Unfolding an 8-high Square, and Other New Wrinkles

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A geometric dissection is a cutting of one or more geometric figures into pieces that we can rearrange to form one or more other geometric figures [3]. When applied to two-dimensional figures, dissections are not only demonstrations of the equivalence of area but also a source of enchantment since ancient times. In the last hundred years, there has been an increasing emphasis on finding dissections that use the fewest possible number of pieces.

Various dissections have remarkable properties. One such property is the hingeability of pieces, which has been explored in some depth in [4]: Swing the pieces one way on their hinges to form one figure, say an equilateral triangle; swing them another way to form another figure, say a square. Different types of hinges effect different types of motion, ranging from swinging to twisting, and most recently, to folding [5].

In this article, we focus on a *piano hinge*, which connects two pieces along a shared edge and allows a folding motion. In [5], we defined a *piano-hinged dissection* as a cutting of one or more figures that are two levels high into pieces that fit together to form another such set of one or more figures, such that the pieces are connected by piano hinges. Here we shall define a related type of dissection, which still uses piano hinges but places somewhat different constraints on the dissections in which they are used.

Consistent with [4], piano-hinged dissections require that whenever two or more figures are dissected to one, then there is one hinged assemblage for each of the original figures. Thus a piano-hinged dissection of two squares to one would have two separate assemblages. One assemblage would fold into one square, the other assemblage would fold into a second identical square, and the pair would together fold into a larger square. Each of the figures would be two levels thick.

The question that we raise here is whether we can create just one assemblage rather than two, that would fold to form a square that is one level thick, and alternatively fold to form a square two levels thick but half the cross-sectional area. We thus define a *stack-folding dissection* to be a dissection of one figure that is p levels thick to a second figure that is $q \neq p$ levels thick, such that all pieces are connected into one assemblage using piano hinges. An example of such a dissection was given in [1], in which a 1-high regular hexagon was stack-folded into a 2-high equilateral triangle.

Specifically, we require that each level in a stack-folding dissection have a given positive thickness. Furthermore, for any smaller positive thickness, the dissection must work, in the sense that its assemblage will fold from the p-high figure to the q-high figure without obstruction.

To identify the location of piano hinges in the diagrams depicting stack-folding dissections, we shall adopt the following convention. A dotted line will denote a hinge between two pieces that are adjacent on the same level. In the case that one piece is hinged to another piece on a level either above it or below it, the hinge will be denoted by a row of dots next to the shared edge on each of the two levels of the hinged pieces. In some cases two separate hinges may be in the same position on nearby levels. (See for example the hinges between pieces B and A, and between pieces C and D, in Figure 12.) To determine which pieces hinge with which, pair the rows of dots starting at the top of the figure.

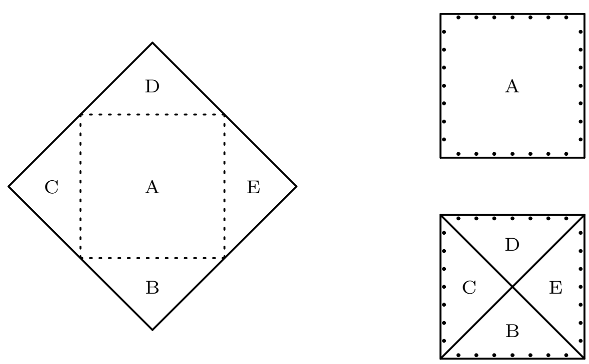

Figure 1: Stack-folding dissection of a 1-high square to a 2-high square

Let's now get started, with the help of one of the simplest of the stack-folding dissections. We see a 5-piece stack-folding dissection of a 1-high

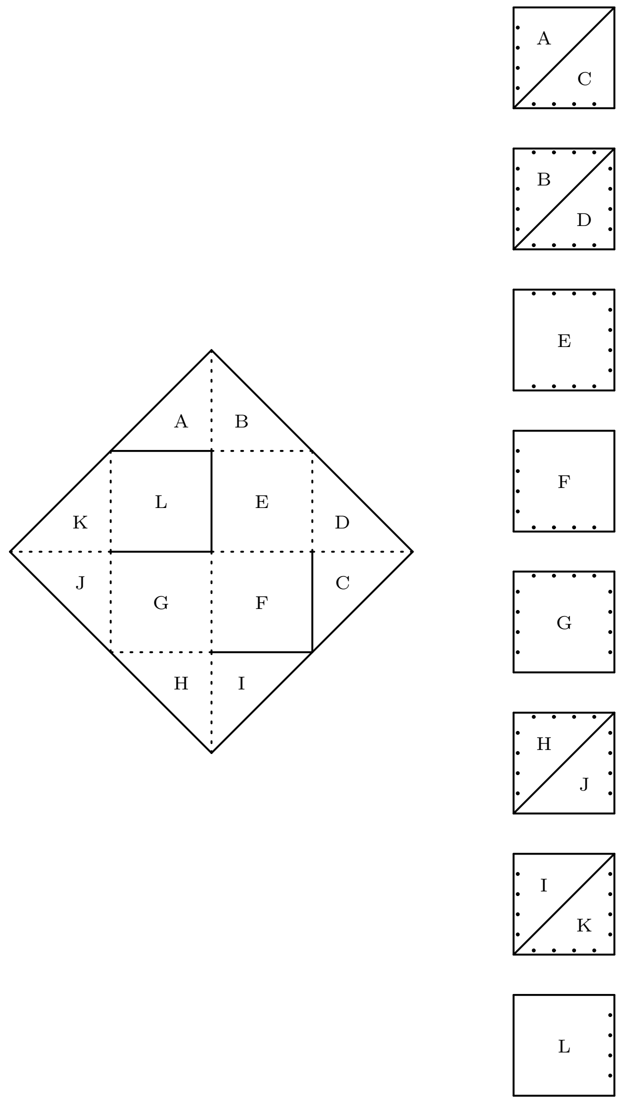

Figure 2: Stack-folding dissection of a 1-high square to an 8-high square

square to a 2-high square in Figure 1. It is quite simple and in fact has already appeared as part of the dissection in Figure 8.11 of [5] and by itself in a paper by Lyle Pagnucco and Jim Hirstein [9].

