A More General Proof System

Definition 1. [Tautological Implication, Valid Consequence] *Let Δ be a set (may be infinite) of expressions and ϕ be another expression. We say Δ tautologically implies ϕ (written $\Delta \models \phi$) iff any model that satisfies every member of Δ also satisfies ϕ.*

Definition 2. [Deduction System] *Let Δ be a set of expressions. We say that a finite sequence of first-order expressions $S = (\phi_1, \phi_2, \ldots, \phi_n)$ is a* proof of expression ϕ_n from Δ *if for each expression ϕ_i in the sequence, $1 \leq i \leq n$, either:*

(a) ϕ_i is a **logical axiom** *or*

(b) $\phi_i \in \Delta$ or

*(c) there are two expressions of the form ψ, $\psi \Rightarrow \phi_i$ among the expressions $\phi_1, \ldots, \phi_{i-1}$. (***Modus Ponens***)*

ϕ_n is a **Δ-first-order** *theorem and we write $\Delta \vdash \phi_n$.*

Soundness and Completeness of First-Order Logic

Theorem 1. [Soundness] *If* $\Delta \vdash \phi$ *then* $\Delta \models \phi$.

Theorem 2. [Godel's Completeness Theorem] *If* $\Delta \models \phi$ *then* $\Delta \vdash \phi$.

Applications of the Completeness Theorem

VALIDITY: Given a first-order expression ϕ, is it valid?

Theorem 3. *VALIDITY is r.e.*

Proof: By Godel's completeness theorem VALIDITY is the same as THEOREMHOOD, i.e. $\models \phi$ iff $\vdash \phi$.

Compactness Theorem for First-Order Logic

Theorem 4. [Compactness Theorem] *If all finite subsets of a set of expressions Δ are satisfiable then Δ is satisfiable.*

Proof:

Suppose that Δ is not satisfiable, but all of its finite subsets are satisfiable.

Then, $\Delta \vdash \phi \wedge \neg\phi$.

The proof of above employs finitely many expressions from Δ.

Therefore, there is a finite subset of Δ that is unsatisfiable. Contradiction.

Lowenheim-Skolem Theorem

Theorem 5. [Lowenheim-Skolem Theorem] *If sentence ϕ has finite models of arbitrarily large cardinality, then it has an infinite model.*

Proof: Suppose that ϕ has arbitrarily large models, but no infinite model.

Let sentence $\psi_k = \exists x_1 \ldots \exists x_k \wedge_{1 \leq i \leq j \leq k} \neg(x_i = x_j)$, where $k > 1$ is an integer.

Consider the set of sentences $\Delta = \{\phi\} \cup \{\psi_k : k = 2, 3, \ldots\}$.

Then Δ has no model.

By the compactness theorem, there is a finite set $D \subset \Delta$ that has no model.

D must contain ϕ (otherwise, any sufficiently large model would satisfy all ψ_k's in D).

Let k be the largest integer such that $\psi_k \in D$.

But ϕ has a finite model of cardinality larger than k.

This model satisfies all sentences in D. Contradiction.