Theorem 1. *For any Horn existential second-order expression $\exists P\phi$, the problem $\exists P\phi$-GRAPHS is in P.*

Proof

Let $\exists P \phi = \exists P \forall x_1 \ldots \forall x_k \eta$

where η is the conjunction of Horn clauses and the arity of P is r.

Is there a relation $P \subseteq V^r$ such that ϕ is satisfied?

Let $V = \{1, 2, \ldots, n\}$.

We can substitute all possible values of x_i in ϕ to get:

$\bigwedge_{v_1,\ldots,v_k=1}^{n} \eta[x_1 \leftarrow v_1, \ldots, x_k \leftarrow v_k]$

The above expression contains exactly hn^k clauses, where h is the number of clauses in η.

The atomic expression in each clause of above can be of only 3 kinds: $G(v_i, v_j)$, $v_i = v_j$, or $P(v_{i_1}, \ldots, v_{i_r})$.

The above boil downs to determining the satisfaction of a conjunction of at most hn^k clauses, each of which is the disjunction of atomic expressions of the form $P(v_{i_1}, \ldots, v_{i_r})$ and their negations.

Each of these atomic expressions can be true or false.

Hence treat each of them as a boolean variable.

We can solve for the satisfiability of the above formula in polynomial time.

Axiomatizing Number Theory

Can we come up with a set of axioms for number theory that is sound and complete?

Consider the following set of axioms NT:

$NT1 : \forall x(\sigma(x) \neq 0)$

$NT2 : \forall x \forall y(\sigma(x) = \sigma(y) \Rightarrow x = y)$

$NT3 : \forall x(x = 0 \vee \exists y \sigma(y) = x)$

$NT4 : \forall x(x + 0 = x)$

$NT5 : \forall x \forall y(x + \sigma(y) = \sigma(x + y))$

$NT6 : \forall x(x \times 0 = 0)$

$NT7 : \forall x \forall y(x \times \sigma(y) = (x \times y) + x)$

$NT8 : \forall x(x \uparrow 0 = \sigma(0))$

$NT9 : \forall x \forall y(x \uparrow \sigma(y) = (x \uparrow y) \times x)$

$NT10 : \forall x(x < \sigma(x))$

$NT11 : \forall x \forall y(x < y \Rightarrow \sigma(x) \leq y)$

$NT12 : \forall x \forall y(\neg(x < y) \Rightarrow y \leq x)$

$NT13 : \forall x \forall y \forall z(((x < y) \wedge (y < z)) \Rightarrow x < z)$

$NT14 : \forall x \forall y \forall z \forall z'(mod(x, y, z) \wedge mod(x, y, z') \Rightarrow z = z')$

($mod(x, y, z)$ stands for $\exists w(x = y \times w + z \wedge z < y)$).