

Thresholding Random Geometric Graph Properties Motivated by Ad Hoc Sensor Networks ^{*}

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Abstract

We study the emerging phenomenon of ad hoc, sensor-based communication networks. The communication is modeled by the random geometric graph model $G(n, r, \ell)$ where n points randomly placed within $[0, \ell]^d$ form the nodes, and any two nodes that correspond to points at most distance r away from each other are connected. We study fundamental properties of $G(n, r, \ell)$ of interest: connectivity, coverage, and routing-stretch. We use a technique that we call *bin-covering* that we apply uniformly to get (asymptotically) tight thresholds for each of these properties. Typically, in the past, random geometric graph analyses involved sophisticated methods from continuum percolation theory; on contrast, our bin-covering approach is discrete and very simple, yet it gives us tight threshold bounds. The technique also yields algorithmic benefits as illustrated by a simple local routing algorithm for finding paths with low stretch. Our specific results should prove interesting to the sensor networking community that has seen a recent increase in the study of random geometric graphs motivated by engineering ad hoc networks.

Keywords: Random geometric graphs, Thresholds, Sensor network models, Local algorithm, Coverage, Stretch, Connectivity.

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1 Introduction and Motivation

We study properties of certain random graphs. There is the classical random graph $G(n, p)$ [8] that has been studied for over 50 years now. There are many other random graphs, some motivated by physical phenomenon. For example, there are random graph models for studying social networks [24] and those for studying web graphs [1, 25] where a physical phenomenon is modeled as an appropriate random graph and understanding the structural properties of random graphs provides insights into the behavior of the underlying physical phenomenon. In this paper, we focus on the physical phenomenon due to ad hoc sensor networks, a technology in the horizon that may transform our lives significantly. The structural properties we study are motivated by engineering such networks—connectivity, coverage, routing, etc — and we prove bounds on the thresholds for emergence of these key properties in suitable random graphs.

1.1 Random Geometric Graph Models

We study the random geometric graph model $G(n, r, \ell) = (V, E)$ where n vertices uniformly and randomly placed within $[0, \ell]^d$ form the nodes ¹ in V , and $(u, v) \in E$ iff $D(u, v) \leq r$, for some $0 < r \leq \ell$. The distance ² between points $D([u_1, \dots, u_d], [v_1, \dots, v_d]) = \max_{1 \leq i \leq d} \{|u_i - v_i|\}$. We call this the *Bernoulli model*. We also study its *Poisson* version where the number of points in $[0, \ell]^d$ is given by the Poisson distribution with mean n . The main advantage of the Poisson model is that the distribution of nodes in disjoint regions are independent; this simplifies analyses in some cases. Of our interest in this paper would be $d = 1, 2$, that is, placement of nodes along a line or on the plane, although our results can be generalized to higher dimensions.

The more widely-studied geometric random graph model is $G(n, r)$ where n nodes are distributed in $[0, 1]^d$. Typically, thresholds in $G(n, r)$ are obtained when $n \rightarrow \infty$ while $r = f(n) \rightarrow 0$. Thus, the model and the thresholds are better suited for *dense* random graphs [35]. The $G(n, r, \ell)$ model on the other hand is more detailed and general: the node density n/ℓ^d can converge to zero, or to a constant $c > 0$, or diverge as $\ell \rightarrow \infty$, depending on the relative values of r, n , and ℓ ; thus, this model can be applied both to dense as well as sparse networks [35]. While there is a lot of literature on $G(n, r)$ ([5, 6, 7, 13, 16, 23, 31, 32, 19, 9, 14] and see e.g., the book by Penrose [30] or the course by Aldous [4] or a short overview at [12]), work on the more-detailed model $G(n, r, \ell)$ is fewer and more recent [35, 33, 34]. In this paper, we study the $G(n, r, \ell)$ and give the first known asymptotically tight thresholds, but our methods when applied to the $G(n, r)$ model will provide results that either give the best known ones (area coverage) or give the first known bounds (stretch) and will be significantly simpler than prior analyses.

1.2 Motivation

Given the history of random geometric graphs and the nice techniques from percolation theory (see e.g., [28, 18]) used to analyze them, we believe that the study of the more detailed model $G(n, r, \ell)$ is well-justified in its own right. However, we arrived at the specific problems we study because of the rising motivation in networking community to understand and use threshold properties in *ad hoc sensor networks* (ASNs) based on these geometric random graph models.

In the router networks in the Internet, we use as few routers as needed; we carefully optimize the infrastructure that connects them which is typically by wired means; and, we manage the network of the routers often in centralized, careful manner. In contrast, with ASNs, the expectation is to use a lot of sensors; to “sprinkle them liberally” in areas of interest; to let them communicate with each other wirelessly; and, to

¹Henceforth, a node refers to one of n vertices of the graph and a point refers to a real-valued coordinate in the d -dimensional square.

²We use the L_∞ norm. Our results will easily extend to Euclidean norm with small changes in the constants.

manage them typically in a distributed manner. Thus while existing wired networks are engineered carefully — in terms of setting up routing paths, covering the area being served with careful placement of routers, etc — ASNs have an opportunity to make use of the “statistical” aspects, and be more flexibly engineered, i.e., have probabilistic coverage of the region or use several paths simultaneously to route information etc. These statistical aspects emerge from the random graph implicit in sprinkling a region with sensors (nodes) and their ability to communicate by wirelessly within a bounded region r (edges). That is the motivation for the increasing study of geometric random graph models in networking community.

Thus far, bulk of the work in networking community on random graph models for ASNs has been in the $G(n, r)$ model [10, 17, 26, 27, 36].³ Recently, the more-detailed $G(n, r, \ell)$ model has been proposed. In particular, in [35, 21] authors point out that in practice, sensor networks cannot be too dense due to the problem of spatial reuse: when a node transmits, all the nodes within its transmitting range must be silent in order to not corrupt the transmission; thus $G(n, r, \ell)$ random graph may be more suitable to model ASNs [35]. We note that the threshold results of $G(n, r, \ell)$ can be inferred from $G(n, r)$ by suitable “scaling” (i.e., considering $G(n, r/\ell)$). In this paper, we adopt the more general $G(n, r, \ell)$ model (partly motivated by the reasons above); however, our techniques apply equally apply to the $G(n, r)$ as well and we will state some of those results as well.

1.3 Problems of Interest

We seek to study properties of $G(n, r, \ell)$ that are of interest to ASNs.

