Examples

Single-pair Shortest-path problem: Given a weighted directed graph find the shortest path from s to t.

Let $d[v]$ denote the shortest path from s to v.

maximize $d[t]$
subject to
$d[v] \leq d[u] + w(u, v)$ for each edge $(u, v) \in E$.
$d[s] = 0$.

$|V|$ variables and $|E| + 1$ constraints.

Maximum flow

maximize $\sum_{u \in V} f(s,v)$
subject to
$f(u,v) \leq c(u,v)$ for each $u, v \in V$
$f(u,v) = -f(u,v)$ for each $u, v \in V$
$\sum_{u \in V} f(u,v) = 0$ for each $u \in V - \{\mathbf{s}, \mathbf{t}\}$

$|V|^2$ variables and $2|V|^2 + |V| - 2$ constraints.

Weighted vertex cover

Given an undirected graph $G = (V, E)$ in which each vertex $v \in V$ has an associated positive weight $w(v)$. The weight of a cover is the sum of the weights of its vertices. Find a vertex cover of minimum weight.

General Idea to approximate:

1. (Try to) Formulate the problem as an Integer linear program.
2. Relax the ILP by relaxing the integer constraint.
3. Solve the relaxed LP.
4. Round the solution to the LP, making sure you don't lose feasibility.

LP formulation

variable $x(v)$ for each $v \in V$.

minimize $\sum_{v \in V} w(v)x(v)$
subject to
$x(u) + x(v) \geq 1$ for each $(u, v) \in E$
$x(v) \in \{0, 1\}$ for each $v \in V$

LP relaxation:
$x(v) \leq 1$, $x(v) \geq 0$ for each $v \in V$.
An optimal solution to the LP is a **lower bound** on the optimal solution to the ILP.

LP rounding:
if $x(v) \geq 1/2$ **then** $x(v) = 1$ (i.e., include v in the vertex cover.)
else $x(v) = 0$

Theorem 1. *LP relaxation and rounding gives a 2-approximation algorithm for the minimum-weight vertex cover problem.*

Proof. z^* - optimal solution to the LP
C^* - optimal vertex cover
C - rounded vertex cover

$$z^* \leq w(C^*)$$

$$z^* = \sum_{v \in V} w(v)\bar{x}(v) \geq \sum_{v \in V : \bar{x}(v) \geq 1/2} w(v)\bar{x}(v)$$

$$\geq \sum_{v \in V : \bar{x}(v) \geq 1/2} \frac{1}{2} w(v) = \frac{1}{2} w(C)$$

□

Randomized Approximation algorithm

Definition 1. *A randomized approximate algorithm has a* **ratio bound** $\rho(n)$*, if for any input of size* n*, the optimal solution* $C^*(n)$ *and the* **expected cost** *of the algorithm's solution* $C(n)$ *satisfy the relation:*

$\frac{C(n)}{C^*(n)} \leq \rho(n)$ *for minimization problems*

$\frac{C(n)}{C^*(n)} \geq \rho(n)$ *for maximization problems.*

Randomized Rounding

General Idea to approximate:

1. (Try to) Formulate the problem as an Integer linear program.
2. Relax the ILP by relaxing the integer constraint.
3. Solve the relaxed LP.
4. **Randomized Rounding**: Treat the fractional (part) of the solution to the LP as probabilities and round based on the probabilities.
5. Make sure you have feasibility.

Randomized Rounding technique for Minimum Set Cover

The minimum set cover problem: Given a set U with n elements, a collection F of subsets of U , and a cost function $c : F \rightarrow R^+$, find a minimum cost sub-collection of F that covers all elements of U.

variable x_S for each set $S \in F$. The ILP is:
minimize $\sum_{S \in F} c(S) x_S$
subject to

$$\sum_{S: e \in S} x_S \geq 1, \;\; e \in U$$
$$x_S \in \{0, 1\}, \;\; S \in F$$

LP relaxation:
minimize $\sum_{S \in F} c(S) x_S$
subject to

$$\sum_{S: e \in S} x_S \geq 1, \;\;\; e \in U$$
$$x_S \geq 0, \;\; S \in F$$
$$x_S \leq 1, \;\; S \in F$$

Randomized Rounding:
Let x_S, $S \in F$ be an optimal solution to the linear program. Pick a set $S \in F$ with probability x_S.

Theorem 2. *LP relaxation plus randomized rounding gives an $O(\log n)$ approximation algorithm for minimum set cover problem.*

Proof. Let C be the collection of sets picked. The expected cost of C is

$$\sum_{S \in F} \Pr[S \text{ is picked }] c_S = \sum_{S \in F} p_S c_S = LOPT \leq OPT$$

where LOPT and OPT are the optimal solutions to the LP and set cover respectively.

Consider an element $e \in U$. Suppose e occurs in k sets of F. Then

$$\Pr[\text{ e is covered by C }] \geq 1 - (1 - 1/k)^k \geq 1 - \frac{1}{e}$$

Independently, pick $c \log n$ such collections and take their union - call it C'. Then for a suitably large constant c,

$$\Pr[\text{ e is not covered by C' }] \leq \left(\frac{1}{e}\right)^{c \log n} \leq \frac{1}{2n}$$

Summing over all elements $e \in U$

$$\Pr[\text{ some e } \in U \text{ is not covered by C' }] \leq n \times 1/2n \leq 1/2$$

Therefore C' is a set cover with probability at least 1/2. If C' is not a set cover repeat the above procedure.

The expected cost is at most $2 \times OPT \times c \log n = O(\log n) OPT$. $\square$