Reconfigurable Benches from Twist-Hinged Dissections of Polygonal Rings*

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1 Twisting Geometry into Garden Furniture

A geometric dissection is a cutting of a geometric figure into pieces that we can rearrange to form another figure [5, 10]. During the past century, the emphasis was generally on minimizing the number of pieces for any given dissection. As dissection methods became more sophisticated, attention then also focused on special properties. Most notable is the property that all pieces of a dissection can be connected by hinges, so that when the pieces are swung one way on the hinges, they form one figure, and when swung the other way on the hinges, they form the other figure. A hundred years ago, Henry Dudeney demonstrated such a hinged dissection of an equilateral triangle to a square [3]. Since then, enough hinged dissections have been identified to fill a whole book on the subject [6]. The power of hinged dissections can be mesmerizing, as indicated by adapting the triangle-to-square dissection to a hinged set of tables [4].

Other types of hinges have also drawn attention. A *twist hinge* has a point of rotation on the interior of the line segment along which two pieces touch edge-to-edge. It allows one piece to be flipped over relative

*This is an expanded version of a portion of a conference paper [8].

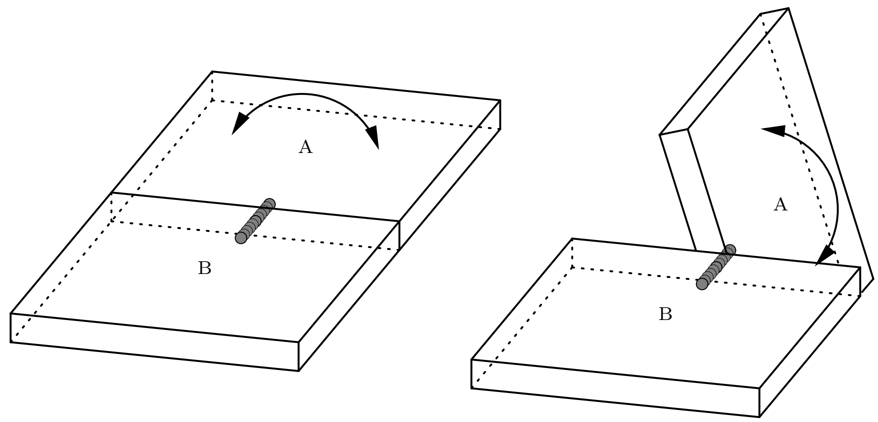

Figure 1: A twist hinge for pieces A and B

to the other, using 180° rotation through the third dimension. Pieces A and B (with exaggerated thickness) are twist-hinged together in Figure 1. The twist-hinged dissection of an ellipse to a heart (Figure 2) is a direct application. We mark any piece that is turned over an odd number of times with an "$*$" on one side and a "$\star$" on the other. A concerted search for twist-hinged dissections has occurred recently [6, 7, 9].

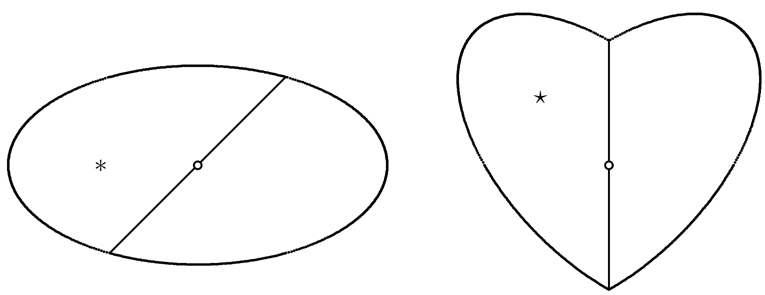

Figure 2: Twist-hinged dissection of an ellipse to a heart

In this article we explore twist-hinged dissections of ringlike figures that are based on regular polygons. We represent a p-sided regular polygon of side length x with the notation x-$\{p\}$. A *polygonal ring* is a regular polygon with a similar, but smaller, regular polygon cut out of it, such that the polygons share the same center and each vertex of the smaller polygon is on a line segment from a vertex of the larger polygon to its center. We represent a polygonal ring based on regular polygon $\{p\}$ of outer side length X and inner side length x with the notation (X, x)-$\{p\}$-ring. We see two examples in Figure 3: a heptagonal ring and an octagonal ring.

