

Proof of Master Theorem

Proof. Draw a “recursion tree” for $T(n)$: $f(n)$ is the root and it has a children each of which is a recursion tree for $T(n/b)$. That is, a recursion tree is a complete a -ary tree where each node at depth i has the value $a^i f(n/b^i)$. The leaves of the tree contains the “base cases” of the recursion. Since we are looking at asymptotic bounds, we can assume without loss of generality (w.l.o.g) that $T(1) = f(1)$. Assuming each level of the tree is full, we have,

$$T(n) = f(n) + af(n/b) + a^2 f(n/b^2) + \dots + a^L f(n/b^L)$$

where L is the depth of the recursion tree.

$$L = \log_b n \text{ and since } f(1) = \Theta(1),$$

$$a^L f(n/b^L) = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a})$$

1) If $f(n)$ is a constant factor larger than $af(n/b)$ then $T(n)$ is a geometric series with largest term $f(n)$. Hence $T(n) = \Theta(f(n))$.

2) If $f(n)$ is a constant factor smaller than $af(n/b)$ then $T(n)$ is a geometric series with largest term $a^L f(n/b^L) = \Theta(n^{\log_b a})$.

3) If $af(n/b) = f(n)$ then there are $L + 1$ levels each level summing to $f(n)$ and hence $\Theta(f(n) \log_b n)$.

□

Geometric Series

For real $x \neq 1$, the summation

$$\sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n$$

is a geometric series and has the value

$$\sum_{k=0}^n x^k = \frac{x^{n+1} - 1}{x - 1}$$

Thus $\sum_{k=0}^n x^k = \Theta(x^n)$ if $x > 1$ (increasing series)

$\sum_{k=0}^n x^k = \Theta(1)$ if $x < 1$ (decreasing series)

If the series is infinite and $|x| < 1$, then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

Other Examples

Some recurrences do not exactly fit the Master Theorem.

1. $T(n) = 2T(n/2) + n/\log n$

Can't apply the Master Theorem directly since $af(n/b) = n/(\log n - 1)$ is not equal to a constant factor times $n/\log n$.

We compute the recurrence directly.

The sum of the nodes in the i th level is $n/(\log n - i)$. Thus, the depth of recursion is at most $\log n - 1$.

$$T(n) = \sum_{i=0}^{\log n - 1} n/(\log n - i) = \sum_{j=1}^{\log n} n/j = nH_{\log n} = \Theta(n \log \log n),$$

where $H_k = \sum_{i=1}^k 1/i$ is the *harmonic function* and $H_k = \Theta(\log k)$.

$$2. T(n) = T(3n/4) + T(n/4) + n$$

Each complete level in the tree adds up to n and each leaf has depth between $\log_4 n$ and $\log_{4/3} n$.

Hence, $T(n) = \Theta(n \log n)$.

Handling Floors and Ceilings

The MergeSort recurrence is really

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$$

We want to show that $T(n) = \Theta(n \log n)$.

We show the upper bound, lower bound is similar.

$$(1) T(n) \leq 2T(\lceil n/2 \rceil) + n \leq 2T(n/2 + 1) + n$$

We do a “domain transformation” as follows.

Let $S(n) = T(n + \alpha)$ where α is a constant chosen such that

$$(2) S(n) \leq 2S(n/2) + \Theta(n).$$

To find α , we compare (1) and (2).

$$T(n + \alpha) \leq 2T((n + \alpha)/2 + 1) + n$$

$$T(n + \alpha) \leq 2T(n/2 + \alpha) + \Theta(n)$$

which gives $\alpha = 2$.

By Master Theorem, $S(n) = O(n \log n)$, and hence $T(n) = S(n - \alpha) = O(n \log n)$.

A Linear Recurrence

Consider the Fibonacci numbers defined as follows:

$$F_0 = 0$$

$$F_1 = 1$$

$$F_i = F_{i-1} + F_{i-2}, i \geq 2.$$

We can show a **closed form** solution for F_i .

$$F_i = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right)$$

How to show this?