A bit more challenging is a stack-folding dissection of a 1-high square

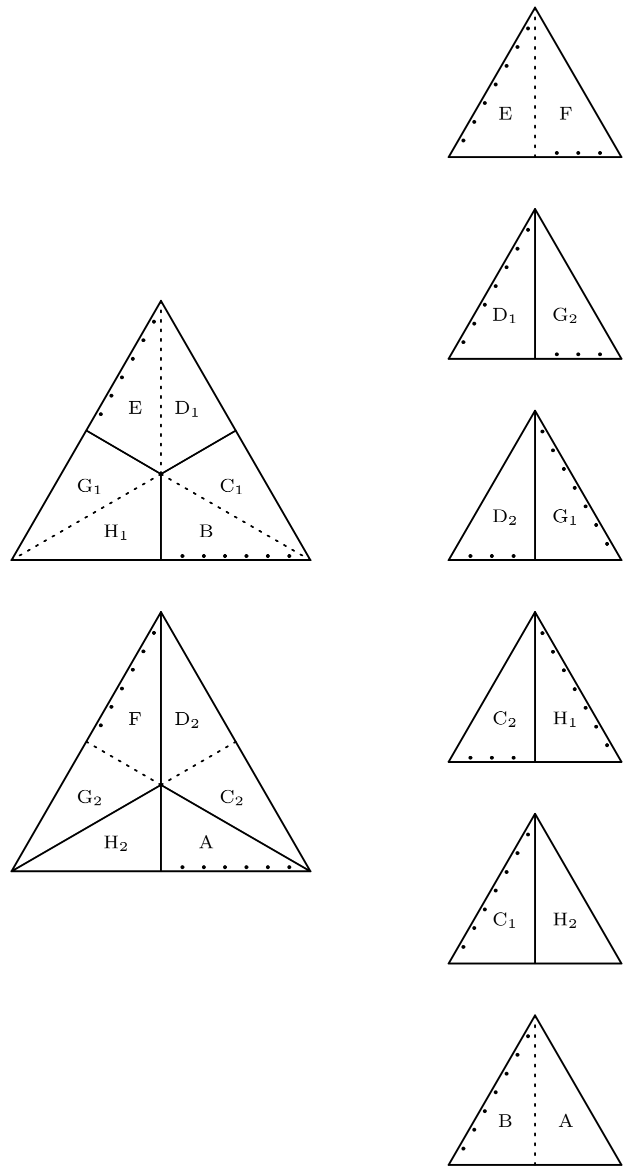

Figure 3: Stack-folding dissection of a 2-high triangle to a 6-high triangle

to an 8-high square. We can adapt a simple 12-piece dissection by Ernest Freese [7], but need to take care. In Figure 2, we take the four pairs of isosceles right triangles in the 1-high square and stack them in the 8-high square. Four of these triangles (pieces A, B, C, and D) are above piece E, and four of them (pieces H, I, J, and K) are below piece G. If we use F to connect pieces E and G, then piece L must connect to either piece K on the bottom or piece A on the top.

It is convenient when one of the figures in a stack-folding dissection is just one level high, but this does not always seem to be attainable. For example, I have not discovered how to fold out a 3-high triangle to form a 1-high triangle. However, we can find a stack-folding dissection of a 2-high triangle to a 6-high triangle. Such a dissection, using eight pieces, appears in Figure 3. Note that four of the pieces (C, D, G and H) are 2-high.

Since there is an elegant standard dissection of three hexagons to one, we expect to get a stack-folding dissection of a 1-high hexagon to a 3-high hexagon. A 6-piece dissection from [8] does not seem to work well, but we can adapt a 7-piece dissection, as in Figure 4.

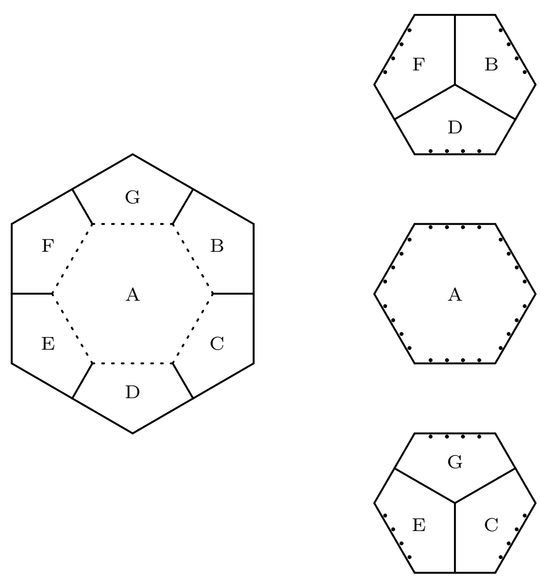

Figure 4: Stack-folding dissection of a 1-high hexagon to a 3-high hexagon

For a 1-high hexagon to a 4-high hexagon, it is not possible to use fewer

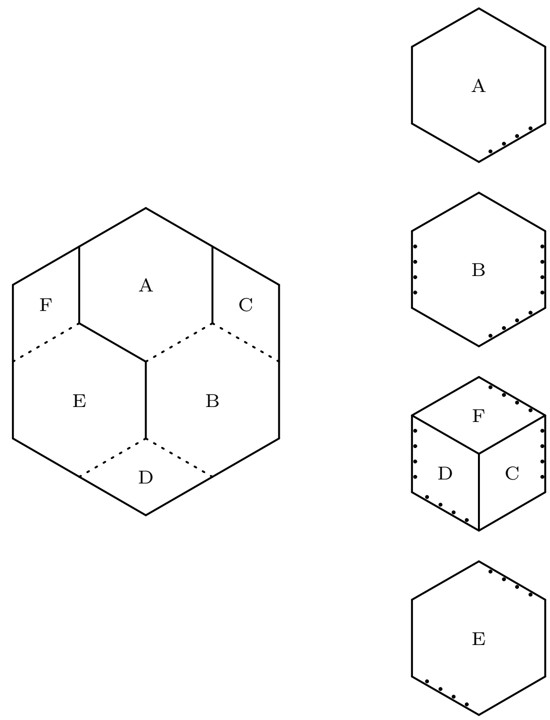

Figure 5: Stack-folding dissection of a 1-high hexagon to a 4-high hexagon

than six pieces, because the cross-section of a 4-high hexagon cannot span more than one sixth of the perimeter of an equivalent 1-high hexagon. Thus the 6-piece dissection in Figure 5 is the best possible.

A stack-folding dissection of a 3-high hexagon to a 4-high hexagon is a bit of a challenge. We can adapt a 12-piece unhinged dissection by Ernest Freese [7] to produce the 11-piece stack-folding dissection in Figure 6. Pieces H and and K are three levels thick, and pieces B and D are two levels thick. Note that pieces A, B, C, and D have a “leaf-cyclic” hinging ([5]) consisting of A to B, B to C, C to D, and D to A—interesting, since they are not all the same height.

