Satisfiability and Validity

Definition 1. *A Boolean expression ϕ is* **satisfiable** *if there is a truth assignment T appropriate to it such that $T \models \phi$.*

ϕ is **valid** *or a* **tautology** *if $T \models \phi$ for all T appropriate to ϕ, written as $\models \phi$.*

Proposition 1. *A Boolean expression is unsatisfiable iff its negation is valid.*

Computational Problems

We encode a boolean expression in binary and its length is the length of the binary encoding.

SAT: Given a Boolean expression in CNF is it satisfiable?

There is a deterministic algorithm that can solve the problem in $O(mn2^n)$ where n is the number of variables and m is the number of clauses.

Can be solved by a nondeterministic algorithm in polynomial time.

Thus $SAT \in NP$.

HORNSAT

Definition 2. A Horn clause *is one that has at most one positive literal.*

HORNSAT: Given a boolean expression in CNF form such that each clause is an Horn clause, is it satisfiable?

Theorem 1. *HORNSAT* $\in P$.

Proof

A Horn clause can be written as an implication or a purely negatively clause.

Consider the following algorithm to satisfy all the implications:

Let T be the set of those variables that are true.

$T = \{\}$. (Initially all variables are false.)

While there is an unsatisfied implication:

Pick an unsatisfied implication $((x_1 \wedge \cdots \wedge x_i) \Rightarrow y)$ and add y to T.

The algorithm will terminate since T gets bigger each step and eventually all implicants will be satisfied.

Claim 1. *If another truth assignment T' satisfies all implications in ϕ then $T \subseteq T'$.*

ϕ is satisfiable iff T satisfies ϕ.

Suppose there is a purely negative clause not satisfied by T – then all its variables should be in T and by Claim no other superset of T can satisfy ϕ.

Boolean Functions

Definition 3. *A n-ary Boolean function is a function $f\{true, false\}^n \rightarrow \{true, false\}$.*

A Boolean expression ϕ with n variables $x_1, \ldots, x_n$ **expresses** *the n-ary boolean function f if, for any tuple of truth values $t = (t_1, \ldots, t_n)$ $f(t)$ is true if $T \models \phi$ and $f(t)$ is false if $T \not\models \phi$, where $T(x_i) = t_i$ for $i = 1, \ldots, n$.*

Proposition 2. *Any n-ary Boolean function f can be expressed as a Boolean expression ϕ_f involving variables $x_1, \ldots, x_n$.*