Random Variable

Let $(\mathcal{S}, Pr)$ be a discrete probability space.

Let V be a set of values.

A random variable X defined on $(\mathcal{S}, Pr)$ is a function

$$X : \mathcal{S} \rightarrow V$$

Let $\mathcal{E}(r) = \{s \in \mathcal{S} \mid X(s) = r\}$

$$Pr(X = r) \;=\; Pr(\mathcal{E}(r)) = \sum_{s \in \mathcal{E}(r)} Pr(s).$$

Two random variables X and Y (defined on the same sample space) are called independent if for all x and y

$$\Pr\{X = x \text{ and } Y = y\} = \Pr\{X = x\} \Pr\{Y = y\}$$

Example 1: In rolling a dice, the number that comes up is a random variable.

Example 2: Consider a gambling game in which a player flips two coins, if he gets head in both coins we wins \$3, else he losses \$1. The payoff of the game is a random variable.

Definition 1. *The* **expectation** *of a discrete random variable X is*

$$E[X] \;=\; \sum_{i \in range(X)} iPr(X=i).$$

The expectation (or mean or average) is a weighted sum over all possible values of the random variable.

Example: The expected value of one dice roll is:

$$E[X] \;=\; \sum_{i=1}^{6} iPr(X=i) \;=\; \sum_{i=1}^{6} \frac{i}{6} \;=\; 3\frac{1}{2}.$$

Consider a game in which a player chooses a number in $[1, ..., 6]$ and then rolls 3 dice.

The player wins $1 for each dice that matches the number, he losses $1 if no dice matches the number.

What is the expected outcome of that game:

$$-1(\frac{5}{6})^3 + 1 \cdot 3(\frac{1}{6})(\frac{5}{6})^2 + 2 \cdot 3(\frac{1}{6})^2(\frac{5}{6}) + 3(\frac{1}{6})^3 = -\frac{17}{216}.$$

Which gambling game would you prefer?

- We flip one coin, you win \$1 if head, loose one \$1 if tail.
- We flip 10 coins, you win $\$2^{10} = 1K$ if all heads, else you pay \$1.
- We flip 20 coins, you win $\$2^{20}/2 = M/2$ if all heads, else you pay \$1.

Linearity of Expectation

Theorem 1. *For any two random variables X and Y*

$$E[X+Y] \; = \; E[X] \; + \; E[Y].$$

Proof.

$$E[X+Y] \; =$$

$$\sum_{i \in range(X)} \sum_{j \in range(Y)} (i+j)Pr((X=i) \cap (Y=j)) =$$

$$\sum_i \sum_j iPr((X=i) \cap (Y=j)) +$$

$$\sum_j \sum_i jPr((Y=j) \cap (Y=j)) =$$

$$\sum_i iPr(X=i) + \sum_j jPr(Y=j).$$

□

(Since we sum over all possible choices of i (j).)

Theorem 2. *If* $E_1, E_2, ..., E_k$ *are disjoint events such that* $\sum_{i=1}^{k} Pr(E_i) = 1$ *then for any event* B*,*

$$\sum_{i=1}^{k} Pr(B \cap E_i) = Pr(B).$$

Examples:

1. The expectation of the sum of two dice is 7, even if they are not independent.

2. Assume that we flip N coins, what is the expected number of heads?

Using linearity of expectation we get $N \cdot \frac{1}{2}$.

By direct summation we get $\sum_{i=0}^{N} i\binom{N}{i}2^{-N}$.

Thus we prove

$$\sum_{i=0}^{N} i\binom{N}{i}2^{-N} = \frac{N}{2}.$$

3. Assume that N people checked coats in a restaurant. The coats are mixed and each person gets a random coat.

How many people got their own coats?

It's hard to compute $E[X] = \sum_{k=0}^{N} kPr(X = k)$. Instead we define N 0-1 random variables X_i, where $X_i = 1$ iff i got his coat.

$$E[X_i] \;=\; 1 \cdot Pr(X_i = 1) \;+\; 0 \cdot Pr(X_i = 0) =$$

$$Pr(X_i = 1) = \frac{1}{N}.$$

$$E[X] \;=\; \sum_{i=1}^{N} E[X_i] \;=\; 1.$$

Randomized Quicksort

Procedure Q_S(S);

Input: A set S.

Output: The set S in sorted order.

1. If $|S| \leq 1$ then return S, else

2. (a) Choose a random element y uniformly from S.
   (b) Compare all elements of S to y. Let

$$S_1 = \{x \in S - \{y\} \mid x \leq y\}$$

$$S_2 = \{x \in S - \{y\} \mid x > y\}.$$

   (Elements in S_1 and S_2 are in the same order as in S.)
   (c) Return the list:

$$Q\_S(S_1), y, Q\_S(S_2).$$

Let T = number of comparisons in a run of QuickSort.

Theorem 3.

$$E[T] \; = \; O(n \log n).$$

Proof:

Let $s_1, ..., s_n$ be the elements of S in sorted order.

For $i = 1, ..., n$, and $j > i$, define 0-1 random variable $X_{i,j}$, s.t.

$X_{i,j} = 1$ iff s_i is compared to s_j in the run of the algorithm.

The number of comparisons in running the algorithm is

$$T = \sum_{i=1}^{n} \sum_{j>i} X_{i,j}.$$

We are interested in $E[T]$.

What is the probability that $X_{i,j} = 1$?

s_i is compared to s_j iff either s_i or s_j is chosen as a "split item" before any of the $j - i - 1$ elements between s_i and s_j are chosen.

Elements are chosen uniformly at random $\rightarrow$ elements in the set $[s_i, s_{i+1}, ..., s_j]$ are chosen uniformly at random.

$$Pr(X_{i,j} = 1) = \frac{2}{j - i + 1}.$$

$$E[X_{i,j}] = \frac{2}{j - i + 1}.$$

$$E[T] \;=\; E[\sum_{i=1}^{n}\sum_{j>i} X_{i,j}] =$$

$$\sum_{i=1}^{n}\sum_{j>i} E[X_{i,j}] = \sum_{i=1}^{n}\sum_{j>i} \frac{2}{j-i+1} \leq$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n-i+1} \frac{2}{k} \leq 2\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{1}{k} = 2nH_n = O(n\log n)$$