Conditioned on E_i we have $\sum_{t=1}^{n} Y_t = \mu_{i+1}$.

Since $\nu_{i+1} \leq \mu_{i+1}$,

$$\Pr(v_{i+1} > k | E_i) \leq \Pr(\mu_{i+1} > k | E_i)$$

$$= \Pr(\sum_{t=1}^{n} Y_t > k | E_i) \leq \Pr(\sum_{t=1}^{n} Y_t > k) / \Pr(E_i)$$
$$\leq \Pr(B(n, (\beta_i/n)^d) > k) / \Pr(E_i)$$

.

$\Pr(\nu_{i+1} > \beta_{i+1} | E_i)$

$$\leq \Pr(B(n, (\beta_i/n)^d) > 2n(\beta_i/n)^d)) / \Pr(E_i)$$

$$\leq e^{-n((\beta_i/n)^d)/3}(1/\Pr(E_i))$$

$\leq \frac{1}{n^2 \Pr(E_i)}$ whenever $n(\beta_i/n)^d \geq 6 \ln n$.

Thus, $\Pr(\neg E_{i+1} | E_i) \leq \frac{1}{n^2 \Pr(E_i)}$ whenever $n(\beta_i/n)^d \geq 6 \ln n$.

Next we remove the conditioning.

$\Pr(\neg E_{i+1})$

$$= \Pr(\neg E_{i+1} | E_i) \Pr(E_i) + \Pr(\neg E_{i+1} | \neg E_i) \Pr(\neg E_i)$$

$$\leq \Pr(\neg E_{i+1} | E_i) \Pr(E_i) + \Pr(\neg E_i) \leq \Pr(\neg E_i) + 1/n^2.$$

Hence, whenever $n(\beta_i/n)^d) \geq 6 \ln n$ and E_i holds, then so does E_{i+1}.

We can show that when $i = i^* = \frac{\ln \ln n}{\ln d} + O(1)$, then $n(\beta_i/n)^d < 6 \ln n$.

Use induction to show that $\beta_{i+4} \leq n/2^{d^i}$.

Thus, $\Pr(\neg E_{i^*}) \leq i^*/n^2$.

We now handle the case where $n(\beta_i/n)^d < 6\ln n$. We have

$$\Pr(\nu_{i^*+1} > 18\ln n | E_{i^*}) \leq \Pr(\mu_{i^*+1} > 18\ln n | E_{i^*})$$

$$\leq \Pr(B(n, 6\ln n/n) \geq 18\ln n)/\Pr(E_{i^*}) \leq \frac{1}{n^2 \Pr(E_{i^*})}.$$

$$\Pr(\nu_{i^*+1} > 18\ln n) \leq \Pr(\neg E_{i^*}) + 1/n^2 \leq (i^* + 1)/n^2.$$

$\Pr(\nu_{i^*+3} \geq 1) \leq \Pr(\mu_{i^*+3} \geq 1) \leq \Pr(\mu_{i^*+2} \geq 2)$.

$\Pr(\mu_{i^*+2} \geq 2 | \nu_{i^*+1} \leq 18 \ln n) \leq \frac{\Pr(B(n,(18 \ln n/n)^d \geq 2)}{\Pr(v_{i^*+1} \leq 18 \ln n)}$

$\leq \frac{\binom{n}{2}(18 \ln n/n)^{2d}}{\Pr(v_{i^*+1} \leq 18 \ln n)}$.

Thus,

$$\Pr(\nu_{i^*+3} \geq 1) \leq \Pr(\mu_{i^*+2} \geq 2)$$

$$\leq \Pr(\mu_{i^*+2} \geq 2 | \nu_{i^*+1} \leq 18 \ln n) \Pr(v_{i^*+1} \leq 18 \ln n)$$

$$+ \Pr(v_{i^*+1} > 18 \ln n)$$

$$\leq \frac{(18 \ln n)^{2d}}{n^{2d-2}} + (i^* + 1)/n^2 = o(1)$$

if $d \geq 2$.

Thus the probability that the maximum loaded bin is more than $i^* + 3 = \ln \ln n / \ln d + O(1)$ is $o(1/n)$.