Sequence Operators

Let's use the notation $\langle a_n \rangle$ ($n \geq 0$) to represent the (infinite) sequence a_0, a_1, a_2, \dots

1. **E** be an **operator** which transforms a sequence by throwing away its first element.

That is, $\mathbf{E} \langle a_n \rangle = \langle a_{n+1} \rangle$.

2. Addition: $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$

3. Subtraction: $\langle a_n \rangle - \langle b_n \rangle = \langle a_n - b_n \rangle$

4. We can multiply a sequence by a constant operator c : $c \langle a_n \rangle = \langle ca_n \rangle$

Composition of Operators:

5. $(\mathbf{X} + \mathbf{Y}) \langle a_n \rangle = \mathbf{X} \langle a_n \rangle + \mathbf{Y} \langle a_n \rangle$

6. $(\mathbf{X} - \mathbf{Y}) \langle a_n \rangle = \mathbf{X} \langle a_n \rangle - \mathbf{Y} \langle a_n \rangle$

7. $\mathbf{XY} \langle a_n \rangle = \mathbf{X}(\mathbf{Y} \langle a_n \rangle)$

8. $\mathbf{E}^k \langle a_n \rangle = \langle a_{i+k} \rangle$

Example

$$T = \langle 2^0, 2^1, 2^2, \dots \rangle = \langle 2^n \rangle$$

$$\mathbf{E}T = \langle 2^1, 2^2, 2^3, \dots \rangle = \langle 2^{n+1} \rangle$$

$$2T = \langle 2^1, 2^2, 2^3, \dots \rangle = \langle 2^{n+1} \rangle$$

$$\mathbf{E}T - 2T = \langle 0, 0, 0, \dots \rangle = \langle 0 \rangle$$

The operator $(\mathbf{E} - 2)$ is an **annihilator** of T .

It can be shown that only $(\mathbf{E} - 2)$ annihilates $\langle 2^n \rangle$ and $(\mathbf{E} - 2)$ annihilates only sequences of the form $\langle c_0 2^n \rangle$ where c_0 is a constant.

If we know the annihilator of the sequence, we *almost* know the sequence.

Example

Suppose we have the recurrence $\langle a_n \rangle$: $a_0 = 1$ and $a_n = 3a_{n-1}$, $i \geq 1$.

Then $\mathbf{E} - 3$ annihilates $\langle a_n \rangle$ because

$$\begin{aligned}(\mathbf{E} - 3) \langle a_n \rangle &= \mathbf{E} \langle a_n \rangle - 3 \langle a_n \rangle \\ &= \langle a_{n+1} \rangle - \langle a_{n+1} \rangle = \langle 0 \rangle\end{aligned}$$

Thus, the closed form solution is $a_n = c_0 3^n$. Using the base case, we can fix $c_0 = 1$.

Combining Annihilators

What is the annihilator for the sequence $\langle 2^n + 3^n \rangle$?

$(\mathbf{E} - 2)$ annihilates 2^n and $(\mathbf{E} - 3)$ annihilates 3^n .

Thus $(\mathbf{E} - 2)(\mathbf{E} - 3)$ annihilates $\langle 2^n + 3^n \rangle$.

1. In general, $(\mathbf{E} - a)(\mathbf{E} - b)$, for integers $a \neq b$, will annihilate any sequence of the form $\langle c_1 a^n + c_2 b^n \rangle$, but nothing else.

One can show that $(\mathbf{E} - a)(\mathbf{E} - b) = \mathbf{E}^2 - (a + b)\mathbf{E} + ab$.

2. If $a = b$, then $(\mathbf{E} - a)^2$ annihilates na^n .

$(\mathbf{E} - a) \langle na^n \rangle = \langle (n + 1)a^{n+1} - na^{n+1} \rangle = \langle a^{n+1} \rangle$

$(\mathbf{E} - a)^2 \langle na^n \rangle = (\mathbf{E} - a) \langle a^{n+1} \rangle = \langle 0 \rangle$

More generally, $(\mathbf{E} - a)^k$ annihilates any sequence $\langle p(n)a^n \rangle$ where $p(n)$ is any polynomial in n of degree $k - 1$.