We study three fundamental classes of problems. At the lowest level, the question is, when does a random set of nodes have the ability to broadcast and “cover” the entire region under consideration? We study this problem for two different notions of coverage. At a higher level, the question is, when does a random set of nodes have a communicating path between any two pair of nodes? We study this formally as the connectivity problem. Finally, at a even higher level, how good are the communicating paths between pairs of nodes? We study this as the quality of paths with respect to the shortest path on the plane. We now present formal definitions of our problems.

1. *Physical Coverage.* There are two distinct notions of a ASN covering a given space. In the first, $area(\cup_{i=1}^n disc(i))$ where $area$ is the area of the region and $disc(i)$ is the set of points in the square within distance r of the node i . (That is, the area covered by a node i is the area of a box — within the unit square — of side $2r$ centered at node i .) We call this *area coverage*. We say there is *full area coverage* if the area coverage is at least $(1 - o(1))(\ell)^d$.

In the second, a box is said to *enclose* a connected component if no other box of smaller area contains all the nodes in the connected component inside. We define *enclosure coverage* as the area of the largest box enclosing a connected component in $G(n, r, \ell)$. We look for threshold for *full enclosure coverage*, i.e., coverage at least $(1 - o(1))(\ell)^d$.

2. *Graph Connectivity.* When does $G(n, r, \ell)$ become connected, that is, there exists a communication path between any two nodes?
3. *Route Stretch.* Given two nodes u, v in $G(n, r, \ell)$, $stretch(u, v)$ is defined as $D_G(u, v)/D(u, v)$, i.e., the ratio of the shortest distance $D_G(u, v)$ between u and v in the graph (assuming that there is a path between u and v in $G(n, r, \ell)$), otherwise the stretch is ∞ and the distance is simply the sum of the

³The classical random graph model is $G(n, p)$ where there are n nodes and $(u, v) \in E$ with probability p independent of all other pairs of vertices. $G(n, p)$ is not a suitable model for communication in ASNs since each sensor node can only transmit to a bounded radial area. The probability of existence of edges $(i, j), (j, k), (i, k)$ is independent in $G(n, p)$, but there is clearly lot of dependence in ASNs based on the geometric distribution of i, j, k and their relative positions. Technically, analyses in $G(n, r)$ or $G(n, r, \ell)$ model become harder because of this dependence.

length of the edges in the path) to the norm distance $D(u, v)$ on the plane. The *stretch* of $G(n, r, \ell)$ is $\max_{(u,v)} \text{stretch}(u, v)$. What is the stretch of $G(n, r, \ell)$ assuming $G(n, r, \ell)$ is connected?

Each of these formal problems is well-motivated. For example, if a sensor wants to send a message from one side of the boundary to the opposite side, we need full enclosure coverage. If we need to monitor every point in a given area we need full area coverage. (The $o(1)$ term is included to ignore boundary effects in $[0, \ell]^d$.) The two notions of coverage are implicit and inherent in several papers on percolation theory [28], and is widely studied as part of “coverage processes” [22]. Connectivity in $G(n, r)$ is among the most extensively studied problems in geometric random graphs [5, 20] and sharp thresholds are known. In the $G(n, r, \ell)$ model, there are gaps [35] which we will close for the one-dimensional case as described in Section 1.4. Stretch is a well-established notion in communication graphs and has been studied in general and Euclidean graphs [29] but we do not know of relevant results for the $G(n, r, \ell)$ or $G(n, r)$ model. All of these problems have been empirically studied in wireless networking community without analyses [10, 17, 26, 27, 36].

1.4 Our Results

Our main technical results are as follows: (We say thresholds are *asymptotically tight* if the lower bound on the threshold for the presence of a property differs from the upper bound by at most a constant; we say they are *sharp* if the upper and lower bounds match). Here we state our results for the $G(n, r, \ell)$ model.

1. *Coverage.* For $d = 1$, we prove that both full enclosure coverage and connectivity occur at the *sharp* threshold of $rn \approx \ell \ln \ell$. This settles the problem left open in [35] of closing the gap between the upper and lower bounds for the connectivity threshold that differed by a factor of 2. For full area coverage, the sharp threshold is at $rn \approx (1/2)\ell \ln \ell$. More precisely, we relate the constant in the threshold to the exponent of r (cf. Theorem 2.1).

For $d = 2$, we prove that full area coverage occurs at the sharp threshold of $r^2n \approx (1/2)\ell^2 \ln \ell$. For full enclosure coverage, threshold is asymptotically tight with full coverage almost surely with $r^2n > 4 \ln 7 \ell^2$ and no full enclosure coverage almost surely with $r^2n < 0.25\ell^2$.

2. *Connectivity.* For $d = 2$, we present asymptotically tight bounds for the threshold. If $r^2n > 2\ell^2 \ln \ell$, graph is almost surely connected, and if $r^2n < 0.25\ell^2 \ln \ell$, graph is almost surely not connected.
3. *Stretch.* For $d = 2$, we show that if $r^2n > (22/\alpha)\ell^2 \ln \ell$, then the stretch is at most $1 + \alpha/2$ almost surely. We also show a simple local algorithm for finding such high quality paths. This is asymptotically optimal because if $r^2n < 0.25\ell^2 \ln \ell$ (from our connectivity bound), then stretch is unbounded.

All the above results are (asymptotically) tight threshold bounds for $G(n, r, \ell)$, and they hold both for the Bernoulli as well as the Poisson models. One way to interpret these technical results is that in the random graph $G(n, r, \ell)$ the “regime” of connectivity is $r^2n = c\ell^2 \ln \ell$ for some c . In the asymptotic vicinity of the connectivity regime, interesting things happen: not only does the graph get connected, but the paths between vertices are of high quality (that is they have small stretch), and the entire domain gets covered. This is a very useful parameterization for network designers to know.

Our technique yields thresholds for all the above problems for the $G(n, r)$ model as well. Some of the results for $G(n, r)$ are not new; our main purpose is to show that results for $G(n, r)$ can be deduced in a fashion similar to the more general $G(n, r, \ell)$ theorems. For the one-dimensional $G(n, r)$ model we get asymptotically sharp thresholds for connectivity, full enclosure coverage and full area coverage. (This result can essentially also be found in Penrose’s monograph [30].) While the threshold for connectivity and full

enclosure coverage occurs at $r = \frac{\ln n}{n}$, the threshold for full area coverage occurs at $r = \frac{1}{2} \frac{\ln n}{n}$ (cf. Theorem 2.2). We note that the same connectivity threshold also follows from the work of [11] (see also [5]), but our derivation here is significantly simpler. For the two-dimensional $G(n, r)$ model, our technique gives a sharp threshold for full area coverage (cf. Theorem 3.3). This problem was also considered in [3][Chapter H] and it was shown that using the Poisson clumping heuristic argument one can approximate the probability of coverage; for a more rigorous and complicated argument see [22][Chapter 3].

