CLIQUE

CLIQUE: Given a graph G, does G contain a complete subgraph of size k.

Theorem 1. *CLIQUE is NP-Complete.*

Proof: CLIQUE $\in NP$: the clique is the certificate.

We reduce 3-SAT to CLIQUE.

Let ϕ be an instance of 3-SAT with k clauses:

$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$.

We construct an instance G as follows:

Each literal in each clause corresponds to a node in G. The nodes are organized into k groups of 3 nodes each called the triples.

There is an edge between all pairs of nodes except between:

1. two nodes in the same triple.

2. two nodes corresponding to opposite literals.

We show that ϕ is satisfiable iff G has a clique of size k.

If ϕ has a satisfying assignment, then choose one true literal in every clause. The corresponding nodes form a k-clique.

If G has a k-clique, then each of the k triples contains exactly one of the k clique nodes. Assign truth values such that each corresponding literal is made true.

VERTEX-COVER

VERTEX-COVER: Given an undirected graph G, is there a k-subset of nodes such that every edge of G is incident on one of the nodes in the subset.

Theorem 2. *VERTEX-COVER is NP-complete.*

Proof:

VERTEX-COVER $\in NP$.

We reduce CLIQUE to VERTEX-COVER.

Let $< G = (V, E), k >$ be an instance of CLIQUE.

Let G^c be the complement graph of G, i.e., contains exactly those edges not in G.

The corresponding instance of VERTEX-COVER is $< G^c, |V| - k >$.

G has a clique of size k iff G^c has a vertex cover of size $|V| - k$.

Let G has a clique V' of size k. Consider an edge (u, v) in G^c.

Then $(u, v) \notin E$ and thus one of u or v should not belong to V'.

So, one of them should belong to $V - V'$.

Thus $V - V'$ is a vertex cover in G^c.

To show the other way, let G^c has a vertex cover V' of size $|V| - k$.

Then for all $u, v \in V - V'$, $(u, v) \notin G^c$ and hence belong to G.

DIRECTED HAMILTONIAN PATH

HAMPATH: Given a *directed* graph G and two nodes $s, t \in V$, is there a path in G from s to t that goes through every node exactly once.

Theorem 3. *HAMPATH is NP-complete.*

Proof:

HAMPATH $\in NP$.

We reduce 3-SAT to HAMPATH.

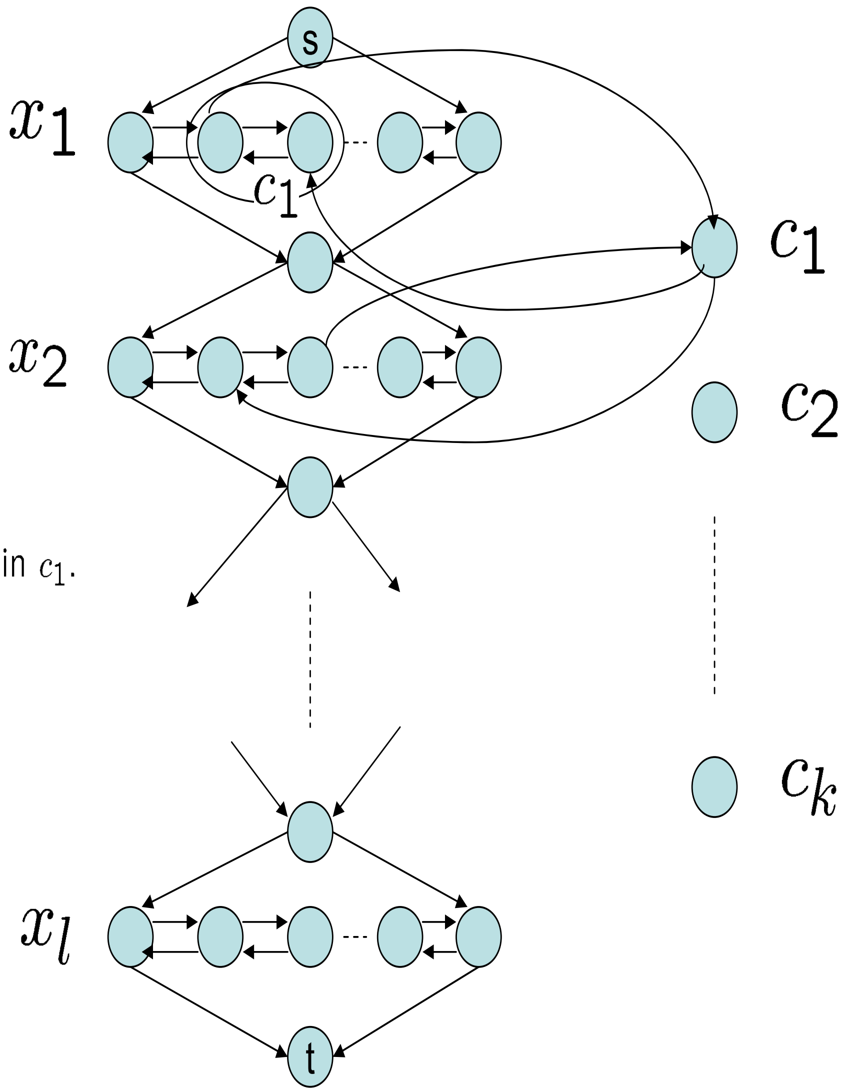
Let ϕ be a 3-SAT instance containing k clauses:

$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$

where each a, b, c is a literal x_i or x_i^c, $1 \leq i \leq l$.

We convert ϕ into a graph instance G as follows.

The high-level structure of G



x_1 and x_2^c appear in c_1.

We show that ϕ is satisfiable iff there exists a Hamiltonian path from s to t.

Suppose that ϕ is satisfiable.

The path begins at s, goes through each diamond (from top to bottom), in turn and ends up at t.

The horizontal nodes are included in the path as follows: if x_i is true then we zig-zag (left-in and right-out from top) and if it is false we zag-zig (right-in and left-out from top) through the diamond.

The clause nodes are included by adding detours at the horizontal nodes. We use one of the true literals of a clause to detour.

To show the other way, let G have a Hamiltonian path from s to t.

If the Hamiltonian path goes through the diamonds in order from top one to the bottom one (except for the detours to the clause nodes), the satisfying assignment is: zig-zag means true and zag-zig means false. By observing the diamond at which the detour is taken, we can determine which of the literals in the corresponding clause is true.

We now argue that any Hamiltonian path in G should be as above.

The only other way is for a path to enter a clause from one diamond and return via another. By our construction of G this is not possible.

UNDIRECTED HAMILTONIAN PATH

UHAMPATH: Given a *undirected* graph G and two nodes $s, t \in V$, is there a path in G from s to t that goes through every node exactly once.

Theorem 4. *UHAMPATH is NP-complete.*

Proof: UHAMPATH $\in NP$.

We reduce from HAMPATH.

Let $< G, s, t >$ be an instance of HAMPATH. Construct an undirected graph G' as follows:

Each node of G, except for s and t is replaced by a triple of nodes u^{in}, u^{mid}, u^{out} in G'. s and t are replaced by s^{out} and t^{in} in G'.

Edges in G' connect u^{mid} with u^{in} and u^{out}. Also, there is an edge between u^{out} and v^{in} in G' if an edge goes from u to v in G.

G has a Hamiltonian path from s to t iff G' has a Hamiltonian path from s^{out} to t^{in}.