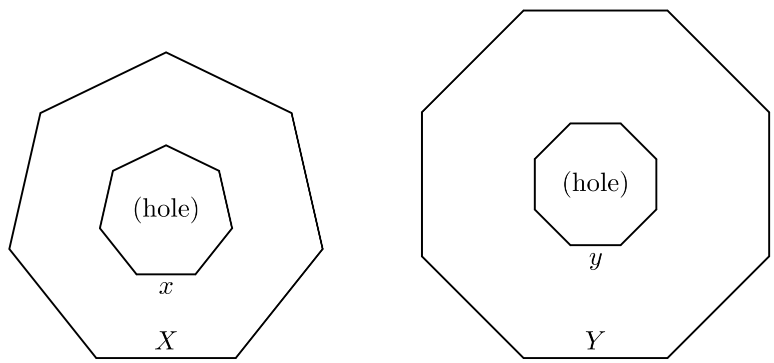

Figure 3: A (heptagonal) (X, x)-$\{7\}$-ring and an (octagonal) (Y, y)-$\{8\}$-ring

Designers of outdoor furniture have long produced benches that can ring a tree trunk or lamp post. Typically, the benches employ either a wrought-iron framework or a lattice-like construction of wood cross-braces. We show how to design, at least implicitly, such benches so that they can be reconfigured as the tree trunk expands, or as alternative seating is desired! These designs are so symmetrical and appealing in their use of twisting motion that they could well be show-stoppers at any garden party.

We assume that the ring benches have no backs, since it would be tricky finding what to do with the backs in the alternative configuration. Yet it seems easy to accommodate the bench's legs. Just place a leg near each corner of a piece not marked by asterisks or stars, whenever that corner is a vertex on the boundary in both figures. Note that a polygonal ring has vertices along the inner boundary as well as the outer boundary.

2 Some Organic Examples

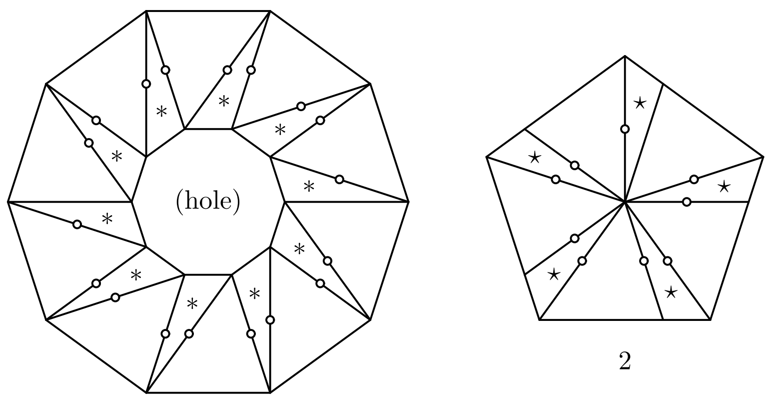

Figure 4: Twist-hinged dissection of a decagonal ring to two pentagons

We start with a lovely family of twist-hinged dissections. Our first example, in Figure 4, is a twist-hinged dissection of a $(1+\phi, 1)$-decagonal ring to two $(2+\phi)$-pentagons. (Recall that ϕ is the golden ratio, which is $(1+\sqrt{5})/2 \approx 1.618$). Our second example, in Figure 5, is a twist-hinged dissection of a $(\sqrt{3}, 1)$-dodecagonal ring to three $(1+\sqrt{3})$-squares.

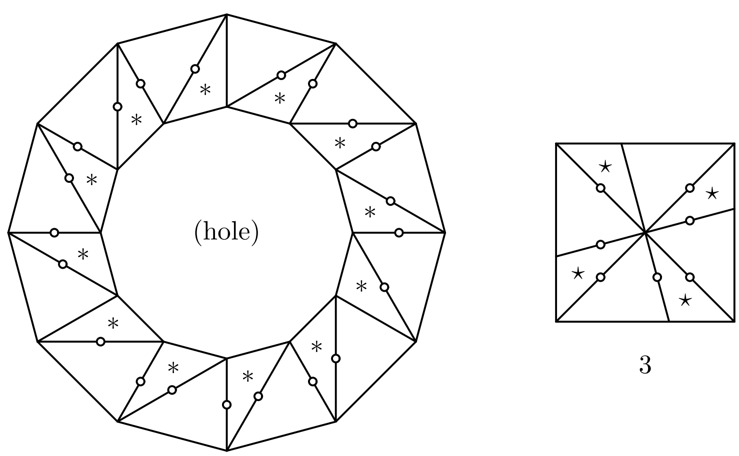

Figure 5: Twist-hinged dissection of a dodecagonal ring to three squares

We can also dissect multiple rings to many other rings. The example in Figure 6 is a twist-hinged dissection of two dodecagonal rings to three octagonal rings. The ratio of the inner side length of the dodecagonal ring to the inner side length of the octagonal ring can vary over a wide range, and the outer side lengths of the polygonal rings depend on these values. In Figure 6, we choose a ratio of 4 : 3 for the ratio of the inner side length of the dodecagonal ring to the inner side length of the octagonal ring.