We note that, interestingly, while thresholds in $G(n, r)$ apply (asymptotically) for dense networks, the thresholds in $G(n, r, \ell)$ can apply to either dense or sparse networks depending on the relative values of r, n and ℓ (see also [35]). For example, consider the threshold region for *full area coverage*. Theorem 3.2 implies that the node density $n/\ell^2 = \Theta(\frac{\ln \ell}{\ell^{2\epsilon} (f(\ell))^2})$ can converge to zero, or a constant $c > 0$, or diverge as $\ell \rightarrow \infty$ depending on ϵ . However, the threshold for the same property for $G(n, r)$ (cf. Theorem 3.3) applies only for dense networks (the node density $\rightarrow \infty$).

1.5 Technical Overview

Our thresholds are obtained by an uniform approach that we call as *bin-covering*. We cover the region with overlapping “bins” with some overlap parameter. We rewrite the desired property as random variables in terms of gaps between points, the number of empty bins etc and optimize the overlap parameter to get the best threshold bound for desired random variables. The whole approach is more detailed than merely bucketing the domain into disjoint regions; in fact, the overlap between bins is crucially used in getting our results. It is equally applicable for both Bernoulli and Poisson models. In fact, the technique is applied in a similar fashion in both models; the independence in the Poisson model simplifies analysis only a little. The technique also works for the $G(n, r)$ model as well.

Previous works on $G(n, r)$ (see [30]) used sophisticated probabilistic results from continuum percolation theory to derive their bounds. In contrast, our bin-covering technique is discrete, quite simple, and it is nearly a cook-book technique to find thresholds for many geometric random graph properties. Taking the approach based on percolation theory might give a more refined analysis in some cases, but we feel that the simplicity and wide applicability of bin-covering approach and its ability to give asymptotically tight bounds (and sharp thresholds in some cases, notably area coverage in both one and two dimensions) makes it readily useful. More importantly, our technique can also yield algorithmic benefits as illustrated by a simple local routing algorithm for finding paths with low stretch (Section 3.4). However, we note that the approach has limitations and does not yield sharp thresholds for all properties, e.g., connectivity or enclosure coverage in two-dimensional case.

In Section 2, we study the one dimensional case. In Sections 3 we study all three problems for the two dimensional case. Extensions and concluding remarks are in Section 4.

2 One-Dimensional Case

We first focus on one dimensional $G(n, r, \ell)$ model: n nodes are placed randomly on a line of length ℓ and two nodes are connected if they are within a distance of r on the line. We denote the corresponding random graph induced to be $G_1(n, r, \ell)$. We apply a technique that we call *bin-covering* to uniformly analyze *connectivity*, *full enclosure coverage*, and *full area coverage* of $G_1(n, r, \ell)$. Our technique yields sharp thresholds for all the three properties.

Note that in the $G(n, r, \ell)$ model, the asymptotic behavior is studied as $\ell \rightarrow \infty$. Since full enclosure coverage is defined in terms of the largest connected component (i.e., the length of the line enclosing the largest connected component is almost the entire line) it turns out that the threshold for connectivity holds for full enclosure coverage also. Still full enclosure coverage is a distinct property: for example, unlike

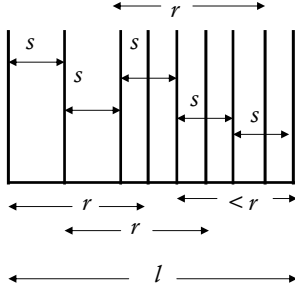


Fig. (a)

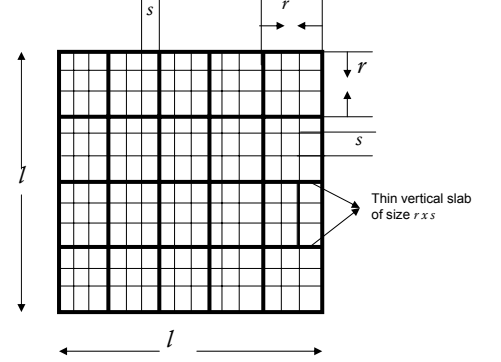


Fig. (b)

Fig. (a): Bin-covering in 1D. Spacing parameter is $s(< r)$, i.e., the line is partitioned into strips of length s . In the above example, $r = 1$, $s = 0.4$, and $l = 2$. Thus there are 5 strips of length s (numbered 1 to 5 from left to right) and there are four bins, all of them of size r , except for the last (which is smaller). A situation when condition 2 mentioned in the proof of Theorem 2.1 occurs is the following: node i lies in strip 1 and node $i + 1$ lies in strip $1 + \lceil r/s \rceil - 1 = 3$ and the distance between i and $i + 1$ is greater than r . **Fig. (b):** Bin covering in 2D. Spacing parameter is $s(< r)$. There is a total of l/s vertical and l/s horizontal slabs; together they induce (approximately) $(l/s)^2$ bins of size $r \times r$. In proof of Theorem 3.2, for area coverage, we partition using blocks of $2r$ instead of r .

connectivity, the presence of isolated vertices does not imply the absence of full enclosure coverage. In fact, in the two dimensional case, the connectivity and full enclosure coverage thresholds are asymptotically different (cf. Section 3).

We assume the Bernoulli model for the following theorem, although the theorem holds for the Poisson model as well, with almost no change in the proof technique.

Theorem 2.1 Consider the $G_1(n, r, \ell)$ model and let $r = r(\ell) = \ell^\epsilon f(\ell)$ for some $0 \leq \epsilon < 1$ and $f(\ell)$ is a function that grows strictly slower than any function of type ℓ^γ where $\gamma > 0$. Let $n = n(\ell) = \omega(1)$.

- $G_1(n, r, \ell)$ is connected and has full enclosure coverage a.a.s.⁴, if $rn \geq \ell((1 - \epsilon) \ln \ell + \ln \ln \ell + h(\ell))$ for any $h(\ell) \rightarrow \infty$.
- $G_1(n, r, \ell)$ is disconnected and has no full enclosure coverage a.a.s. if $rn \leq \ell((1 - \epsilon) \ln \ell + \ln \ln \ell + g(\ell))$ for any $g(\ell) \rightarrow -\infty$.

