Prenex Normal Form

Definition 1. [Prenex Normal Form] *An expression is said to be in* **Prenex Normal Form** *if it consists of a sequence of quantifiers followed by an expression that is free of quantifiers (a Boolean combination of atomic expressions — called the* matrix*).*

Theorem 1. *Any first-order expression can be transformed to an equivalent one in Prenex normal form.*

Proposition 1. *Let ϕ and ψ be arbitrary first-order expressions. Then*

(1) $\forall x(\phi \wedge \psi) \equiv (\forall x\phi \wedge \forall x\psi)$.

(2) If x does not appear free in ψ, $\forall x(\phi \wedge \psi) \equiv (\forall x\phi \wedge \psi)$.

(3) If x does not appear free in ψ, $\forall x(\phi \vee \psi) \equiv (\forall x\phi \vee \psi)$.

(4) If y does not appear in ϕ, $\forall x\phi \equiv \forall y\phi[x \leftarrow y]$.

Example

$$(\forall x(G(x,x) \wedge (\forall y G(x,y) \vee \exists y \neg G(y,y))) \wedge G(x,0))$$

$$(\forall x(G(x,x) \wedge (\forall y G(x,y) \vee \exists z \neg G(z,z))) \wedge G(w,0))$$

$$\forall x((G(x,x) \wedge (\forall y G(x,y) \vee \exists z \neg G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y((G(x,x) \wedge (G(x,y) \vee \exists z \neg G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y((G(x,x) \wedge \neg(\neg G(x,y) \wedge \forall z G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y((G(x,x) \wedge \neg \forall z(\neg G(x,y) \wedge G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y(\neg(\neg G(x,x) \vee \forall z(\neg G(x,y) \wedge G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y(\neg \forall z(\neg G(x,x) \vee (\neg G(x,y) \wedge G(z,z))) \wedge G(w,0))$$

$$\forall x \forall y \neg \forall z((\neg G(x,x) \vee (\neg G(x,y) \wedge G(z,z))) \vee \neg G(w,0))$$

$$\forall x \forall y \exists z \neg((\neg G(x,x) \vee (\neg G(x,y) \wedge G(z,z))) \vee \neg G(w,0))$$

A Proof System

Definition 2. *We say that a finite sequence of first-order expressions* $S = (\phi_1, \phi_2, \ldots, \phi_n)$ *is a* proof *of expression* ϕ_n *if for each expression* ϕ_i *in the sequence,* $1 \leq i \leq n$*, either:*

(a) ϕ_i *is a* **logical axiom** *or*

(b) there are two expressions of the form ψ*,* $\psi \Rightarrow \phi_i$ *among the expressions* $\phi_1, \ldots, \phi_{i-1}$*. (***Modus Ponens***)*

ϕ_n *is a* **first-order** *theorem and we write* $\vdash \phi_n$*.*

Example

Here is a proof for the following first-order theorem: $x = y \Rightarrow y = x$.

1. $\phi_1 = (x = y \wedge x = x) \Rightarrow (x = x \Rightarrow y = x)$ (equality axiom with equality relation)

2. $\phi_2 = (x = x)$ (equality axiom)

3. $\phi_3 = x = x \Rightarrow ((x = y \wedge x = x) \Rightarrow (x = x \Rightarrow y = x)) \Rightarrow (x = y \Rightarrow y = x)$ (Boolean validity)

4. $\phi_4 = ((x = y) \wedge x = x) \Rightarrow (x = x \Rightarrow y = x)) \Rightarrow (x = y \Rightarrow y = x)$ (from ϕ_2 and ϕ_3 by modus ponens)

5. $\phi_5 = (x = y \Rightarrow y = x)$ (from ϕ_1 and ϕ_4 by modus ponens)

Hence $\vdash x = y \Rightarrow y = x$.

THEOREMHOOD

We can encode first-order expressions (in a given fixed vocabulary) as well as proofs in an appropriate alphabet.

THEOREMHOOD: Given an (encoding of an) expression ϕ, is ϕ a first order theorem?

Theorem 2. *THEOREMHOOD is r.e.*

Proof. The TM that accepts the language tries all possible finite sequences of expressions in lexicographic order and accepts if one of them is a proof of ϕ.

(Note that given a string, there is an algorithm to check whether it a proof or not.) $\square$