Proof of Theorem

Use the recursive inseparability of $L_1' = \{M : M(\epsilon) = yes\}$ and $L_2' = \{M : M(\epsilon) = no\}$.

Given any (description of) TM M we show how to construct a sentence ϕ_M such that:

(a) If $M(\epsilon) = yes$ then $NT \vdash \phi_M$

(b) If $M(\epsilon) = no$ then ϕ_M is unsatisfiable.

This will prove the result.

$\phi_M = NT \wedge \psi$ where

$\psi = \exists x(comp_M(x) \wedge ((\forall y < x)\neg comp_M(y)) \wedge mod(x, b^2, b.(|\Sigma| + 1)))$.

Let $M(\epsilon) = yes$.

Then there is an unique integer n such that $N \models comp_M[x \leftarrow n]$ and hence $N \models \exists x comp_M(x)$.

Furthermore since n is unique, $N \models \exists x(comp_M(x) \wedge ((\forall y < x)\neg comp_M(y))$. And the last two digits of n are $|\Sigma| + 1$ and 0.

Hence, $N \models \psi$. Also $NT \vdash \phi$ and hence $NT \vdash \phi_M$.

Let $M(\epsilon) = no$.

Then $N \models \phi'_M$ where

$\phi'_M = \exists x'(comp_M(x') \wedge ((\forall y < x')\neg comp_M(y)) \wedge mod(x', b^2, b.(|\Sigma| + 2)))$.

$NT \vdash \phi'_M$.

We can show that ϕ_M and ϕ'_M are inconsistent.

That is we can prove that $\phi_M \wedge \phi'_M \vdash F$.

Hence ϕ_M is unsatisfiable.

Proof of Godel's Incompleteness Theorem

By contradiction. Assume that there is a r.e set Δ such that for all expressions ϕ, $\Delta \vdash \phi$ iff $N \models \phi$.

Since Δ is r.e., the set of proofs from Δ is also r.e.

Thus there is a TM that enumerates the set of all proofs from Δ.

Hence there is TM that enumerates the set $\{\phi : \Delta \vdash \phi\}$, i.e., the set $\{\phi : N \models \phi\}$.

Hence $\{\phi : N \models \phi\}$ is r.e. and similarly $\{\phi : N \models \neg\phi\}$ is r.e.

This implies that both these sets are recursive. Contradiction.