguish two cases:

1. w is a leaf node in the recursion tree. In that case $j = i + 1$, and the claim trivially holds (because there is an original edge from v_i to v_{i+1}).

2. w is an internal node in the recursion tree. In that case the path of length 3 from v_i to v_j consists of the edge from v_i to the special vertex v_k in the chunk of v_i (which is the rightmost vertex of that chunk), followed by the edge from v_k to the special vertex v_s of the chunk that is just before the chunk of v_j, followed by the edge from v_s to v_j.