Second-Order Logic

Definition 1. **Syntax:** *An expression of* **existential second-order logic** *over a vocabulary* $\Sigma = (\Phi, \Pi, r)$ *is of the form* $\exists P \phi$*, where* ϕ *is a first-order expression over the vocabulary* $\Sigma' = (\Phi, \Pi \cup \{P\}, r)$*.*

Semantics: *A model* M *appropriate for* Σ *satisfies* $\exists P \phi$ *if there is a relation* $P^M \subseteq (U^M)^{r(P)}$ *such that* M *augmented with* P^M *to comprise a model appropriate for* Σ'*, satisfies* ϕ*.*

Examples

1. A second-order expression in number theory:

$\phi = \exists P \forall x((P(x) \vee P(x+1)) \wedge \neg(P(x) \wedge P(x+1)))$.

$N \models \phi$. (Take P^N to be set of even numbers.)

2. A second-order expression in graph theory:

$\exists P \forall x \forall y(P(x,y) \Rightarrow G(x,y))$.

Valid since any graph has a subgraph.

3. UNREACHABILITY in Graph Theory:

$\phi(x,y) = \exists P(\forall u \forall v \forall w((P(u,u)) \wedge (G(u,v) \Rightarrow P(u,v)) \wedge ((P(u,v) \wedge P(v,w)) \Rightarrow P(u,w)) \wedge \neg P(x,y)))$

HAMILTONIAN PATH

Existential second-order logic can also capture properties that are in NP.

Example: The following sentence describes graphs with a Hamiltonian path.

$\phi = \exists P \psi$ where

$\psi = \forall x \forall y ((P(x, y) \vee P(y, x) \vee x = y)) \wedge$

$\forall x \forall y \forall z ((\neg P(x, x)) \wedge ((P(x, y) \wedge P(y, z)) \Rightarrow P(x, z))) \wedge$

$\forall x \forall y ((P(x, y) \wedge \forall z (\neg P(x, z) \vee \neg P(z, y))) \Rightarrow G(x, y))$.

ϕ-GRAPHS same as HAMILTONIAN PATH.

Existential Second-Order Logic is in NP

Theorem 1. *For any existential second-order expression $\exists P\phi$, the problem $\exists P\phi$-GRAPHS is in NP.*

Proof. Given any graph $G = (V, E)$ with n nodes.

The NTM can "guess" a relation (if it exists) $P^M \subseteq V^{r(P)}$ such that G augmented with P^M satisfies ϕ.

Takes polynomial time since there are at most $n^{r(P)}$ elements to guess.

The machine can then verify whether M satisfies ϕ deterministically in polynomial time. $\square$

Horn Existential Second-Order Logic is in P

Definition 2. *An expression in existential second-order logic is said to be an Horn expression if:*

(1) it is in prenex form with only universal first-order quantifiers and

(2) its matrix is the conjunction of clauses each of which contains at most one unnegated atomic formula that involves P (the second-order relation symbol).

Theorem 2. *For any Horn existential second-order expression $\exists P\phi$, the problem $\exists P\phi$-GRAPHS is in P.*