Properties of a Poisson process

1. The expected number of arrivals in an interval of t steps is λt.

2. Sum of Poisson processes is a Poisson process with sum of rates.

3. If a Poisson process is split randomly, the two processes are Poisson.

Exponential Distribution

$X \sim Exp(\mu) \Rightarrow \Pr(X \leq s) = 1 - e^{-\mu s}$

$E[X] = 1/\mu$

The exponential distribution is **memoryless**:

$$\Pr(X > t + \tau | X > t) = \frac{\Pr(X > t + \tau)}{\Pr(X > t)}$$

$$= \frac{e^{-\mu(t+\tau)}}{e^{-\mu t}} = e^{-\mu\tau} = \Pr(X > \tau)$$

Gamma Distribution

A Gamma distribution with parameters (α, β) has a density function given by:

$f(x) = \frac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)}$ if $x > 0$

$= 0$ otherwise

where $\Gamma(\alpha)$ is the gamma function defined by

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ for any real number $z > 0$.

Some properties of the gamma function are:

1. $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$.

2. $\Gamma(k+1) = k!$ for any nonnegative integer k.

Properties of the Gamma Distribution

1. Mean of the Gamma distribution is $\alpha\beta$ and the variance is $\alpha\beta^2$.

2. The $Exp(\beta)$ and $\Gamma(1, 1/\beta)$ are the same.

3. If $X_1, \ldots X_m$ are independent $Exp(\beta)$ r.v.s, then $\sum_{i=1}^{m} X_i$ is $\Gamma(m, 1/\beta)$ – also called the m-*Erlang* distribution.

Interarrival distributions

Let X_n ($n \geq 1$) denote the time from the $(n-1)$th to the nth arrival.

The sequence $\{X_n, n = 1, 2, \ldots\}$ is the **sequence of interarrival times**.

$\Pr(X_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$

Hence X_1 has exponential distribution with mean $1/\lambda$.

$Pr(X_2 > t) = E[\Pr(X_2 > t | X_1)]$

However,

$$\Pr(X_2 > t | X_1 = s) = \Pr(0 \text{ events in } (s, s+t] | X_1 = s)$$

$$= e^{-\lambda t}$$

If the arrival process is Poisson with rate λ, then X_n, $n = 1, 2 \ldots$ are independently and identically distributed exponential random variable with mean $1/\lambda$.

Waiting Time Distribution

S_n: arrival time of the nth event - called the *waiting time* of the nth event.

$$S_n = \sum_{i=1}^{n} X_i \;\; n \geq 1$$

S_n has a *Gamma* distribution with parameters n and λ.

$N(t) \geq n \equiv S_n \leq t$

Hence,

$F_{S_n}(t) = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$

Conditional Distribution of the Arrival Times

Given that exactly one event has taken place in the interval $[0, t]$, determine the distribution of the time at which it occurred.

$$\Pr(X_1 < s | N(t) = 1) = \frac{\Pr((N(s)=1) \cap (N(t)-N(s)=0))}{\Pr(N(t)=1)}$$

$$= \frac{e^{-\lambda s}\lambda s e^{-\lambda(t-s)}}{e^{-\lambda t}\lambda t} = \frac{s}{t}$$

That is, the time of the event is uniformly distributed in $[0, t]$.

If $Y_1, \ldots Y_n$ are n random variables, then $Y_{(1)}, \ldots Y_{(n)}$ are the *order statistics* corresponding to $Y_1, Y_2, \ldots, Y_n$ if $Y_{(k)}$ is the kth smallest value among $Y_1, \ldots, Y_n$, $k = 1, \ldots, n$.

Theorem 1. *Given that $N(t) = n$), the n arrival times $S_1, \ldots S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.*