Proof of (c): Reachability Method

Given a k-string NTM M with I/O which decides L within space $f(n)$.

We will deterministically simulate M on input x of length n within time $c^{\log n + f(n)}$ for some constant c depending on M.

A configuration of M can be represented by a $(2k-2)$-tuple: $(q, i, w_2, u_2, \ldots, w_k, u_k)$ where $0 \leq i \leq n$ is an integer which points to the position of the cursor of the input string.

There are at most $|K|(n+1)|\Sigma|^{(2k-2)f(n)} \leq nc_1^{f(n)} = c_2^{\log n + f(n)}$ configurations where K is the set of states and Σ is the alphabet set, and for some constants c_1 and c_2.

Consider the graph $G(M, x)$ — **configuration graph** of M on input x:

nodes — set of configurations and there is an edge between two nodes C_1 and C_2 iff $C_1 \rightarrow^M C_2$.

$x \in L$ iff there is a path in $G(M, x)$ from the initial configuration to some configuration which halts in the "yes" state.

This is simply the REACHABILITY problem which can be solved in at most $c_3 c_2^{2(\log n + f(n))}$ steps.

PSPACE = NPSPACE

Theorem 1. [Savitch's Theorem] *REACHABILITY* $\in SPACE(\log^2 n)$.

Proof: Let $G = (V, E)$ be a graph with n nodes, and let $x, y \in V$ and $i \geq 0$.

$PATH(x, y, i)$: There is a path from x to y in G of length at most 2^i.

REACHABILITY can be solved by computing $PATH(x, y, \lceil \log n \rceil)$.

We design a 3-string TM to decide $PATH(x, y, i)$ for any i.

Assume that the first (input) string contains the adjacency matrix.

The second string contains the triple (x, y, i).

The third string will be used for scratch space.

Algorithm for computing $PATH(x, y, i)$

If $i = 0$: check whether there is an edge between x and y.

If $i \geq 1$: for all nodes z check whether $PATH(x, z, i - 1)$ and $PATH(z, y, i - 1)$.

Generate all nodes z, one after another, reusing space. Once a new z is generated, we *add* $(x, z, i - 1)$ on the second string — solve $PATH(x, z, i - 1)$ recursively.

If $PATH(x, z, i-1)$ is false then erase $(x, z, i-1)$ and try the next z.

Otherwise, erase $(x, z, i - 1)$ and write $(z, y, i - 1)$ and solve $PATH(z, y, i - 1)$ recursively.

If this is also positive then REACHABILITY is true, otherwise we erase and try next z.

Total space at any time in the second string is $O(\log^2 n)$ and in the third string is $O(\log n)$.

PSPACE = NPSPACE

Corollary 1. $NSPACE(f(n)) \subseteq SPACE(f^2(n))$ *for any proper complexity function* $f(n) \geq \log n$*.*

Proof: The proof is similar to Savitch's Theorem.

Solve REACHABILITY in a graph of size $c^{f(n)}$, for some constant c.