⁴We say that an event E_ℓ describing a property of a random structure depending on a parameter ℓ holds asymptotically almost surely (a.a.s.) if $\Pr(E_\ell) \rightarrow 1$ as $\ell \rightarrow \infty$.

Also, $G_1(n, r, \ell)$ has full area coverage a.a.s. if $rn \geq \frac{1}{2}\ell((1 - \epsilon) \ln \ell + \ln \ln \ell + h(\ell))$ for any $h(\ell) \rightarrow \infty$, and has no full area coverage a.a.s if $rn \leq \frac{1}{2}\ell((1 - \epsilon) \ln \ell + \ln \ln \ell + g(\ell))$ for any $g(\ell) \rightarrow -\infty$.

We observe that the conditions on the magnitude of r and n are not restrictive. In fact, if $r = \Omega(\ell)$, then every node can directly transmit to most other nodes and connectedness, full enclosure coverage, and full area coverage are ensured independently of n . And if $n = O(1)$ then, r should be $\Omega(\ell)$ for these properties to be satisfied.

Proof: Choose a spacing parameter $s < r$ (whose value will be fixed appropriately later) and split up the line by strips of length s . Number the strips with index $j, 1 \leq j \leq \lceil \ell/s \rceil$, starting from the left end. Refer Figure (a). For every $i = js, 0 \leq j \leq \lceil (\ell - r)/s \rceil$, identify a bin of size r (except for the last bin which may be possibly smaller) starting from point i from the left end. Thus, there are a total of $\lceil (\ell - r)/s \rceil + 1$ bins where each bin overlaps with at most $2\lceil r/s \rceil$ other such bins. Let r.v. $X_i, 1 \leq i \leq \lceil (\ell - r)/s \rceil + 1$ be an indicator for an empty bin numbered i (starting from the left end). Let r.v. $X = \sum_i X_i$ denote the total number of empty bins.

It is clear that there will be full enclosure coverage and connectivity if the following two conditions are true: if there are no empty bins and there is no gap greater than r between any two consecutive nodes in the line. By our bin-covering scheme the above two conditions are ensured if:

1. There are no empty bins, and
2. For every node $i, 1 \leq i \leq n - 1$ (assume that nodes are numbered according to their value on the line) the following configuration does not occur: i occurs in some strip $j, i + 1$ occurs in strip $j + \lceil r/s \rceil - 1$, and the distance between i and $i + 1$ is greater than r . (Refer Figure (a) for an illustration.)

We will use r.v. $Y_i, 1 \leq i \leq n - 1$ to indicate if the configuration mentioned in condition (2) occurs between nodes i and $i + 1$. Let $Y = \sum_{i=1}^{n-1} Y_i$. Let the r.v. $Z = X + Y$. We will upper bound $E[Z]$ and show that it tends to zero for an appropriate threshold function. Then, by Markov's inequality, it will follow that $P[Z = 0] \rightarrow 1$, i.e., there will be full coverage a.a.s. We have $E[X] \leq \lceil \ell/s \rceil (1 - r/\ell)^n$. We upper bound $E[Y]$ by using the sum $\sum_{a,b} Y_{a,b}$, where $Y_{a,b}$ is defined as follows. $\Pr(Y_{a,b} = 1)$ if node a occurs in some strip j , node b occurs in strip $j' = j + \lceil r/s \rceil - 1$ and there is (this specific) gap of length r between a and b starting from a . By conditioning on strip j , we have

$$\Pr(Y_{a,b} = 1) \leq \sum_{j=1}^{\lceil \ell/s \rceil} (s/\ell)(s/\ell)(1 - r/\ell)^{n-2} \leq s/\ell(1 - r/\ell)^{n-2}(1 + o(1)).$$

Thus, $E[Y] \leq (1 - r/\ell)^{n-2}(n^2 s/\ell)(1 + o(1))$.

$$\begin{aligned} E[Z] &= E[X] + E[Y] \\ &\leq \lceil \ell/s \rceil (1 - r/\ell)^n + n^2 s/\ell (1 - r/\ell)^{n-2} (1 + o(1)) \\ &\leq \left(\frac{\ell}{s} + s/\ell\right) e^{-rn/\ell} + \frac{n^2 s}{\ell} e^{-r(n-2)/\ell} (1 + o(1)). \end{aligned}$$

Substituting $rn = k\ell \ln \ell$, where $k = k(\ell)$, we have,

$$E[Z] \leq (1 + o(1)) \frac{1}{s\ell^{k-1}} + \frac{k^2 (\ln \ell)^2 s}{\ell^{k-1} r^2}.$$

The value of s that minimizes the above is $s = \frac{r}{k \ln \ell}$. Let $r = \ell^\epsilon f(\ell)$ as in the theorem. Substituting the above value of s , we have

$$E(Z) \leq \frac{2k\ell^{1-\epsilon-k} \ln \ell}{f(\ell)}.$$

Then $E(Z)$ goes to zero $k \geq (1 - \epsilon) + \frac{\ln \ln \ell + h(\ell)}{\ln \ell}$ for any $h(\ell) \rightarrow \infty$ as $\ell \rightarrow \infty$.

To show the lower bound, we use the second moment method [2]. Let $\delta = \delta(\ell) = \Theta(r/(k\ell \ln \ell))$. We partition this part of the line into strips and bins as before. That is, let s be a spacing parameter; split up the line by strips of length s . We will set $s = r/k \ln \ell$ (this is same value that was used in the upper bound proof above). Number the strips with index j , $1 \leq j \leq \lceil \ell(1 - 2\delta)/s \rceil$, starting from the left end. For every $i = js$, $0 \leq j \leq \lceil (\ell(1 - 2\delta) - r)/s \rceil$, identify a bin of size r (except for the last bin which may be possibly smaller) starting from point i from the left end. Thus, there are a total of $\lceil (\ell(1 - 2\delta) - r)/s \rceil + 1$ bins where each bin overlaps with at most $2\lceil r/s \rceil$ other such bins. We show that there is an empty bin in this middle part of the line asymptotically almost surely (if the condition of the Theorem holds) implying that there can not be full coverage. Let $X = \sum_i X_i$ be the total number of empty bins, as defined earlier; we show that $\Pr(X = 0) \rightarrow 0$, i.e., a.s. there is an empty bin.