There’s more challenge with a 1-high hexagon to a 9-high hexagon. We could fill in the 1-high hexagon with six copies of a small hexagon, and fill

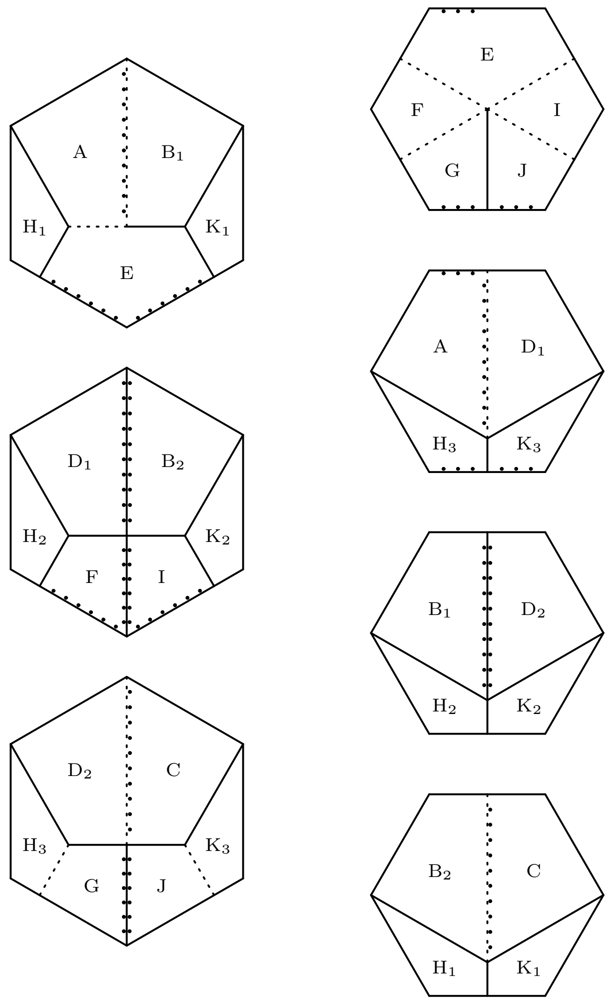

Figure 6: Stack-folding dissection of a 3-high hexagon to a 4-high hexagon

in the top corner, lower right corner, and lower left corner each with two half-hexagons. This is the 12-piece unhinged dissection that Ernest Freese described [7]. Yet placing the folds and cuts appropriately seems difficult, until we cut three of the six copies of hexagons in half. This gives a 15-piece stack-folding dissection as in Figure 7. It was difficult discovering how to stack the half-hexagons on top of each other in the 9-high hexagon, but that

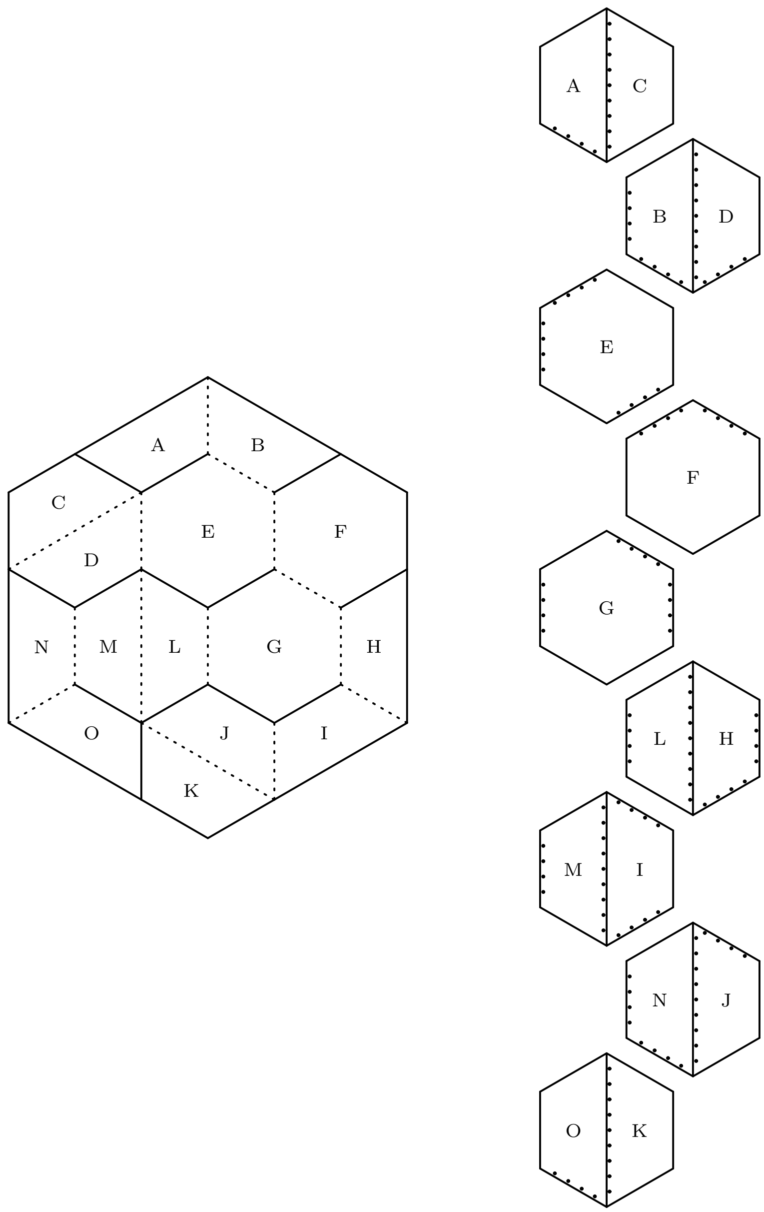

Figure 7: Stack-folding dissection of a 1-high hexagon to a 9-high hexagon

trick seems necessary.