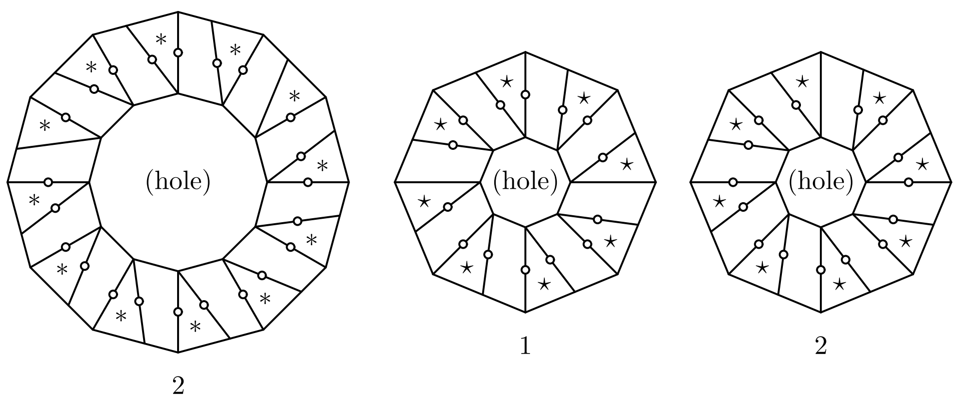

Figure 6: Twist-hinged two dodecagonal rings to three octagonal rings

To create such a dissection, first identify the number of sides in each of the two different polygonal rings: Choose p and q to be natural numbers with $p > q$. Next determine the multiplicity of each type of polygonal ring. Let $g = \gcd(p, q)$, the greatest common divisor of p and q. There will be q/g $\{p\}$-rings and p/g $\{q\}$-rings. Then fix the lengths of the inner sides of the polygonal rings: Choose x to be the inner side length of the $\{p\}$-rings, and y to be the inner side length of the $\{q\}$-rings, with $x > y \geq 0$.

Let $h = (x-y)/(\tan(\pi/q) - \tan(\pi/p))$. Set $z = (x-y) + 2h\tan(\pi/p)$. Determine the outer side lengths of the polygonal rings: $X = y + z$ is the outer side length of the $\{p\}$-rings, and $Y = x + z$ is the outer side length of the $\{q\}$-rings. Cut the pieces as in Figure 6, and then hinge in a greedy fashion: Starting with the first $\{p\}$-ring and the first $\{q\}$-ring, hinge as many pieces as possible from what remains of the current $\{q\}$-ring to fill up as completely as possible what remains of the current $\{p\}$-ring. The number of twist-hinged assemblages will be one less than the total number of polygonal rings of both types.

In Figure 6, $p = 12$ and $q = 8$, and $g = \gcd(12, 8) = 4$. Thus there are $8/4 = 2$ dodecagonal rings and $12/4 = 3$ octagonal rings. Once we choose x and y, we can compute values h, z, X, and Y. There are 4 twist-hinged assemblages: 2 from one octagonal ring, and one each from the other two octagonal rings.

When $y = 0$, the corresponding polygonal rings are simple polygons. This is the case for either of the first two examples. For the second example, of dodecagonal rings and squares, $\tan(\pi/4) = 1$ and $\tan(\pi/12) = 2-\sqrt{3}$. When $x = 1$, we have $X = \sqrt{3}$ and $Y = 1+\sqrt{3}$.

Something may seem wrong when you compare Figures 4 and 5 with Figure 6: The pieces that are not turned over in the former figures do not share sides with the inner boundary of the rings, whereas those pieces that are not turned over in the latter figure do share sides with the inner boundary of the rings. The reason is that in the latter figure, we switched which pieces get turned over, so as to not turn over the pieces of larger area. This choice makes sense if you want to sit on the ring benches that you build!