$$\begin{aligned}
E[X] &= (1 + \lceil \frac{\ell(1 - 2\delta) - r}{s} \rceil) (1 - \frac{r}{\ell})^n \sim \frac{\ell(1 - 2\delta)}{s} (1 - \frac{r}{\ell})^n \\
E[X^2] &= E[X] + 2 \sum_{i \neq j} E[X_i X_j] \\
&= \sum_{\text{bins } i \text{ and } j \text{ don't overlap}} E[X_i X_j] + \\
&\quad \sum_{\text{bins } i \text{ and } j \text{ overlap}} E[X_i X_j]. \\
&\sim \frac{\ell(1 - 2\delta)}{s} \frac{\ell(1 - 2\delta) - r}{s} (1 - 2r/\ell)^n + \frac{2\ell(1 - 2\delta)}{s} \sum_{i=0}^{r/s-1} (1 - ((r + is)/\ell)^n) (\ell(1 - 2\delta)/s) \\
&\sim \frac{\ell(1 - 2\delta)}{s} \frac{\ell(1 - 2\delta) - r}{s} (1 - 2r/\ell)^n + \frac{2\ell(1 - 2\delta)}{s} \sum_{i=0}^{r/s-1} e^{-(r+is)n/\ell} \\
&\sim \frac{\ell(1 - 2\delta)}{s} \frac{\ell(1 - 2\delta) - r}{s} (1 - 2r/\ell)^n + \frac{2\ell(1 - 2\delta)}{s} e^{-rn/\ell} \frac{1 - e^{-sn/\ell}}{1 - e^{-sn/\ell}}
\end{aligned}$$

Substituting $s = r/(k \ln \ell)$ and $rn = k\ell \ln \ell$ (where $k = k(\ell)$), and using the fact that $r = \ell^\epsilon f(\ell)$, we have,

$$\begin{aligned}
\frac{E[X^2]}{E^2[X]} &\sim 1 + \frac{2(1 - 1/e)(1 - 1/\ell^k)\ell^k f(\ell)}{\ell^{1-\epsilon} k \ln \ell (1 - 2\delta)} \\
&\rightarrow 1
\end{aligned}$$

if $k \geq (1 - \epsilon) + \frac{\ln \ln \ell + g(\ell)}{\ln \ell}$ where $g(\ell) \rightarrow -\infty$. In other words, $rn \leq \ell((1 - \epsilon) \ln \ell + \ln \ln \ell + g(\ell))$. Since, by Chebyshev's inequality, $\Pr(X = 0) \leq \frac{E[X^2]}{E^2[X]} - 1$, there is at least one empty bin in $(\delta\ell, (1 - \delta)\ell)$; the lower bound for full enclosure coverage follows. To see the same for connectivity, we note that there will be at least one node a.s. on either side of the empty bin since $(1 - 2\delta/\ell)^n \sim e^{-2\delta k \ln \ell / r} \sim \Theta(1)$, by our choice of δ ; we conclude that the graph will be also disconnected a.s.

To obtain thresholds for full area coverage we divide the line into bins of size $2r$ (instead of r as was done above). We will have full area coverage if there are no empty bins and there is no gap greater than $2r$ between any two consecutive nodes in the line. Then the analysis is similar as above. ■

We could have tried to get threshold bounds by simply dividing the line into a bins of size, say $r/2$ (for upper bound) or r (for lower bound), but this would give us worse bounds. Intuitively, it seems better to

have bins of size r but all bins being non-empty does not imply full enclosure coverage. In our bin-covering technique, we allow overlap of bins of suitable size (say r) and we use a spacing parameter to control the amount of overlap. Note that we could also have considered bins of size $r - 2s$ (instead of r) and forcing that situation that there be no empty bins. This would have spared us from invoking the r.v. Y . The reader can verify that the first moment step yields the same upper bound but finding the best value of s (which consequently determines the threshold) becomes a matter of trial and error and it is difficult to optimize.

Bin-covering Technique The approach used in the above proof consists of the following steps (works for both the Bernoulli and the Poisson model):

1. Cover the space using many overlapping bins, the amount of overlap being determined by the spacing parameter s .
2. Define a random variable (Z as in proof of Theorem 2.1) which represents the number of gaps of desired length. Upper bound Z by the number of empty bins (X) and the number of gaps between points (Y).
3. $E[X]$ will be inversely proportional to s while $E[Y]$ will be directly proportional to s . Thus, there is a value of s which minimizes the function. Plug that value of s in the above upper bound and guess the form of the function of r (for example, $rn = k\ell \ln \ell$ as in the proof of Theorem 2.1).
4. Determine the best value of the threshold function which makes $E[Z]$ go to zero asymptotically.
5. To show the lower threshold, we cover the space as before using spacing parameter s . We use the value of s as determined in step 3. Guess the form of the function of r and show that $\Pr(X = 0) \rightarrow 0$, i.e., a.a.s there is an empty bin.

Applying the same approach to the one-dimensional version of the $G(n, r)$ model yields *sharp* thresholds for connectivity, coverage, and full area coverage shown in the theorem below.

Theorem 2.2 $G_1(n, r)$ has full enclosure coverage and connectivity a.a.s. if $r \geq \frac{\ln n + g(n)}{n}$ where $g(n) \rightarrow \infty$ and no full enclosure coverage and is disconnected a.a.s. if $r \leq \frac{\ln n + h(n)}{n}$, where $h(n) \rightarrow -\infty$. Also, $G_1(n, r)$ has full area coverage a.a.s. if $r \geq \frac{\ln n + g(n)}{2n}$ and has no full area coverage a.a.s if $r \leq \frac{\ln n + h(n)}{2n}$.

3 Two-Dimensional Case

We now focus on the 2-dimensional case, i.e., $G_2(n, r, \ell)$ model (or just $G(n, r, \ell)$ for convenience). We assume that the corners of the $\ell \times \ell$ square are $(0, 0)$, $(\ell, 0)$, (ℓ, ℓ) , $(0, \ell)$ (listed counterclockwise). The left, right, top, bottom sides of the square have the obvious interpretation. We assume the Bernoulli model for the proofs unless otherwise stated; the theorems also hold for the Poisson model with little change in the proofs.