The best known unhinged dissection of three hexagrams to one, by Ernest

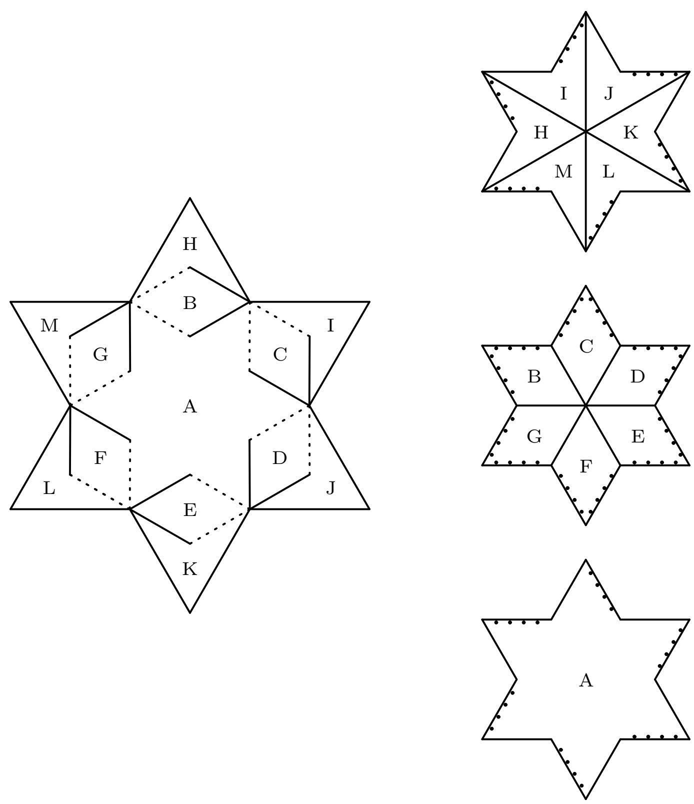

Figure 8: Stack-folding dissection of a 1-high hexagram to a 3-high hexagram

Freese, uses twelve pieces. The stack-folding dissection in Figure 8 uses just thirteen pieces and has such a lovely symmetry.

Because the Greek Cross is comprised of five squares, it seems natural to dissect two of them to one. Henry Dudeney gave a 5-piece dissection for the unhinged version of this problem [2, August 26, 1900]. There is a 20-piece piano-hinged dissection in [5]. An economical 13-piece stack-folding dissection of a 1-high Greek Cross to a 2-high Greek Cross is possible (Figure 9). The approach is to have the center of the bottom level of the 2-high cross coincide with the center of the 1-high cross, with the arms of one cross at 45° angles to the other cross. We can then cut the top level into four pieces that fold out and bring along the excess from the bottom level. To accomplish the folding without obstruction, it seems necessary to fold simultaneously along all hinges attached to piece A.

For a stack-folding dissection of a 1-high Latin Cross to a 2-high Latin Cross, we can mimic the approach from the Greek Cross, if we take "center" to mean the center of the Greek Cross that we would get if we cut one square from the Latin Cross. We then cut the long arm of the 2-high Latin Cross so that it unfolds to fill out the long arm of the 1-high Latin Cross. The resulting 14-piece stack-folding dissection is in Figure 10.

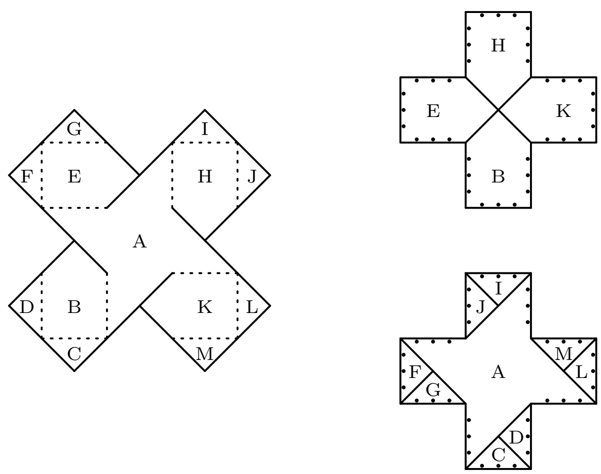

Figure 9: Stack-folding dissection of a 1-high Greek Cross to a 2-high Greek Cross

Emboldened by our success with Greek and Latin Crosses, we can try for a stack-folding dissection of a 1-high Cross of Lorraine to a 2-high Cross of Lorraine. For this cross, we have two possible "centers", but never mind! If we choose one candidate position for the centers, and try to work out the details in an appropriate fashion, then the other candidate position for the centers will align automatically! The resulting 25-piece stack-folding dissection of a 1-high Cross of Lorraine to a 2-high Cross of Lorraine is in Figure 11. Performing the folding is a bit trickier, because now there

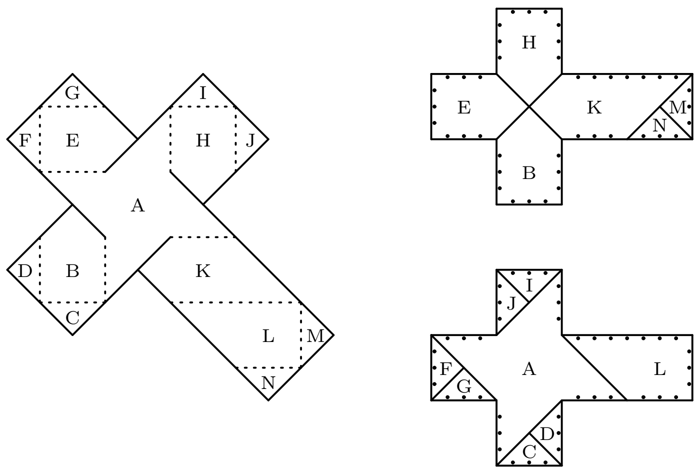

Figure 10: Stack-folding dissection of a 1-high Latin Cross to a 2-high Latin Cross

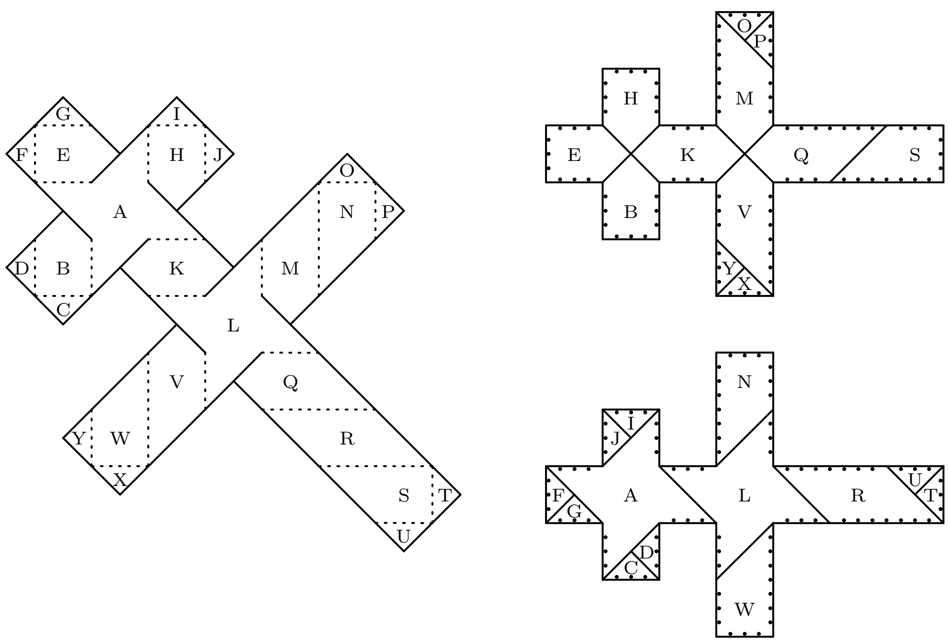

Figure 11: Stack-folding 1-high Cross of Lorraine to a 2-high Cross of Lorraine

is another piece L like piece A. First fold simultaneously along all hinges attached to piece L, and then do the same for piece A.