3 Transformed into a Growth Industry

There is another way to view the dissections of the previous section, which leads to a considerable expansion of the family. Note that the width of the ring, i.e., the distance from the outside to the inside, does not change. We could thus derive the cuts of the dissection in the following manner: Cut a $\{p\}$-ring up into p isosceles trapezoids and then connect with twist-hinges, so that we can twist from the ring to a parallelogram if p is even (as in Figure 7), or to a trapezoid if p is odd.

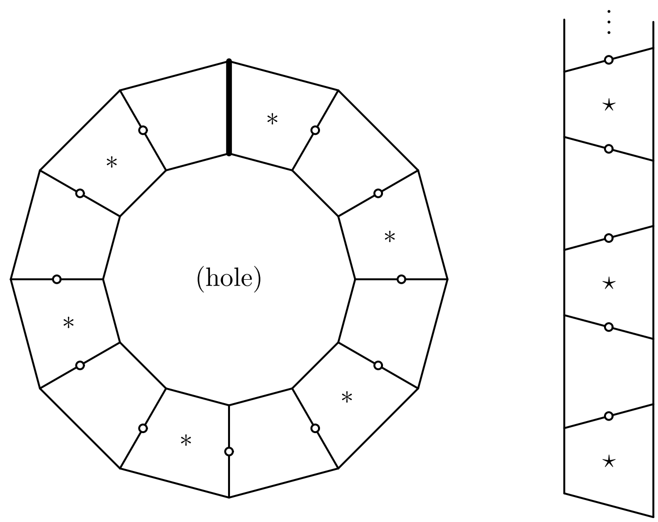

Figure 7: Twist-hinging a ring to a parallelogram or a trapezoid

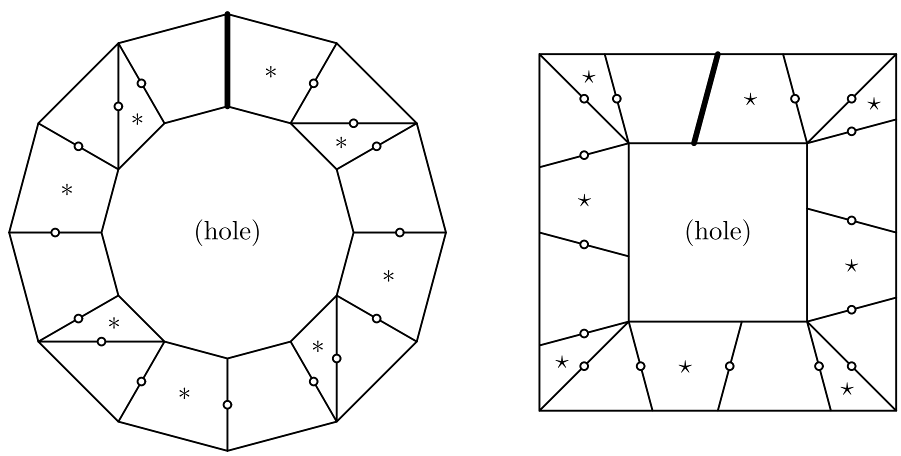

Figure 8: Twist-hinged dodecagonal ring to a "square ring"

Suppose that we wish to dissect the $\{p\}$-ring into a $\{q\}$-ring, where $q > 2$ divides evenly into p and the width of the ring is the same. Then cut up the $\{q\}$-ring into q isosceles trapezoids, so that we get a differently dissected parallelogram (or trapezoid) but of the same perimeter and the same height.

Note that we can convert a trapezoid to a parallelogram by one more cut with a twist hinge in the middle. Overlaying the two parallelograms then gives a combined set of cuts. For example, we take the dodecagonal ring from Figure 7 and dissect it into a square ring. This gives us the 16-piece twist-hinged dissection in Figure 8.

