Resolution

A method to determine whether a Boolean expression ϕ in CNF form is satisfiable.

0. $\phi^* = \phi$.

1. Repeat until possible:

   Let $(x \vee C)$ and $(\neg x \vee D)$ be two clauses in ϕ^*.

   Add the clause $(C \vee D)$ (*resolvent*) to ϕ^*.

2. If ϕ^* contains the empty clause then ϕ is not satisfiable else satisfiable.

Soundness and Completeness of Resolution

Theorem 1. [Resolution] *ϕ is satisfiable iff ϕ^* does not contain the empty clause.*

Soundness: If ϕ^* contains the empty clause then ϕ is unsatisfiable.

Completeness: If ϕ is unsatisfiable then ϕ^* should contain an empty clause.

A Deduction System

Definition 1. [Tautological Implication, Valid Consequence] *Let Σ be a set of boolean expressions. We say Σ tautologically implies τ (written $\Sigma \models \tau$) iff every truth assignment that satisfies every member of Σ also satisfies τ.*

Definition 2. [Deduction System] *Let Σ be a set of boolean expressions. A* **deduction** *from Σ is a finite sequence $< \alpha_0, \ldots, \alpha_n >$ of boolean expressions such that for each $k \leq n$, either*

(a) α_k is a tautology or

(b) $\alpha_k \in \Sigma$ or

(c) for some i and j less than k, α_i is $(\alpha_j \Rightarrow \alpha_k)$ (i.e., α_k is obtained by **modus ponens** *from α_i and α_j).*

Soundness and Completeness of Boolean Logic

Theorem 2. [Soundness] *Let* $< \alpha_0, \ldots, \alpha_n >$ *be a deduction from a set* Σ *of boolean expressions. Then* $\Sigma \models \alpha_k$ *for each* $k \leq n$.

Theorem 3. [Completeness] *Whenever* $\Sigma \models \tau$, *then there exists a deduction from* Σ, *the last component of which is* τ.

First-Order Logic: Syntax

Definition 3. [Vocabulary, Variables] *A* **vocabulary** $\Sigma = (\Phi, \Pi, r)$ *consists of two disjoint countable sets: a set* Φ *of function symbols, and a set* Π *of relation symbols,* r *is the arity function, mapping* $\Phi \cup \Pi$ *to the nonnegative integers.*

A function symbol $f \in \Phi$ *with* $r(f) = k$ *is called a* k*-ary function symbol.*

A relation symbol $R \in \Pi$ *with* $r(R) = k$ *is called a* k*-ary relation symbol.*

A 0*-ary function symbol is a constant.*

We assume that Π *always contains the binary equality relation "="*.

We also have a fixed, countable set of variables $V = \{x, y, z, ...\}$ *(which will take values from a universe of "discourse").*

Terms and Expressions

Definition 4. [Term] *A* **term** *over a vocabulary* Σ *can be:*

1. a variable $\in V$*.*

2. if $f \in \Phi$ *is a* k*-ary function symbol and* $t_1, \ldots, t_k$ *are terms, then* $f(t_1, \ldots, t_k)$ *is a term. (A constant is a term.)*

Definition 5. [Expression] *An first order expression over the vocabulary* Σ*:*

1. If $R \in \Pi$ *is a* k*-ary relation symbol and* $t_1, \ldots, t_k$ *are terms, then* $R(t_1, \ldots, t_k)$ *is a first-order expression (called an* **atomic** *expression).*

2. If ϕ *and* ψ *are expressions, then* $\neg\phi$*,* $\phi \vee \psi$*, and* $\phi \wedge \psi$ *are first order expressions.*

3. if ϕ *is a first-order expression and* x *is any variable, then* $\forall x \phi$ *is a first-order expression. (We also use* $\exists x \phi$ *as shorthand for* $\neg(\forall x \neg\phi)$*.*

Example: Number Theory

Vocabulary $\Sigma_N = (\Phi_N, \Pi_N, r_N)$, defined below.

$\Phi_N = \{0, \sigma, +, \times, \uparrow\}$.

$\Pi_N = \{=, <\}$. (binary relations)

Example Expressions:

1. $((1 \times 3) + 1) = (2 \uparrow 2)$
2. $\forall x (x < x + 1)$.
3. $\exists y \forall x (x = y + 1) \Rightarrow \forall w \forall z (w = z)$