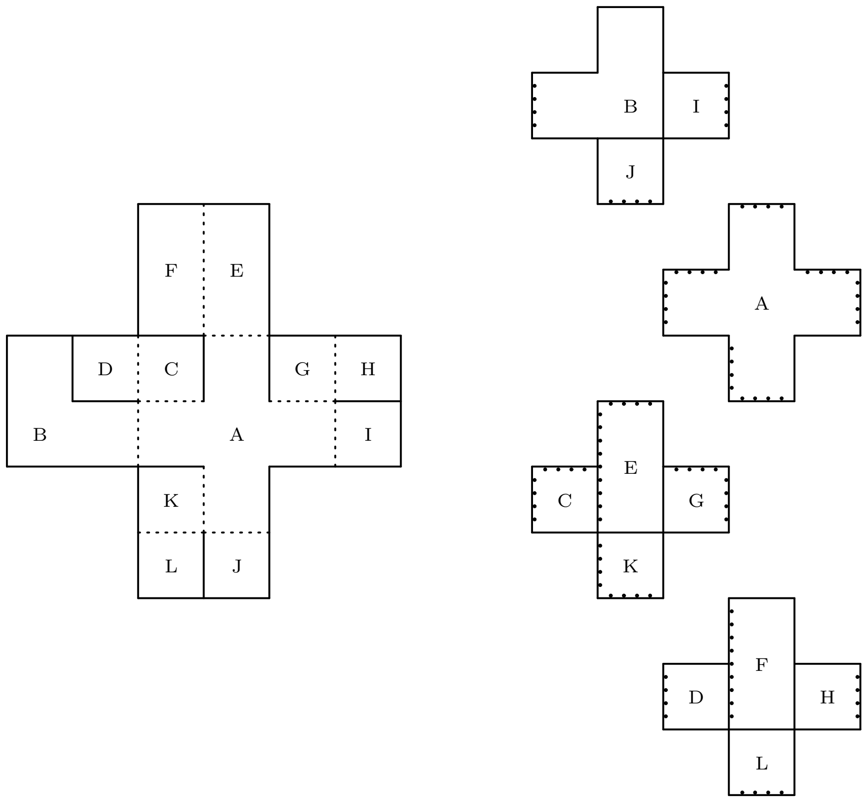

Figure 12: Stack-folding a 1-high Greek Cross to a 4-high Greek Cross

There's a 16-piece piano-hinged dissection of four Greek Crosses to one in [5]. How about trying a stack-folding dissection of a 1-high Greek Cross to a 4-high Greek Cross? We respond with the 12-piece dissection in Figure 12. Without cutting, we can fit one whole level of the 4-high Greek Cross within the boundaries of the 1-high cross. It then becomes a modest challenge to see how to cut and fold out the remaining three levels to fill the remainder of the 1-high cross.

If we handle this dissection appropriately, then we can easily generalize it into a 12-piece stack-folding dissection of a 1-high Latin Cross to a 4-high

Latin Cross, as we see in Figure 13.

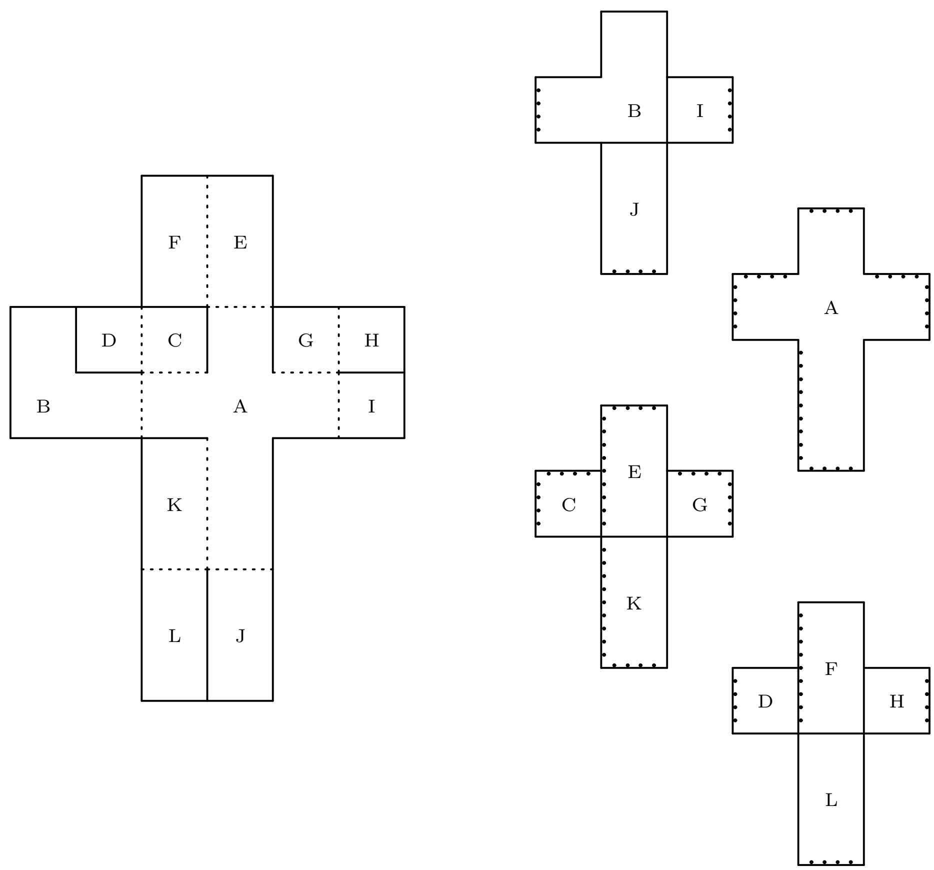

Figure 13: Stack-folding dissection of a 1-high Latin Cross to a 4-high Latin Cross

After the successes with the Greek Cross and the Latin Cross, there should be a real chance of a stack-folding dissection of a 1-high Cross of Lorraine to a 4-high Cross of Lorraine. Yet, several failed attempts suggested that I was perhaps over-reaching on this one. Finally, I found the 19-piece dissection in Figure 14. If you want a challenge, cover up the figure and see if you can discover it on your own. It's not so easy! If you construct this dissection out of a material such as wood, then you can do so without "rounding" the edges. However, you will need to be careful with the order in which you fold the pieces.