3.1 Enclosure Coverage

We first analyze enclosure coverage. For this, we need the following concept (a similar concept is used in continuum percolation [28]). A *left-to-right (L-R) crossing* in $G(n, r, \ell)$ is a path starting from a node within a distance of $o(\ell)$ from the left side of the unit square and ending in a node within a distance of $o(\ell)$ from the right side of the square. A *Top-to-Bottom (T-B) crossing* is defined similarly. It is easy to show the following Lemma:

Lemma 3.1 *The number of nodes in any L-R crossing or T-B crossing in $G(n, r, \ell)$ is at least $\ell(1 - o(1))/r$. There is full enclosure coverage in $G(n, r, \ell)$ if and only if there is a L-R crossing and a T-B crossing.*

We also need the following. Divide the square into blocks of size $r/2 \times r/2$. Let the blocks be numbered by their row and column numbers (rows are numbered from top to bottom and columns are numbered left to right starting from 0 to $\lfloor 2/r \rfloor$). Define the following (undirected) graph H : vertices are the set of empty blocks (i.e., no nodes of G in the blocks) and there is an edge between two empty blocks if they touch each other (either at the sides or at the corners, thus each vertex has a maximum of 8 neighbors).

Lemma 3.2 *If there is no simple path of length $\lfloor 2\ell/r \rfloor - 1$ in H then there exists a T-B crossing in $G(n, r, \ell)$.*

Proof: We show the contrapositive. Assume that there exists no T-B crossing. Consider the natural “grid” graph induced by partitioning the unit square into $r/2 \times r/2$ blocks, i.e., the vertices of the grid are the corners of the block squares and the edges are the grid edges. Associate with each node the left bottom-corner of the block to which it belongs. Consider the subgraph G' of the grid graph induced by those vertices with *no* associated (sensor) nodes (ignore the vertices of the top and right sides — they will not be associated with any nodes). Since there exists no T-B crossing, there should be a path in G' of size $\lfloor 2\ell/r \rfloor - 1$. We note that the dual of G' (which is planar) is H . Thus H has a path of length $\lfloor 2\ell/r \rfloor - 1$. ■

Theorem 3.1 *There is no full enclosure coverage a.a.s. in $G(n, r, \ell)$ when $r^2 n < \ell^2/4$ and $r = o(\ell)$, and there is full enclosure coverage a.a.s. when $r^2 n > (4 \ln 7)\ell^2$ and $r = o(\ell)$.*

Proof: For the upper threshold, we show that the probability of an L-R crossing in $G(n, r, \ell)$ tends to zero as ℓ tends to ∞ . We do so by constructing an appropriate branching process.

Let p be an arbitrary node within a distance of $o(\ell)$ from the left side. We note that the number of neighbors of any node is stochastically dominated by the random variable $Z \sim B(n, 4r^2/\ell^2)$. Now consider the following sequence of random variables defined by: $Y_0 = 1, Y_i = Y_{i-1} + Z_i - 1$, where $Z_i \sim Z$. Thus Y_i stochastically upper bounds the number of nodes in step i (in step 0, we have just node p). The Y_i 's and Z_i 's can be naturally interpreted as a branching process (for example, see [2]). Let T be the least t for which $Y_t = 0$ ($T = \infty$ if no such t exists). From Lemma 3.1, a necessary condition for an L-R crossing to exist in $G(n, r, \ell)$ is that the $T > \ell(1 - o(1))/r$. Since $Z_1 + \dots + Z_t$ has a binomial distribution with mean $nt(4r^2)/\ell^2 < t$ we have by Chernoff bound,

$$\Pr(T > t) \leq \Pr(Y_t > 0) = \Pr(Z_1 + \dots + Z_t \geq t) \leq e^{-\delta t}$$

for some constant (independent of n, r , and ℓ) $\delta > 0$. Thus the probability that an L-R crossing exists from p is at most $e^{-\ell(1-o(1))\delta/r}$; and the probability that L-R crossing exists at all (from any starting point within a distance of $o(\ell)$ from the left) is at most $ne^{-\ell(1-o(1))\delta/r} \rightarrow 0$ as $\ell \rightarrow \infty$.

For the lower threshold, Lemma 3.2 gives a sufficient condition for the existence of a T-B crossing. The number of paths of length $\lfloor 2\ell/r \rfloor - 1$ starting from the left is upper bounded by $\frac{2\ell}{r} 7^{2\ell/r}$. Thus the expected number of paths having all of its $\lfloor 2\ell/r \rfloor$ blocks empty is

$$\leq \frac{2\ell}{r} 7^{\frac{2\ell}{r}} \left(1 - \frac{r}{2\ell}\right)^n \sim e^{\ln(\frac{2\ell}{r}) + \frac{2\ell}{r} \ln 7 - \frac{rn}{2\ell}}$$

Substituting $n = \frac{k\ell^2}{r^2}$, we see that the above tends to zero for a constant $k > 4 \ln 7 \approx 7.78$. A similar argument holds good for the existence of a L-R crossing. ■

3.2 Area Coverage

The following theorem shows a threshold for full area coverage in two dimensions. Again, dividing the square into $r \times r$ -size bins (totaling ℓ^2/r^2 approximately) and arguing that every bin should contain a node gives a weaker bound. On the other hand, if we use bins of size $2r \times 2r$, then even if every bin contains a node, this does not imply full coverage. We use 2D bin-covering to show tight threshold for full area coverage.

Theorem 3.2 *Consider the $G(n, r, \ell)$ model and let $r = r(\ell) = \ell^\epsilon f(\ell)$, for some $0 \leq \epsilon < 1$, and $f(\ell)$ is a function which grows strictly slower than any function of type ℓ^γ where $\gamma > 0$. Let $n = n(\ell) = \omega(1)$.*

- $G(n, r, \ell)$ has full area coverage a.a.s. if $r^2 n \geq \ell^2 \left(\left(\frac{1}{2} - \frac{1}{2}\epsilon \right) \ln \ell + \frac{1}{2} \ln \ln \ell + h(\ell) \right)$, for any $h(\ell) \rightarrow \infty$.
- $G(n, r, \ell)$ does not have full area coverage a.a.s. if $r^2 n \leq \ell^2 \left(\left(\frac{1}{2} - \frac{1}{2}\epsilon \right) \ln \ell + \frac{1}{2} \ln \ln \ell + g(\ell) \right)$, for any $g(\ell) \rightarrow -\infty$.

Proof: Our approach is analogous to that of Theorem 2.1, but adapted to two dimensions. We show that we can tightly cover the area by squares of dimension $2r \times 2r$ in such a way there is no “hole” of dimension $2r \times 2r$. This will ensure that every point in the unit square is covered by some node.