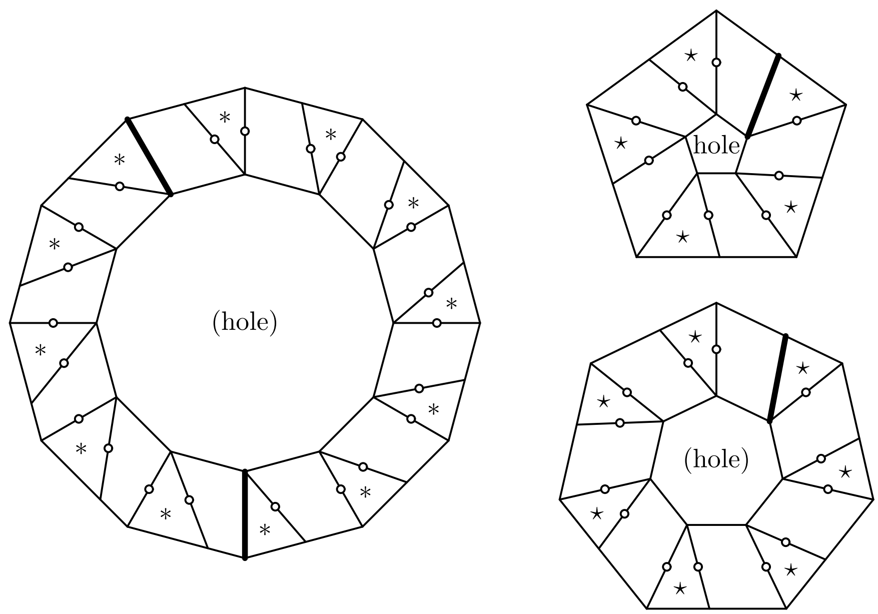

Figure 9: Twist-hinged dodecagonal ring to pentagonal and heptagonal rings

We can extend this method by dissecting a parallelogram or trapezoid obtained as in Figure 7 into a several parallelograms or trapezoids. We can then dissect a $\{p\}$-ring into a $\{q\}$-ring and an $\{r\}$-ring of appropriate dimensions. An example is shown in Figure 9, where $p = 12$, $q = 5$, and $r = 7$. Clearly, this approach extends to a set R of rings into a set R' of rings, where the total number of sides in the rings of R equals the total number of sides in the rings of R'.

4 Continued Evolution in the Garden

Our final family of dissections is of a $\{2p\}$-ring to a $\{p\}$-ring, but where the widths of the rings are not necessarily the same. Each dissection is based on a $(2p+1)$-piece twist-hinged dissection of a $\{2p\}$ to a $\{p\}$ from [6]. An example is the dissection of a regular octagon to a square in Figure 10.

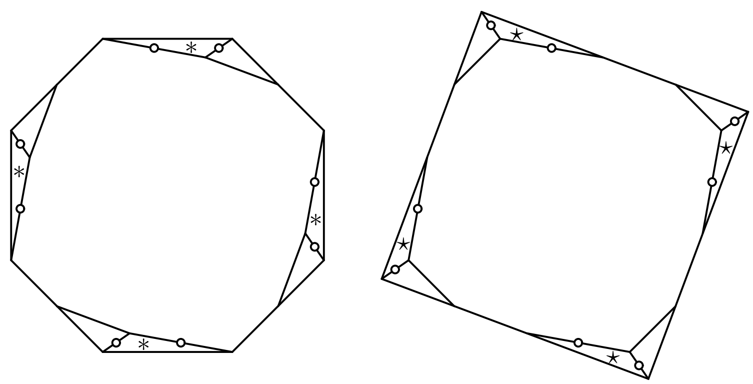

Figure 10: Twist-hinged octagon to a square

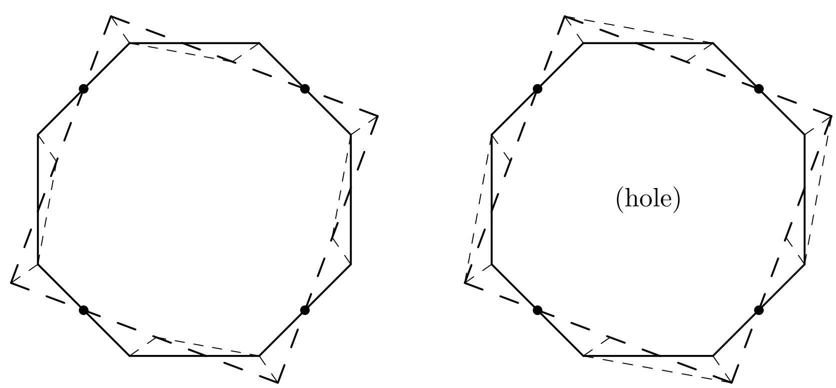

Figure 11: Derivations: (a) octagon to square, (b) octagonal hole to square hole

We see a derivation of this dissection in Figure 11(a). The square (in thick dashes) and the regular octagon (in solid line segments) are overlaid so that their centers coincide, and each side of the square intersects a side of the octagon at its midpoint. (Black dots indicate the midpoints.) Additional line segments (in thin dashes) identify cuts in either the square or the octagon.