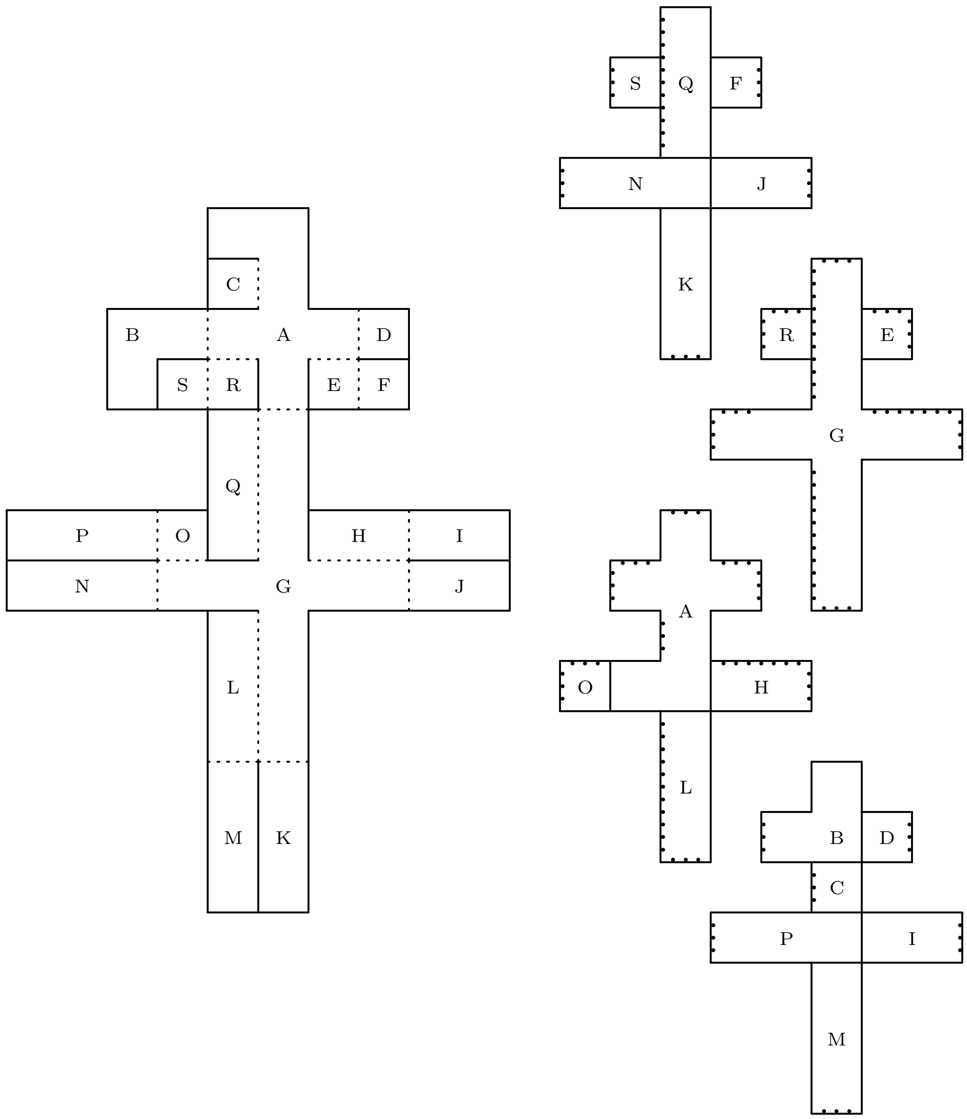

Figure 14: Stack-folding a 1-high Cross of Lorraine to a 4-high Cross of Lorraine

With good results for 4-high crosses to 1-high crosses, can we hope to do anything with 9-high crosses? Serendipitously, the answer is yes, as we see in the 27-piece dissection in Figure 15.