Divide the $\ell \times \ell$ square into blocks of size $2r \times 2r$. (Refer Figure (b).) Fix a spacing parameter s . For every $i = js$, $0 \leq j \leq \lceil \ell/s \rceil$, starting from point i from the left side of the square, split up the (entire) square into equally spaced vertical bins of size $\ell \times 2r$. Analogous to the one dimensional case, the square can be viewed as being covered by vertical slabs of size $\ell \times s$ – a total of $\lceil \ell/s \rceil$. Similarly, for every $i = js$, $0 \leq j \leq \lceil \ell/s \rceil$, starting from point i from the bottom side of the square, split up the square into equally spaced horizontal bins of area $2r \times \ell$. Similarly, the square can also be viewed as being covered by horizontal slabs of size $s \times \ell$ – a total of $\lceil \ell/s \rceil$. The vertical and horizontal slabs induce a total of $\lceil \ell^2/s^2 \rceil$ bins of size $2r \times 2r$.

We will assume an ordering among the nodes based on their coordinates: we will say that $u \leq v$, if the x -coordinate of u is less than the x -coordinate of v , and if they are equal, then whichever with the smaller y coordinate is smaller. (We assume that the origin $(0, 0)$ is at the bottom-left corner of the $\ell \times \ell$ square as before).

There will be full coverage if (1) there are no empty bins among all the $\lceil \ell^2/s^2 \rceil$ bins (2) the following configuration does not occur: there exists nodes u, v, w , and x , ($u \leq v \leq w \leq x$) such that u occurs in horizontal slab b_1 , v occurs in horizontal slab b_2 such that $|b_1 - b_2| = \lceil 2r/s \rceil - 1$ and w occurs in vertical slab b_3 and x occurs in vertical slab b_4 , such that $|b_3 - b_4| = \lceil 2r/s \rceil - 1$ and there is (this specific) empty square of size $2r \times 2r$ between the points (i.e., the line joining u, v and w, x goes through the empty bin) whose bottom corner is the smallest.

Let the random variable X denote the total number of empty bins. Let random variable Y_i , $1 \leq i \leq \binom{n}{2}$ be the indicator for the event of (2) between a pair of points. Let $Z = X + Y$. Then, $E[X] \leq \frac{\ell^2}{s^2} (1 - \frac{4r^2}{\ell^2})^n (1 + o(1))$ and $E[Y] \leq n(n-1)(n-2)(n-3)(4rs/\ell^2)(2rs/\ell^2)(2r/s)(2rs/\ell^2)(1 - 4r^2/\ell^2)^{n-4} (1 + o(1))$. We obtain the bound for $E[Y]$ as follows. Let $Y_{u,v,w,x} = 1$ if the configuration in (2) occurs. Then conditioning on the occurrence of u, v can occur in an area of size at most $4rs$ for it to occur in a slab b_2 such that $|b_2 - b_1| = \lceil 2r/s \rceil - 1$ (two horizontal slabs of dimensions $r \times s$ — one above and one below); conditioning on the occurrences of u and v , w can occur in an area of size at most $(2r/s)(2rs) = 4r$ (this is needed since the line joining u, v and w, x must cross); conditioning on the occurrences of u, v, w, x can occur in an area of dimension at most $2rs$; and conditioning on the occurrences of these four points, no other point can occur in the empty square of area $2r \times 2r$ whose corner is closest to u . Then $\Pr(Y_{u,v,w,x} = 1) \leq (4rs/\ell^2)(2rs/\ell^2)(2r/s)(2rs/\ell^2)(1 - 4r^2/\ell^2)^{n-4} (1 + o(1))$ and $E[Y] = \sum_{u,v,w,x} \Pr(Y_{u,v,w,x} = 1)$.

Choosing $s = \Theta(\frac{\ell^2}{nr})$ (found by minimizing $E[Z]$ as a function of s) and letting $r^2n = k\ell^2 \ln \ell$, shows that the expectation tends to zero as $\ell \rightarrow \infty$, for any $k = k(\ell) \geq \frac{1}{2} - \frac{1}{2}\epsilon + \frac{1}{2} \frac{\ln \ln \ell + h(\ell)}{\ln \ell} + h(\ell)$.

The lower bound is shown by applying the second moment method — the approach is similar to Theorem 2.2. \blacksquare

A similar approach yields the following result which gives a *sharp* threshold for area coverage for the 2-dimensional $G(n, r)$ model.

Theorem 3.3 $G(n, r)$ has full area coverage a.a.s. if $r^2 \geq \frac{1}{4n}(\ln n + \ln \ln n + h(n))$, for any $h(n) \rightarrow \infty$ and no full area coverage a.a.s. if $r \leq \frac{1}{4n}(\ln n + \ln \ln n + g(n))$ for any $g(n) \rightarrow -\infty$.

3.3 Connectivity

The following theorem establishes the asymptotic connectivity regime via bin-covering. However, in this case, it does not yield sharp thresholds. We note that using a sharp connectivity threshold result for the $G(n, r)$ model (see [5] for a sophisticated proof) one may infer a sharp threshold for the $G(n, r, \ell)$ model as well; we will not go into this, as our aim here is to simply establish the asymptotic connectivity regime which is useful in understanding the threshold for stretch (next Section).

Theorem 3.4 Consider the $G(n, r, \ell)$ model and let $r = r(\ell) = \Theta(\ell^\epsilon f(\ell))$, for some $0 \leq \epsilon < 1$, and $f(\ell)$ is a function which grows strictly slower than any function of type ℓ^γ where $\gamma > 0$. Let $n = n(\ell) = \omega(1)$. Given any two constants $c_1 > 2 - 2\epsilon$ and $c_0 < \frac{1}{2} - \frac{1}{2}\epsilon$,

- $G(n, r, \ell)$ is connected a.a.s. if $r^2n \geq c_1\ell^2 \ln \ell$, and
- $G(n, r, \ell)$ is disconnected a.a.s. if $r^2n \leq c_0\ell^2 \ln \ell$.

Proof: The graph is connected if we can tightly cover the whole area by $r \times r$ bins such that there is no “hole” of size $r \times r$. The analysis is similar to that of full area coverage, except that we cover by bins of size $r \times r$ instead of $2r \times 2r$. This yields the upper threshold for connectivity. A necessary condition for connectivity is the absence of isolated nodes in $G(n, r, \ell)$. Let r.v. X denote the number of isolated nodes. Applying the second moment method to enforce that $\Pr(X = 0) \rightarrow 1$ yields the lower threshold for connectivity. \blacksquare

3.4 Stretch

We study the stretch of $G(n, r, \ell)$ in the asymptotic connectivity regime, i.e., $r^2n \geq k\ell^2 \ln \ell$ for some constant $k > \frac{1}{2} - \frac{1}{2}\epsilon$. This is reasonable since the stretch is unbounded if $G(n, r, \ell)$ is not connected.

