3-SAT and CLIQUE

A boolean formula is in 3-CNF form if it is in CNF form and each clause has exactly three literals per clause.

3-SAT: Given a 3-CNF formula F is there a satisfying assignment for F?

CLIQUE: Given a graph G, does G contain a complete subgraph of size k?

Theorem 1. *3-SAT is polynomial time reducible to CLIQUE.*

Proof

We give a polynomial time reduction from 3-SAT to CLIQUE.

Let ϕ be an instance of 3-SAT with k clauses:

$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \cdots \wedge (a_k \vee b_k \vee c_k)$.

We construct an instance G as follows:

Each literal in each clause corresponds to a node in G. The nodes are organized into k groups of 3 nodes each called the triples.

There is an edge between all pairs of nodes except between:

1. two nodes in the same triple.

2. two nodes corresponding to opposite literals.

We show that ϕ is satisfiable iff G has a clique of size k.

If ϕ has a satisfying assignment, then choose one true literal in every clause. The corresponding nodes form a k-clique.

If G has a k-clique, then each of the k triples contains exactly one of the k clique nodes. Assign truth values such that each corresponding literal is made true.

VERTEX-COVER

VERTEX-COVER: Given an undirected graph G, is there a k-subset of nodes such that every edge of G is incident on one of the nodes in the subset.

Theorem 2. *VERTEX-COVER is NP-complete.*

Proof:

VERTEX-COVER $\in NP$.

We reduce CLIQUE to VERTEX-COVER.

Let $< G = (V, E), k >$ be an instance of CLIQUE.

Let G^c be the complement graph of G, i.e., contains exactly those edges not in G.

The corresponding instance of VERTEX-COVER is $< G^c, |V| - k >$.

G has a clique of size k iff G^c has a vertex cover of size $|V| - k$.

Let G has a clique V' of size k. Consider an edge (u, v) in G^c.

Then $(u, v) \notin E$ and thus one of u or v should not belong to V'.

So, one of them should belong to $V - V'$.

Thus $V - V'$ is a vertex cover in G^c.

To show the other way, let G^c has a vertex cover V' of size $|V| - k$.

Then for all $u, v \in V - V'$, $(u, v) \notin G^c$ and hence belong to G.