# Recursion Theorem

**Theorem 1.** *Let $T$ be a TM that computes a function $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$.*

*There is a TM $R$ that computes a function $r : \Sigma^* \rightarrow \Sigma^*$, where for every $w$,*

$$r(w) = t(< R >, w).$$

To make a TM $R$ that can obtain its own description **and** then compute with it, we need only to make a machine $T$ that takes an extra input that receives the description of the machine.

# Proof

TM $R$ has three parts $A$, $B$, and $T$.

1. $A = P_{<BT>}$. (After $A$ runs the tape contains $< BT >$.)

2. $B = q(< BT >)$ (Applies $q$ to the output of $A$ to get $< A >$).

$B$ then combines $A$, $B$, $T$ into a single machine, writes its description on the tape.

3. $T$ takes the output of $B$ and the input $w$ and computes with it.

# Applications

**Theorem 2.** *$H$ is undecidable.*

**Proof:** Suppose not. Then, let TM $M_H$ decide $H$.

Construct the following TM $B$:

On input $w$:

1. Obtain, via the recursion theorem, own description $< B >$.

2. Simulate $M_H$ on input $< B, w >$.

3. Accept if $M_H$ rejects and reject if $M_H$ accepts.

Running $B$ on $w$ does the opposite of what $H$ says it does. Therefore $H$ cannot decide $H$.

# Another Non-R.E. language

**Definition 1.** *If $M$ is a TM, then we say that the* **length** *of the description $< M >$ of $M$ is the number of symbols in the string describing $M$.*

*We say that $M$ is* **minimal** *if there is no TM equivalent to $M$ (in the language accepting sense) that has a shorter description.*

*Let $MIN_{TM} = \{M | M$ is a minimal $TM\}$.*

**Theorem 3.** *$MIN_{TM}$ is not r.e.*

# Proof

Assume that some TM $E$ enumerates $MIN_{TM}$ and we will obtain a contradiction.

Construct the following TM $C$:

On input $w$:

1. Obtain own description $< C >$.

2. Run the enumerator $E$ until a machine $D$ appears with a longer description than that of $C$.

3. Simulate $D$ on $w$.

Let $D$ be a TM with a longer description than $C$ in $E$'s list.

$C$ simulates $D$, but is shorter than $D$, hence $D$ cannot be minimal.

But $D$ appears on $E$'s list. Contradiction.