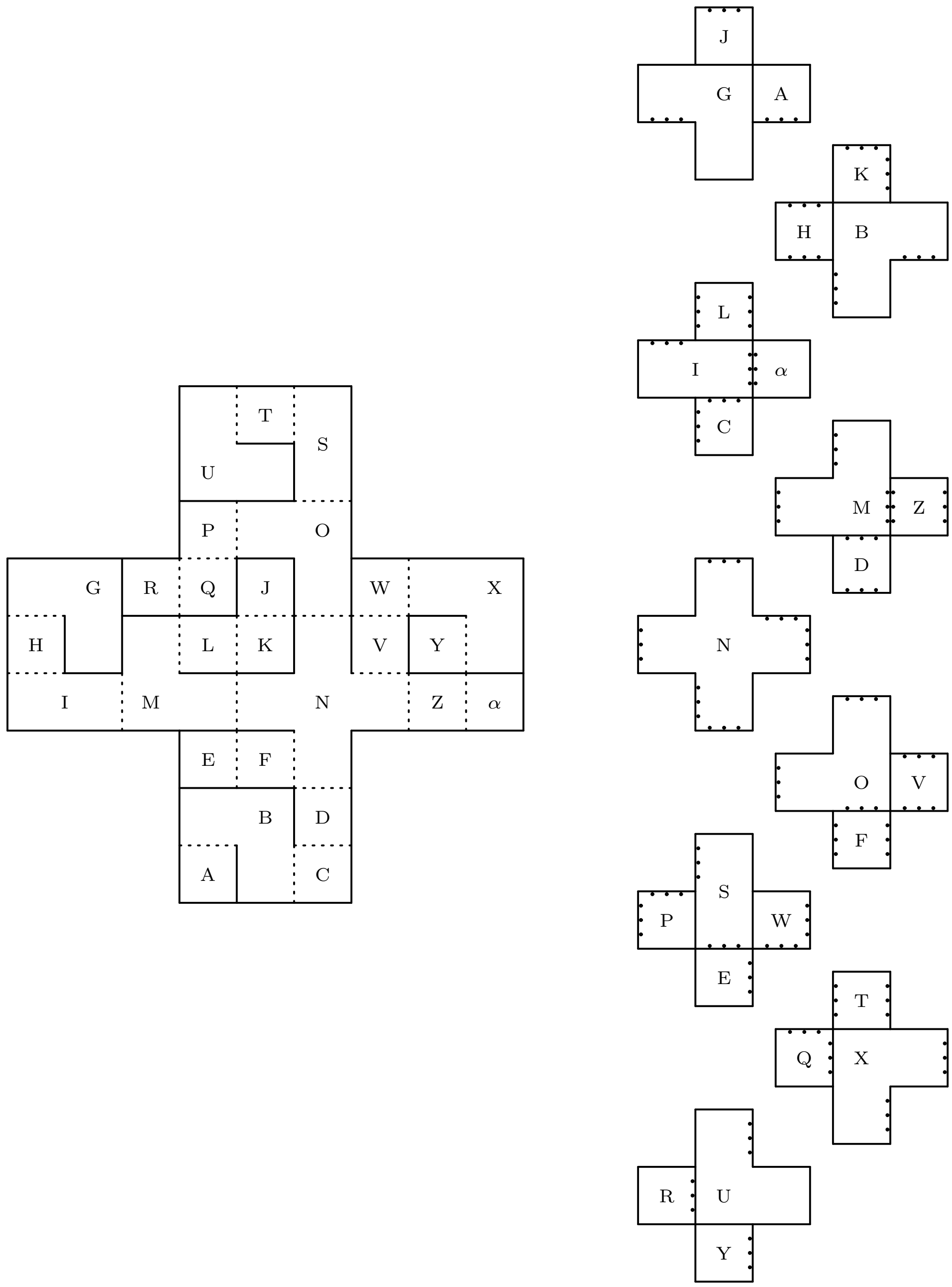

Figure 15: Stack-folding a 1-high Greek Cross to a 9-high Greek Cross

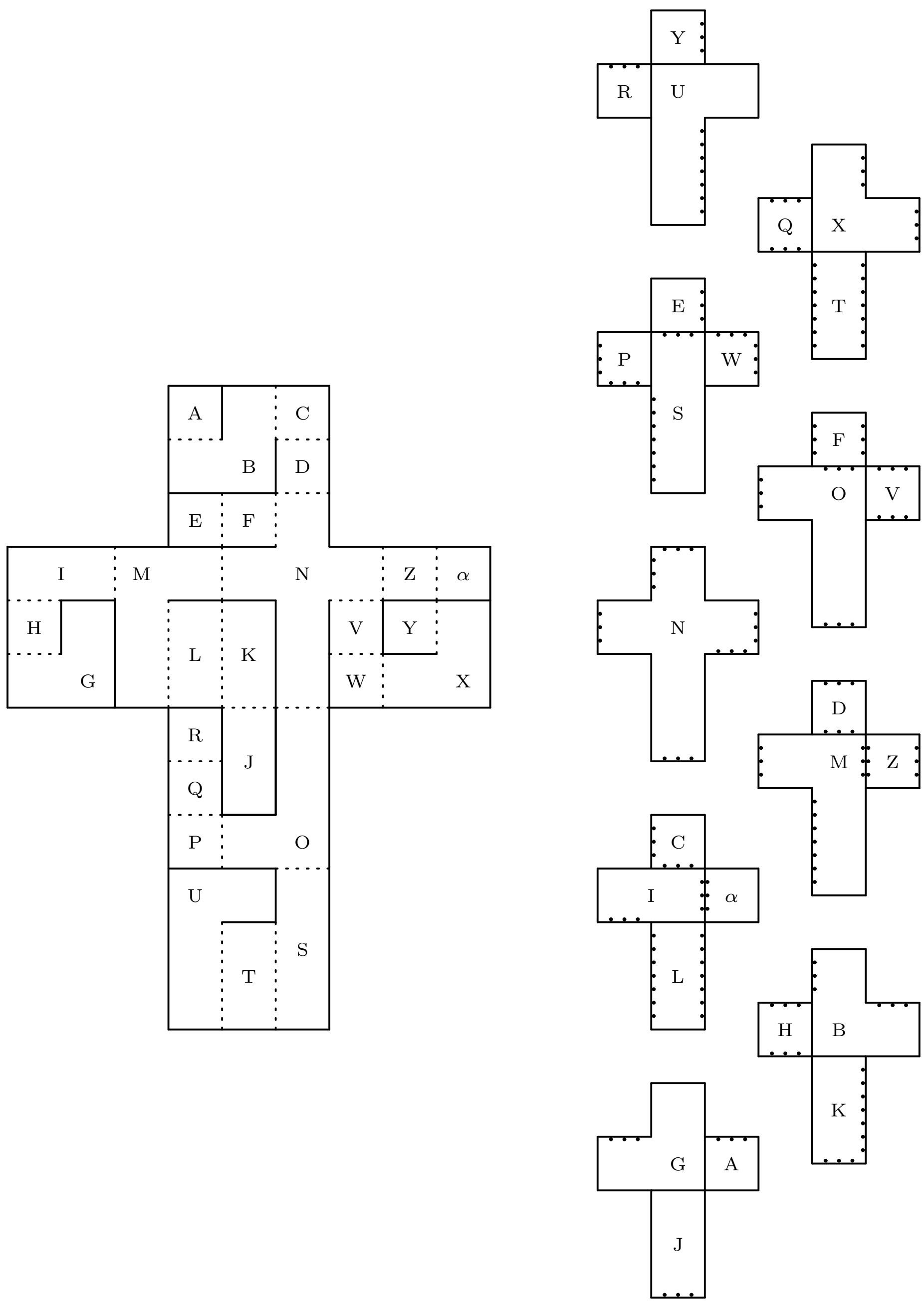

Figure 16: Stack-folding a 1-high Latin Cross to a 9-high Latin Cross

This dissection started as a 31-piece dissection obtained as follows. Place the middle level of the 9-high cross so that it extends into both the right and the upper arms of the 1-high cross. Use the right arm of the 9-high cross to fill in the right arm of the 1-high cross, and similarly for the lower arm. Fill in the left arm and the rest of the center of the 1-high cross with the rest of the top four levels of the 9-high cross, and similarly for the upper arm and bottom four levels. Finally, I swapped and combined pieces to find a 4-piece improvement. This dissection generalizes to the 27-piece dissection of a 1-high Latin Cross to a 9-high Latin Cross in Figure 16.