Examples

1. $(E - 1)^3$ annihilates $\langle c_0 + c_1n + c_2n^2 \rangle$.
2. $(E - 3)(E - 2)(E - 1)$ annihilates $\langle c_0 + c_12^n + c_23^n \rangle$.
3. $(E - 3)^2(E - 2)(E - 1)$ annihilates $\langle c_0 + c_12^n + c_23^n + c_3n3^n \rangle$.

Cookbook

Annihilator	Sequence
$(\mathbf{E} - 1)$	$\langle c_0 \rangle$
$(\mathbf{E} - a)$	$\langle c_0 a^n \rangle$
$(\mathbf{E} - a_0) \dots (\mathbf{E} - a_j)$	$\langle c_0 a_0^n + \dots + c_j a_j^n \rangle$
$(\mathbf{E} - 1)^2$	$\langle c_1 n + c_2 \rangle$
$(\mathbf{E} - a)^2$	$\langle (c_1 n + c_2) a^n \rangle$
$(\mathbf{E} - a)^2 (\mathbf{E} - b)$	$\langle (c_1 n + c_2) a^n + c_3 b^n \rangle$
$(\mathbf{E} - a)^i$	$\langle (c_0 + c_1 n + c_2 n^2 + \dots + c_{i-1} n^{i-1}) a^n \rangle$

If X annihilates $\langle a_n \rangle$ then X annihilates $c \langle a_n \rangle$ for any constant c .

If X annihilates $\langle a_n \rangle$ and Y annihilates $\langle b_n \rangle$ then XY annihilates $\langle a_n \pm b_n \rangle$.

Solving the Fibonacci Recurrence

The recurrence $F_i = F_{i-1} + F_{i-2}$ is annihilated by $\mathbf{E}^2 - \mathbf{E} - 1$.

And $\mathbf{E}^2 - \mathbf{E} - 1 = (\mathbf{E} - r_1)(\mathbf{E} - r_2)$ where

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

Thus, $F_i = c_1 r_1^n + c_2 r_2^n$ for some constants c_1, c_2 .

Use boundary conditions to solve for c_1 and c_2 .

$$0 = c_1 + c_2$$

$$1 = c_1 r_1 + c_2 r_2.$$

Solving Homogeneous Linear Recurrences with Constant Coefficients

Recurrences of the form:

$$p_0 a^n + p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_k a_{n-k} = 0.$$

1. Write down the annihilator.

$$p_0 \mathbf{E}^k + p_1 \mathbf{E}^{k-1} + \dots + p_k.$$

2. Factor the annihilator.

3. Find the sequences annihilated by each factor.

4. Combine the sequences.

5. Solve for constants using the boundary conditions.

Solving Nonhomogeneous Equations

Recurrences of the form:

$$p_0 a^n + p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_k a_{n-k} = f(n).$$

where $f(n)$ is a nonzero function of n .

Examples:

1. $a_n - 5a_{n-1} + 6a_{n-2} = 4, a_0 = 5, a_1 = 7.$

The operator $(\mathbf{E}^2 - 5\mathbf{E} + 6)$ annihilates the “homogeneous” part of the recurrence.

That is, $(\mathbf{E}^2 - 5\mathbf{E} + 6) \langle a_n \rangle = \langle a_{n+2} - 5a_{n+1} + 6a_n \rangle = \langle 4 \rangle.$

To annihilate $\langle 4 \rangle$ we apply $(\mathbf{E} - 1)$. Thus, the final annihilator is

$$(\mathbf{E}^2 - 5\mathbf{E} + 6)(\mathbf{E} - 1) = (E - 2)(E - 3)(E - 1).$$

Thus $a_n = c_1 + c_2 2^n + c_3 3^n.$

We solve for constants:

$$a_2 - 5a_1 + 6a_0 = 4, \quad a_0 = 5, \quad a_1 = 7.$$

2. Solve $a_n - 2a_{n-1} = 2^n - 1$, for $n \geq 1$.

$$a_0 = 0.$$

$(\mathbf{E} - 2)$ annihilates the homogeneous part and $(E - 2)(E - 1)$ annihilates the right hand side.