Sentence

A variable in an expression can be **free** or **bound**.

In $\forall x \phi$, any occurrence of x in ϕ is bound.

All other occurrences are free.

Example: $\forall x(x + y > 0) \wedge (x = 0)$ has both a free and a bound occurrence of x.

A **sentence** is an expression without free variables.

Example: $\forall x(\forall y(\exists z(G(x, z) \wedge G(z, y)) \Rightarrow G(x, y)))$.

Model

The analog of a truth assignment in first-order logic.

Definition 1. [Model] *Given a vocabulary* Σ*, a* **model appropriate to** Σ *is a pair* $M = (U, \mu)$*:*

1. U *is a (non-empty) set called the* **universe** *of* M*.*

2. μ *is a function assigning to each variable, function symbol, and relation symbol in* $V \cup \Phi \cup \Pi$ *actual objects in* U*.*

For each variable x*,* μ *assigns an element* $x^M \in U$*.*

For each k*-ary function symbol* $f \in \Phi$*,* μ *assigns an actual function* $f^M : U^k \rightarrow U$*.*

For each k*-ary relation symbol* $R \in \Pi$*,* μ *assigns an actual relation* $R^M \subseteq U^k$*.*

For equality relation $=$*,* μ *assigns the relation* $=^M$ *which is (always)* $\{(u, u) : u \in U\}$*.*

Examples

1. A Model N appropriate to Σ_N (number theory):

The universe is the set of all nonnegative whole numbers.

N assigns the number 0 to constant 0: $0^N = 0$.

N assigns the unary function $\sigma^N(n) = n + 1$ to σ.

$+, \times, \uparrow$ are assigned the usual arithmetic operations.

$<$ is assigned the "less than" relation.

N maps all variables to elements in U.

2. A model appropriate to Σ_G is a graph G (graph theory).

The universe is the set of all nodes in G.

Relation G captures whether two nodes are connected by an edge.

First-Order Logic: Semantics

Definition 2. [Meaning of Terms] *Let ϕ be an expression over vocabulary Σ and let M be a model appropriate to Σ.*

For an arbitrary term t over Σ, its **meaning under** M, t^M *is defined as:*

1. If t is a variable or a constant, t^M is defined (explicitly) by μ.

2. If $t = f(t_1, \ldots, t_k)$, where f is a k-ary function symbol, and $t_1, \ldots, t_k$ are terms, then t^M is defined to be $f^M(t_1^M, \ldots, t_k^M)$ (an element of U).

First-Order Logic: Semantics

Definition 3. [Satisfaction] *Let ϕ be an expression over vocabulary Σ and let M be a model appropriate to Σ. M satisfies ϕ ($M \models \phi$):*

1. ϕ is an atomic expression, $\phi = R(t_1, \ldots, t_k)$ where $t_1, \ldots, t_k$ are terms: $(t_1^M, \ldots, t_k^M) \in R^M$.

2. $\phi = \neg\psi$: $M \not\models \phi$.

3. $\phi = \psi_1 \vee \psi_2$: $M \models \psi_1$ or $M \models \psi_2$.

4. $\phi = \psi_1 \wedge \psi_2$: $M \models \psi_1$ and $M \models \psi_2$.

5. $\phi = \forall x \psi$: for any $u \in U$, let $M_{x=u}$ be the model that is identical to M in everything except that $x^{M_{x=u}} = u$. Then for all $u \in U$, $M_{x=u} \models \psi$.

Examples

1. Number Theory:

a. $N \models \forall x (x < x + 1)$.

b. $N \not\models \forall x \exists y (x = y + y)$.

2. Graph Theory:

a.

$G \models \forall x (\forall y (G(x,y) \Rightarrow G(y,x)))$.

A symmetric graph G satisfies.

b.

$G \models (\forall x \exists y G(x,y) \wedge \forall x \forall y \forall z ((G(x,y) \wedge G(x,z)) \Rightarrow y = z))$.

G in which all nodes have outdegree 1 satisfies.

A sentence can be seen as a description of a set of models.

Checking Graph Model Satisfiability

ϕ-GRAPHS: Given a model G for ϕ (not necessarily a sentence) does $G \models \phi$?

Theorem 1. *For any expression ϕ over Σ_G, the problem ϕ-graphs is in P.*