Inexpressibility of REACHABILITY in First-Order Logic

REACHABILITY: Given a graph G with two nodes x and y of G, is there a path from x to y?

Theorem 1. *There is no first-order expression ϕ (with two free variables x and y) such that ϕ-graphs is the same as REACHABILITY.*

Proof

Suppose there is such an expression ϕ.

Consider $\psi_0 = \forall x \forall y \phi$ states that G is strongly connected.

Let $\psi_1 = (\forall x \exists y G(x,y) \wedge \forall x \forall y \forall z ((G(x,y) \wedge G(x,z)) \Rightarrow y = z))$.

Let $\psi_2 = (\forall x \exists y G(y,x) \wedge \forall x \forall y \forall z ((G(y,x) \wedge G(z,x)) \Rightarrow y = z))$.

Then $\psi = \psi_0 \wedge \psi_1 \wedge \psi_2$ states that G is a cycle.

ψ has arbitrarily finite models.

Hence by the Lowenheim-Skolem Theorem it has an infinite model — G_∞.

But infinite cycles cannot exist. Contradiction.