Randomized Rounding

General Idea to approximate:

1. (Try to) Formulate the problem as an Integer linear program.

2. Relax the ILP by relaxing the integer constraint.

3. Solve the relaxed LP.

4. **Randomized Rounding**: Treat the fractional (part) of the solution to the LP as probabilities and round based on the probabilities.

5. Make sure you have feasibility.

Randomized Rounding - Basic Idea

1. Assume that each source generates one unit of commodity.
2. Run the *relaxed* linear programming problem for $0 \leq F_e^i \leq 1$.
3. The flow of commodity i is divided between some m different (not always disjoint) paths. Let $0 \leq X_j^i \leq 1$ be the amount of commodity using path j.
4. Send ALL commodity i through one of the m paths. Choose the path randomly with probabilities $X_1^i, ..., X_m^i$.

For a given edge e the relaxed linear programming solution gives $\sum_{i=1}^{s} F_e^i \leq T$.

The expected flow through an edge in the rounded solution is $\leq T$.

The integer solution is the sum of 0-1 independent random variables.

Using the Chernoff bound the probability that any of the N edges has flow more than $O(T + \log N)$ is bounded by $1/N$.

The expected rounded flow of each commodity equal the relaxed flow. Thus,

$$E[\tilde{G}] = G$$

Randomized Rounding - Algorithm

1. Assume that we have one unit of each commodity.

2. Run the *relaxed* linear programming problem for $0 \leq F_e^i \leq 1$.

3. The integer solution is constructed as follows:

   (a) If v is a source, the commodity exits through vertex e with probability F_e^i.

   (b) If $\tilde{F_e^i} = 1$ and e leads to vertex v: The commodity exits vertex v using edge e' chosen with probability

$$\frac{F_{e'}^i}{\sum_{e \in OUT(v)} F_e^i}.$$

Conditional Expectation

Consider the following game: A player rolls a dice. If the outcome of the dice is i it flips i independent coins. The payoff of the game, X, is the number of heads in the i coin flips.

What is the expected payoff of this game?

Let Y be the outcome of the dice.

$$E[X \mid Y] = Y/2$$

$$E[X] = E[E[X \mid Y]] = E[Y/2] = \sum_{i=1}^{6} (\frac{i}{2})(\frac{1}{6})$$

The conditional expectation of a random variable is defined as

$$E[Y|Z = z] = \sum_y \Pr(Y = y|Z = z)$$

where the summation is over all y in the range of Y.

A useful identity is

$$E[X] = \sum_y \Pr(Y = y)E[X|Y = y]$$

The expression $E[Y|Z]$ is a random variable $f(Z)$ that takes the value $E[Y|Z = z]$ when $Z = z$.

Theorem 1.

$$E[X] = E[E[X \mid Y]]$$

Proof.

$$E[X] = \sum_i iPr(X = i)$$

$$= \sum_i i(\sum_j Pr(X = i \mid Y = j)Pr(Y = j))$$

$$= \sum_j (\sum_i iPr(X = i \mid Y = j))Pr(Y = j) = E[E[X \mid Y]]$$

□

We'll show by induction on the distance from source i that for any edge e, $E[\tilde{F}_e^i] = F_e^i$.

The claim is clearly true for the edges adjacent to source i.

Assuming the the claim is true for all edges leading to vertex v, then the expected flow of commodity i through vertex v in the rounded solution is equal to the flow in the relaxed solution.

Let $X_v^i = 0, 1$ be the rounded flow of commodity i through vertex v. Let e' be an edge leading out of vertex v.

$$E[\tilde{F}_{e'}^i \mid X_v^i] = X_v^i \frac{F_{e'}^i}{\sum_{e \in OUT(v)} F_e^i}$$

$$E[\tilde{F}_{e'}^i] = E[E[\tilde{F}_e^i \mid X_v^i]]$$
$$= E[X_v^i \frac{F_{e'}^i}{\sum_{e \in OUT(v)} F_e^i}] = F_{e'}^i$$

since

$$E[X_v^i] = \sum_{e \in IN(v)} F_e^i = \sum_{e \in OUT(v)} F_e^i$$

Thus, $E[\tilde{G}] = G$.

What is the probability that the rounded flow on any of the N original edges is more than $6T + 2\log N$?

$$N2^{-(6T+2\log N)} \leq \frac{1}{N}$$