TMs with Multiple Strings

Definition 1. *A k-string TM, where $k \geq 1$ is an integer, is a like a (ordinary) TM except that the machine can read/write on k strings at the same time.*

That is, δ is a function from $K \times \Sigma^k$ to $(K \cup \{h, yes, no\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$.

Thus, a transition is written as:

$$\delta(q, \sigma_1, \ldots, \sigma_k) = (p, \rho_1, D_1, \ldots, \rho_k, D_k).$$

Initially, all strings start with a $\triangleright$.

The input is in the first string and the output can be read from the last string (if the machine computes a function).

Configurations and "yields" are straightforward generalizations.

Time Complexity of a TM

The time complexity of a TM M on a input is the number of steps (transitions) to halting.

We say M operates **within time bound** $f(n)$ if, for any string x, the time required by M on x is at most $f(|x|)$.

If $L \subset (\Sigma - \{\sqcup\})^*$ is decided by a **multistring** TM operating in time $f(n)$, then we say $L \in TIME(f(n))$, i.e., belongs to complexity class $TIME(f(n))$.

Example: PALINDROME $\in TIME(O(n))$.

Robustness of TMs

Theorem 1. *Given any k-string TM M operating within time $f(n)$, we can construct a TM M' operating within time $O(f(n)^2)$ and such that, for any input x, $M(x) = M(x')$.*

M has the same "power" of M' (may be with a polynomial loss in efficiency).

Proof

M' simulates M as follows:

1. It simulates the k strings by concatenating them (with special delimiters) as:

$\triangleright w'_1 u_1 \# w'_2 u_2 \# \ldots w_k u_k \#\#$.

2. The first symbol of w'_i is has a special symbol $\#'$ and the last symbol is "marked".

3. To simulate a move of M, M' scans **twice** its string from left to right and back. The first scan gathers information on the k currently scanned symbols in M and in the second scan changes the information based on the corresponding transitions of M.

4. If "overflow" occurs in any substring $\# \ldots \#$, a (marked) blank is inserted.

Time complexity of M' is $O(k^2 f(|x|)^2)$ for an input of length x, where $f(|x|)$ is the time complexity of M.

TMs with Input/Output

Definition 2. *Let k-string ($k > 2$) TM with I/O is an (ordinary) k-string TM where*

1. the first (input) string is read-only and cursor never goes "outside" the input.

2. the last string is write-only (cursor never moves left).

Theorem 2. *For any k-string TM M operating within time bound $f(n)$ there is a $(k+2)$-string TM M' with I/O which operates within time bound $O(f(n))$.*

Space Complexity

Definition 3. *Suppose that, for a k-string TM and an input x:*

$$(s, \triangleright, x, \ldots, \triangleright, \epsilon) \quad \to^{M^*} \quad (H, w_1, u_1, \ldots, w_k, u_k)$$

where $H = \{h, yes, no\}$. Then the space required by M on input x is $\sum_{i=1}^{k} |w_i u_i|$.

If M is a TM with I/O, then space required is $\sum_{i=2}^{k-1} |w_i u_i|$.

A TM M operates within space bound $f(n)$, if for any input x, M requires space at most $f(|x|)$.

Let L be a language. If there is a TM with I/O that can decide L and operates within space bound $f(n)$ we say $L \in SPACE(f(n))$.

Example: PALINDROMES $\in SPACE(\log n)$.

Church-Turing Thesis

"Intuitive notion of algorithms = Turing machine algorithms"

"Turing machines can implement arbitrary algorithms"

A Stronger Thesis:

" Any reasonable model of algorithms and their time bound can be implemented by a TM within a polynomial (loss) of efficiency"