To dissect a $\{2p\}$-ring to a $\{p\}$-ring, we will use a similar technique to dissect around the hole. Figure 11(b) illustrates the derivation of the hole dissection technique, where all cuts are outside of the octagonal or square holes, whose centers coincide. We thus get the "hole dissection" in Figure 12. The hole technique creates $2p$ additional pieces.

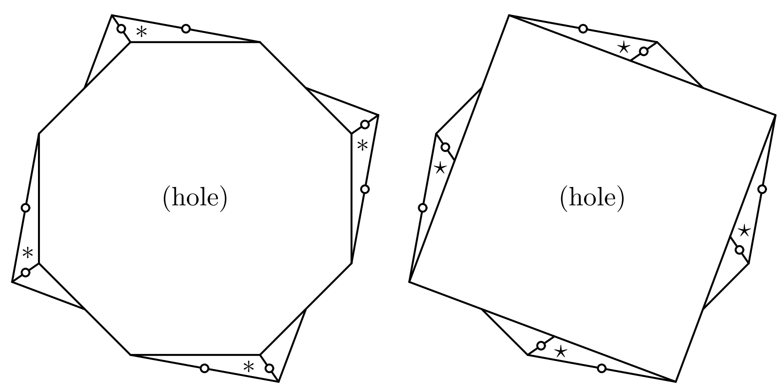

Figure 12: Twist-hinged octagonal hole to a square hole

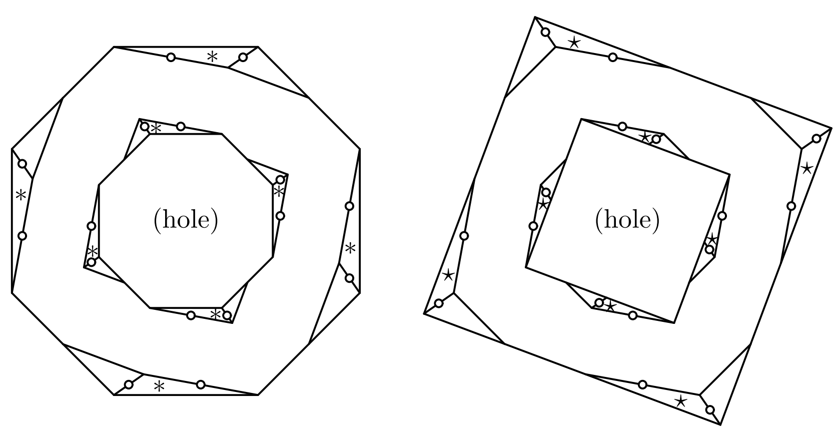

Figure 13: Twist-hinged octagonal ring to a square ring

We then employ the techniques from both Figures 10 and 12. Figure 13 displays the final 17-piece dissection of the octagonal ring to the square ring. The corresponding dissection for a $\{2p\}$-ring to a $\{p\}$-ring will have $4p + 1$ pieces.

5 Musing About Mathematics and the Landscape

We have described three families of twist-hinged dissections upon which to base the design of ring benches. Practical considerations favor the first two families: With convex pieces and fewer sharp angles, the benches are easier to construct. With compact pieces and hinges closer to the extremities of the pieces, there is less torque on the individual hinges. And with no single piece containing a hole, it's easier to encircle a lamp post or a tree.

For each dissection, it is instructive to think through the sequence of twists that takes the polygonal ring or rings to their alternative figures. Not just any sequence will work, because it is possible to have one piece collide with another when a bad sequence is chosen. Animations of many of the dissections in this paper, realized as reconfigurable benches, are given at:

http://www.cs.purdue.edu/homes/gnf/book2/bd_anims_amm.html

Videos showing physical models of Figure 5 produced by architecture students of Dirk Huylebrouck at Sint Lucas (Brussels, Belgium) are given at:

http://www.cs.purdue.edu/homes/gnf/book2/huylebstud.html

An interesting feature of the dissections in the first two families is that not only is the area dissected, but also the boundary. Such dissections have been called *complete dissections* [2]. The authors of that article, in their concluding remarks, suggest that no such dissections had previously been discovered, but examples had already appeared in [8].

Acknowledgements

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