Nondeterministic Turing Machines (NTM)

A NTM $N = (K, \Sigma, \Delta, s)$ is like a ordinary TM except that instead of a transition function we have a transition **relation** Δ.

$\Delta \subset (K \times \Sigma) \times [(K \cup \{h, yes, no\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}]$.

Many (or none) next steps for each state-symbol combination.

"yields" is a relation.

Definition 1. *We say that a NTM N* **decides** *a language L if for any $x \in \Sigma^*$, the following is true: $x \in L$ iff $(s, \triangleright, x) \rightarrow^{N^*} (yes, w, u)$ for some strings w and u.*

N decides L in time $f(n)$ if N decides L, and for any $x \in \Sigma^*$, if $(s, \triangleright, x) \rightarrow^{N^k} (q, w, u)$, then $k \leq f(|x|)$.

NTIME

The set of languages decided by NTMs within time f is called $NTIME(f(n))$.

NP is the set of languages that can be a decided by a NTM in polynomial time.

$P \subseteq NP$.

Example: HAMILTONIAN CYCLE $\in NP$.

Can be decided by a (2-string) NTM in $O(n^2)$ time.

Simulating NTMs by TMs

Theorem 1. *Let L be decided by a NTM N in time $f(n)$. Then it is decided by a 3-string TM M in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on N. That is,*

$$NTIME(f(n)) \subseteq \cup_{c>1} TIME(c^{f(n)}).$$

Proof. Let $N = (K, \Sigma, \Delta, s)$.

For each $(q, \sigma) \in K \times \Sigma$, consider

$$C_{q,\sigma} = \{(q', \sigma', D) : (q, \sigma), (q', \sigma', D)) \in \Delta\}.$$

Let $d = \max_{q,\sigma} |C_{q,\sigma}|$ — "degree of nondeterminism".

Assume $d > 1$.

M considers all paths in the computation tree in order of increasing length and simulates N on each path.

M maintains computation paths in its second string.

M uses the third string to generate computation paths.

The total time is bounded by $\sum_{t=1}^{f(n)} (O(d))^t = (O(d))^{(f(n))}$

Space Complexity of NTMs

Definition 2. *Given an k-string NTM with input and output $N = (K, \Sigma, \Delta, s)$ we say that N decides language L within space $f(n)$ if N decides L and moreover, for any $x \in (\Sigma - \{\sqcup\})^*$, if $(s, \triangleright, x, \ldots, \triangleright, \epsilon) \rightarrow^{N^*} (q, w_1, u_1, \ldots, w_k, u_k)$, then*

$\sum_{j=2}^{k-1} |w_i u_i| \leq f(|x|)$.

Let L be a language. If there is a NTM with I/O that can decide L and operates within space bound $f(n)$ we say $L \in NSPACE(f(n))$.

Example: REACHABILITY $\in NSPACE(O(\log n))$.

Universal Turing Machines

Takes as input a description of another TM M and an input x for M and simulates the behavior of M on x, i.e.,

$$U(M;x) = M(x).$$