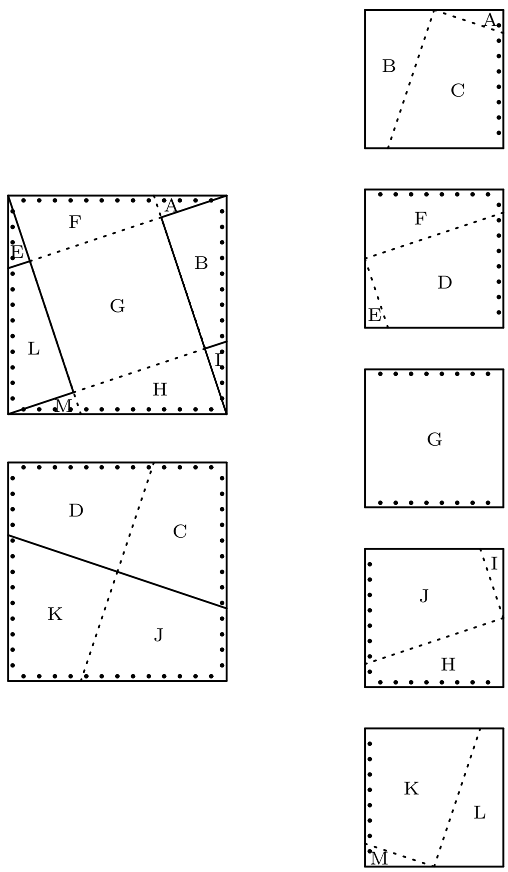

Figure 17: Stack-folding 2-high square to 5-high square

In [5], I gave a 26-piece piano-hinged dissection of two squares to five squares, based on a 12-piece unhinged dissection by David Paterson [10]. I have not been able to adapt Paterson's dissection to stack-fold a 2-high square into a 5-high square. However, a (non-optimal) 13-piece dissection adapts readily to give the 13-piece stack-folding dissection in Figure 17. A lovely feature of the dissection is that there are two cycles of pieces that are flat-cyclicly hinged, namely (A, C, D, F) and (H, J, K, M).

Ernest Freese [6] identified a 10-piece dissection of a $\{10/3\}$ to two pentagrams. Harry Lindgren [8] independently discovered this dissection and furthermore identified a special trigonometric relationship, which I listed in my first book, between a $\{(4n+2)/(2n-1)\}$ and a $\{(2n+1)/n\}$, for any positive integer n. For $n = 2$, it describes the dissection of a $\{10/3\}$ to two pentagrams. This relationship leads to a $(4n+2)$-piece dissection of a $\{(4n+2)/(2n-1)\}$ to two $\{(2n+1)/n\}$s. In my book on piano-hinged dissections [5] I showed that this leads to an $(8n+4)$-piece piano-hinged dissection. For stack-folding dissections we can produce a $(6n+3)$-piece dissection.

We see the case of the 15-piece dissection of a 2-high $\{10/3\}$ to 4-high pentagram in Figure 18. Every 72° around the $\{10/3\}$ we glue what were a pair of pieces together into one piece that is 2-high. We proceed clockwise through the pieces of the $\{10/3\}$, alternating between two on the top and two on the bottom as we hinge them together. When we form the 4-high pentagram, we proceed clockwise through the pieces of the pentagram, moving alternately from top to bottom and then bottom to top. The 2-high pieces occupy the second and third levels of the 4-high pentagram.

As in the piano-hinged dissection of a $\{10/3\}$ to two pentagrams, this new dissection has the inside-out property. The surfaces of the pieces that are exposed on the outside of the 2-high $\{10/3\}$, ten on the top and ten on the bottom, are hidden between the first and second levels and between the third and fourth levels in the 4-high pentagram. Conversely, the surfaces of the pieces that are exposed on the outside of the 4-high pentagram, five on the top and five on the bottom, are hidden between the first and second levels of the 2-high $\{10/3\}$.

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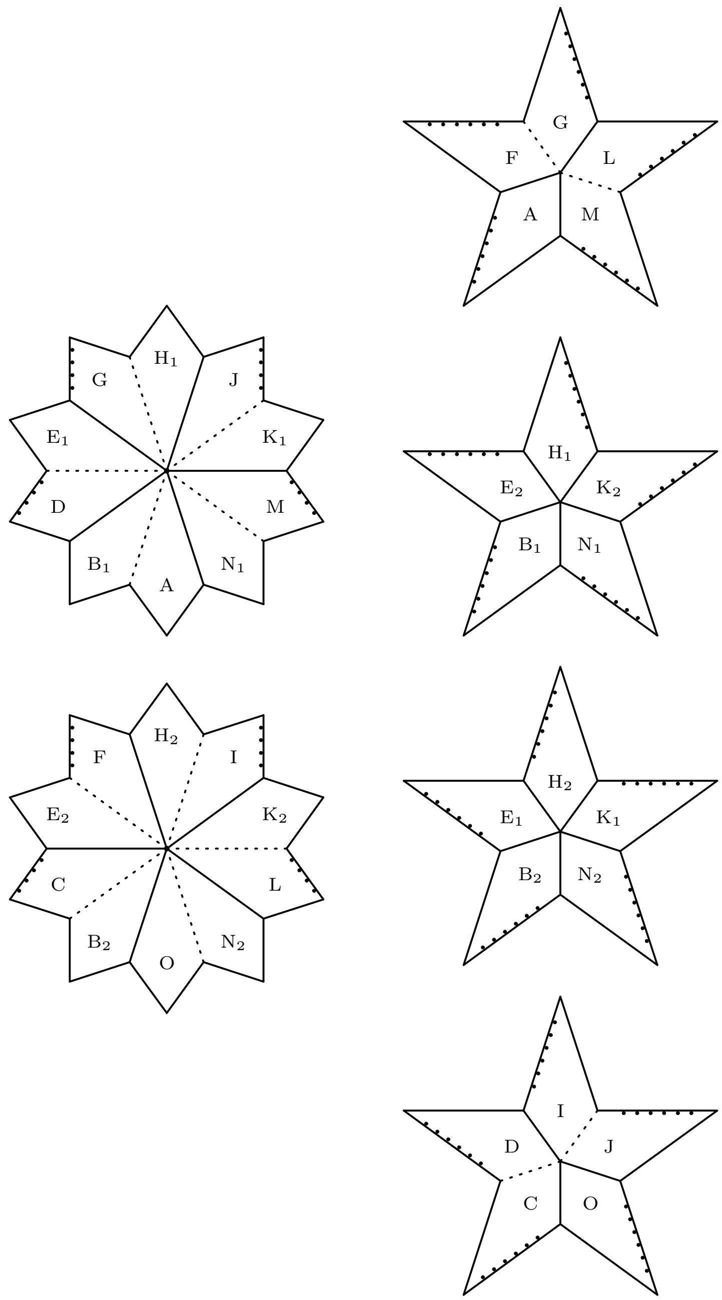

Figure 18: Stack-folding 2-high {10/3} to 4-high {5/2}