Stable Marriage

Consider a society of n men and n women. A marriage is a 1-1 function between men to women.

Each person (man or woman) has an ordered list of preferences.

A marriage in unstable if there are two couples x_1, y_1 and x_2, y_2 such that x_1 prefers y_2 on y_1 and y_2 prefers x_1 to x_2. Else the marriage is **stable**.

Stable Marriage Algorithm:

1. Start with all men and women unmarried.

2. **Repeat** till all men are married.

   (a) An unmarried man proposes to the first woman on his list that hasn't rejected him yet.

   (b) The proposed woman accepts the proposal if she is currently unmarried, or if she is married but prefers the current proposer on her current mate (in that case the old mate becomes unmarried).

Theorem 1. *The algorithm always terminates in $O(n^2)$ steps with a stable marriage.*

Proof. Once a woman is married she remains married, when she changes mates her preference of the mates only improves.

If a man is not married there is an available woman in his list, since all the women he proposed to are married and there are n women. Thus, $O(n^2)$ time.

Assume that output marriage is not stable, i.e. x_1 married to y_1 and prefers y_2 and y_2 prefers x_1 on x_2. We prove a contradiction.

If x_1 prefers y_2 he must have proposed y_2 before y_1. Since y_2 rejected x_1 at some point in the algorithm it must be married to a more desirable mate.

□

Theorem 2. *Assuming that the lists of preferences are chosen uniformly and independently at random, the expected number of proposals made by the algorithm is $O(n \log n)$, the maximum number of proposal any woman receives is bounded by $O(\log n)$.*

Proof.

Instead of having a pre-selected preference lists we can assume that a man chooses a random woman that did not rejected him till now.

We simplify the analysis by assuming that each time a man proposed he proposed to a random woman chosen uniformly from among **all** women.

Since all the "added" proposals are rejected the number of proposals in the new algorithm **stochastically dominates** the number of proposal in the original algorithm.

$$\forall x \quad Pr(T_{new} > x) \geq Pr(T > x).$$

The algorithm terminates once all women receive at least one proposal. □