The following theorem shows the existence of *high quality paths* (i.e. stretch is at most $1 + \epsilon$) between any two nodes in the network for a value of k (as in $r^2n = k\ell^2 \ln \ell$) only a bit larger than that needed for connectivity. The theorem actually gives a tradeoff result between k and stretch.

Theorem 3.5 In $G(n, r, \ell)$ let $r^2n = k\ell^2 \ln \ell$, and $r = r(\ell) = \Theta(\ell^\epsilon f(\ell))$, for some $0 \leq \epsilon < 1$, as before. Let $0 < \alpha \leq 1$ be a fixed constant. Then for any constant $k > \frac{22(1-\epsilon)}{\alpha}$, the stretch is $1 + \alpha/2$ a.a.s. Further, if we consider only the subset F of nodes such that $D(u, v) = \omega(r)$ (i.e., strictly larger than r) for all $u, v \in F$ then the stretch restricted to this subset is 1 a.a.s.

Proof: We show that there is always a path of length less than $D(u, v) + \alpha/2$, for every pair of vertices u and v .

For a given pair of nodes u, v we apply the bin-covering technique as follows. Consider the line joining u and v and let the angle made by this line to the horizontal be less than 45 degrees i.e., $D(u, v)$ is determined

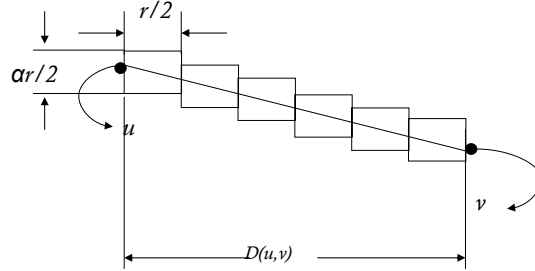


Fig. (c): Illustration for proof of Theorem 3.5.

by the difference in the horizontal coordinates (the proof will be similar if this angle is greater than 45 degrees with the width and length of the boxes interchanged). We cover this line by equally spaced bins (boxes) of dimensions $\alpha r/2 \times r/2$ centered by the line (i.e., the center of the box lies in the line — refer Figure (c)) with spacing parameter s — a total of $\lceil D(u, v)/s \rceil$ bins. We call this a *strip* of width α .

Assume that there is no empty box and there is no “hole” of dimension $\alpha r/2 \times r/2$ in the strip, i.e., there is no configuration such that we can slide an empty box of dimension $\alpha r/2 \times r/2$ between consecutive points. We now show that there will be a path between u and v of length less than $D(u, v) + \alpha$. We construct such a path $P : u = x_0, x_1, \dots, x_{k-1}, v = x_k$ as follows: x_i is the neighbor of x_{i-1} which is closest to v and is contained in a box. We show by induction that for $0 \leq j < k - 1$, $D(x_j, x_{j+1}) \geq r/2$. Clearly $D(x_0, x_1) \geq r/2$. Assume $D(x_{j-1}, x_j) \geq r/2$ for $2 \leq j \leq i$. Then $D(x_i, x_{i+2}) \geq r$ (otherwise x_{i+2} will be chosen as the neighbor of x_i). If $D(x_i, x_{i+1}) < r/2$ then $D(x_{i+1}, x_{i+2}) > r/2$. Since there is no empty box (or a hole) of length $r/2$ between x_{i+1}, x_{i+2} (by our assumption), there must be a node (say y) of within a distance of $r/2$ of x_{i+1} implying that $D(x_i, y) < r$. This means that y is a neighbor of x_i which closer to v than x_{i+1} , contradicting the choice of x_{i+1} . Note that since the width of all the boxes is less than $r/2$, $D(x_i, x_{i+1})$, for $0 \leq i < k - 1$ is the horizontal distance covered by the path towards v . Thus the length of P is bounded by the $D(u, v) + D(x_{k-1}, v) \leq D(u, v) + \alpha r/2$. Since, $D(u, v) \geq r$ (otherwise the stretch is 1) the stretch of P is bounded by $1 + \alpha/2$. If $D(u, v) = \omega(r)$ then stretch is $1 + o(1)$.

Repeating this argument for all $\binom{n}{2}$ pairs of nodes we bound the total number of empty bins and the number of “hole” configurations:

$$\binom{n}{2} \left(\frac{\max_{(u,v)} D(u, v)}{s} \left(1 - \frac{\alpha r^2}{4\ell^2}\right)^n + \left(1 - \frac{\alpha r^2}{4\ell^2}\right)^n \frac{n^2 \alpha s r}{2\ell^2} \right)$$

As usual, we choose the best value of s and letting $r^2 n = k\ell^2 \ln \ell$, the RHS tends to zero as $\ell \rightarrow \infty$ if $k > \frac{22(1-\epsilon)}{\alpha}$. ■

A byproduct of the proof of the above theorem (i.e., in computing the path P) is that it gives an efficient and *local* routing algorithm for finding a path of low stretch between any two nodes in the sensor network. We note that the algorithm is optimal with respect to space and number of sensors visited and can be easily implemented in a local distributed fashion.

Corollary 3.1 *Let $G(n, r, \ell)$ have parameters as defined in Theorem 3.5. Then given two nodes u and v , there is a local algorithm that (a.a.s.) finds a path between u and v of stretch at most $1 + \alpha/2$ that takes $O(1)$ space (per vertex), and visits $2D(u, v)/r$ nodes altogether.*

4 Concluding Remarks

We used an approach to uniformly analyze threshold bounds for fundamental properties like coverage, connectivity and stretch of random geometric graphs. Our bin-covering approach may be used to analyze other properties of interest, e.g., the number of multiple paths of high quality and the size of dominating sets.

There are many open questions in the $G(n, r, \ell)/G(n, r)$ model. Can we show sharp thresholds for enclosure coverage and stretch for the two-dimensional case?

$G(n, r)$ and more generally $G(n, r, \ell)$ are used in networking community to give rough parameters to understand ASNs. To what extent is $G(n, r, \ell)$ a faithful model for ASNs? One of the disadvantages of $G(n, r, \ell)$ is that in practice, an ad hoc network need not be uniformly distributed; also, r may differ from node to node. Any model with suitable non-uniform spatial distribution of nodes would be of great interest. It will be interesting to extend our approach to analyze more general models.

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