Independent Sets

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices with $dn/2$ edges. Then G has an independent set with at least $n/2d$ vertices.*

Proof:

Consider the following randomized algorithm:

1. Delete each vertex of G (and its incident edges) independently with probability $1 - 1/d$.

2. For each remaining edge, remove it and one of its adjacent vertices.

"Sample and Modify" Technique.

Let X denote the number of vertices that survive step 1.

$E[X] = n/d$.

Let Y be the number of edges that survive the first step.

$E[Y] = \frac{nd}{2}(\frac{1}{d})^2 = \frac{n}{2d}$

The algorithm outputs a set of size at least $X - Y$, so

$E[X - Y] = \frac{n}{2d}$

Dominating Set

A dominating set of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that ever vertex $v \in V - U$ has at least one neighbor in U.

Given an arbitrary graph, finding the minimum dominating set is NP-complete.

Theorem 2. *Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of size at most $n\frac{1+\ln(\delta+1)}{\delta+1}$.*

Proof:

For some $p \in [0, 1]$ pick randomly and independently each vertex of V with probability p.

Let X be the set of vertices picked.

$E[|X|] = np$.

Let Y_X denote the set of vertices in $V - X$ that do not have any neighbor in X.

$$E[|Y_X|] \leq n(1-p)^{\delta+1}$$

$$E[|X| + |Y_X|] \leq np + n(1-p)^{\delta+1}$$

Thus there is at least one choice of $X \subseteq V$ such that $U = X \cup Y_X$ which is a dominating set of at most this size.

We can optimize p. A crude bound is:

$$|U| \leq np + ne^{-p(\delta+1)}$$

The RHS is minimized for $p = \frac{\ln(\delta+1)}{\delta+1}$.

High Girth and High Chromatic Number

Girth is the length of the smallest cycle in the graph.

Chromatic number is the minimum number of colors needed to color the vertices of the graph such that no two adjacent vertices have the same color.

Theorem 3. *For all k, l there exists a graph G with $girth(G) > l$ and $\chi(G) > k$.*

Proof: If $\alpha(G)$ is the size of the largest independent set of G then $\chi(G)\alpha(G) \geq n$. We want to show that there exists a graph with small α and large girth.

Fix $\theta < 1/l$ and choose a random graph $G \in G(n,p)$ with $p = n^{\theta-1}$.

Let X be the number of cycles of size at most l.

$E[X] = \sum_{i=3}^{l} \binom{n}{i} \frac{(i-1)!}{2} p^i$

$\leq \sum_{i=3}^{l} n^i p^i = \sum_{i=3}^{l} n^{\theta i} = o(n)$

since $\theta l < 1$.

$\Pr(X \geq n/2) = o(1)$

Set $x = \frac{3}{p} \ln n$.

$\Pr(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < \left[ne^{-p(x-1)/2}\right]^x = o(1)$

Let n be sufficiently large so that both these events happen with probability less than 1/2.

Then there exists a graph G with less than $n/2$ cycles of length at most l and with $\alpha(G) < \frac{3}{p} \ln n = 3n^{1-\theta} \ln n$.

Remove from G a vertex from each cycle of length at most l. This gives a graph G' with at least $n/2$ vertices.

G' has girth $> l$ and $\alpha(G') \leq \alpha(G)$. Thus

$\chi(G') \geq \frac{n/2}{3n^{1-\theta} \ln n} = \frac{n^\theta}{6 \ln n} > k$

for a sufficiently large n.

First and Second Moment Method

Theorem 4. *Let X be a non-negative integer-valued r.v. Then*

$$\Pr(X \geq 1) \leq E(X)$$

$$\Pr(X = 0) \leq \frac{Var(X)}{(E(X))^2} = \frac{E(X^2)}{(E(X))^2} - 1$$

Proof:

$$\Pr(X = 0) \leq \Pr(|X - E(X)| \geq E(X)) \leq \frac{Var(X)}{(E(X))^2}$$

These help in showing $\Pr(X = 0) \rightarrow 1$ and $\Pr(X = 0) \rightarrow 0$.

Thresholds in Random Graphs

Theorem 5. *For the $G(n, p)$ random graph model, let $p = c\frac{\log n}{n}$. If $c > 1$ then almost all graphs have no isolated vertices and if $c < 1$ almost all graphs have at least one isolated vertex.*

Proof:

Upper Threshold: Let X denote the number of isolated vertices in a random $G \in G(n, p)$.

Let X_i be the indicator r.v. for a vertex to be isolated.

$E[X] = \sum_{i=1}^{n} X_i = n(1-p)^{n-1}$

$(1-p)^n = e^{n\log(1-p)} = e^{n(-p-p^2/2-p^3/3\ldots)}$

$= e^{-np}e^{-np^2(1/2+p/3+\ldots)}$

$\sim e^{-np}$ provided $np^2 \to 0$.

$E[X] = n(1-p)^{n-1} \sim ne^{-np} \sim n^{1-c}$

Thus, $E(X) \to 0$ if $c > 1$ and

$E(X) \to \infty$ if $c < 1$.

Lower Threshold:

$E[X^2] = \sum_{i=1}^{n} E(X^2) + \sum_{i \neq j} X_i X_j$

$= E(X) + n(n-1)E(X_1 X_2)$

$= E(X) + n(n-1)(1-p)^{2(n-2)+1}$

$\Pr(X = 0) \leq \frac{E(X^2)}{(E(X))^2} - 1$

$= \frac{1}{E(X)} + \frac{n(n-1)(1-p)^{2(n-2)+1}}{n^2(1-p)^{2(n-1)}} - 1$

$\sim \frac{1}{E(X)} \to 0$ if $c < 1$.