A New Wrinkle on an Old Folding Problem

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A problem that has become a staple in calculus textbooks is the *box problem*: Determine an open box of largest volume that we can form from a rectangular sheet by cutting squares out of the corners, folding up the sides, and then gluing or soldering the joints. Isaac Todhunter [**19**] included it as an exercise in his calculus textbook a century and a half ago. The problem is a variation of a considerably older problem, which corresponds to the task of forming one quarter of an open box. That older problem was posed by the seventeenth century French mathematician Pierre de Fermat [**3**] and solved by the Dutch mathematician Frans van Schooten [**20**].

The box problem entered the mathematical puzzle literature when it appeared in 1903 in Henry Dudeney's puzzle column in *The Weekly Dispatch* [**5**]. In 1908 he rephrased it for his puzzle column in *Cassell's Magazine* [**6**], adding a picturesque illustration by Paul Hardy, before he included the revised puzzle in his book *Amusements in Mathematics* [**7**]. The original version is delightfully quaint:

> **No. 525.–HOW TO MAKE CISTERNS.**
>
> Here is a little puzzle that will elucidate a point of considerable importance to cistern makers, ironmongers, plumbers, cardboard-box makers, and the public generally.
>
> Our friend the cistern-maker has an interesting task before him. He has a large sheet of zinc, measuring eight feet by three feet, and he proposes to cut out square pieces from the four corners (all, of course, of the same size), then fold up the sides, join them with solder, and make a cistern.
>
> So far, the work appears to be pretty obvious and easy. But the point that puzzles him is this: What is the exact size for the square pieces that he must cut out if the cistern is to contain the greatest possible quantity of water?
>
> Call the feet inches, and take a piece of cardboard or paper eight inches long and three inches wide. By experimenting with this you will soon see that a great deal depends on the size of those squares. To get the greatest contents you have to avoid cutting those squares too small on the one hand and too large on the other. How are you going to get at the right dimensions?
>
> **I SHALL AWARD OUR WEEKLY HALF-GUINEA PRIZE.**
>
> for a correct answer. State the dimensions of the squares and try to find a rule that the intelligent working man may understand.

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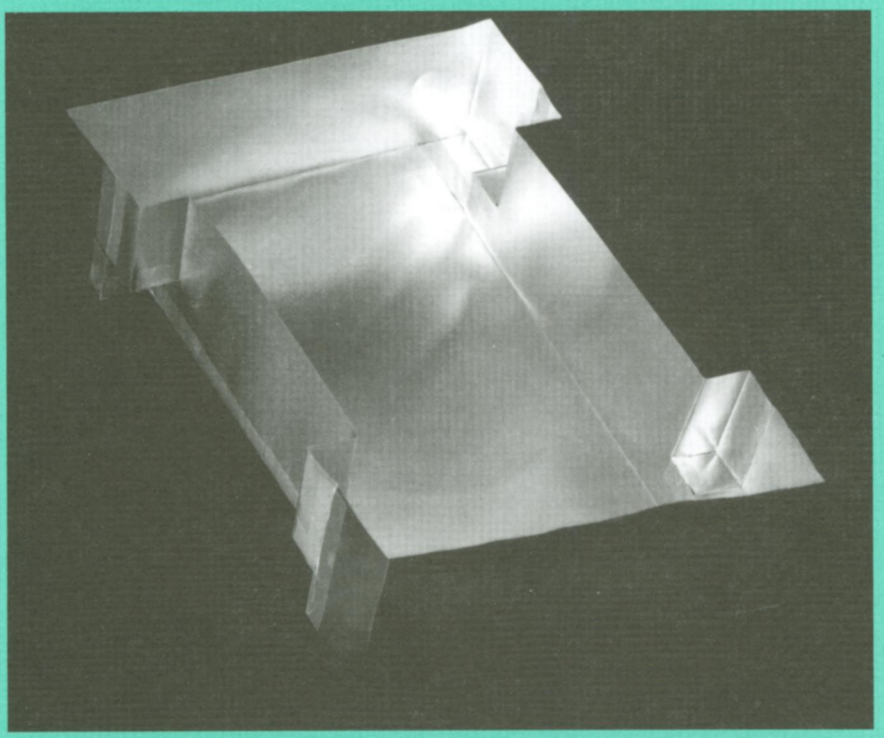

A bigger box

IN THIS ISSUE:

- Folding boxes, perhaps for fun and profit
- $f^{-1}(x) = (f(x))^{-1}$?
- Another characterization of conic sections
- The origin of a formula for π

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Figure 1. Illustration from *Cassell's Magazine*

The illustration from the later version of the puzzle appears in Figure 1. Readers are invited to guess why the two men look so unhappy. The solution to the original version of the puzzle appeared two weeks later in *The Weekly Dispatch* [**5**]:

> This was a little puzzle of a very practical and useful character. Given an oblong sheet of zinc, how should the workman cut out a square piece from each corner so that the four sides fold up and make a cistern that shall contain the largest possible quantity of water? The rule is simply this: (1) Deduct the product of the sides from the sum of their squares; (2) find the square root of the remainder; (3) deduct this square root from the sum of the sides; and (4) divide the remainder by 6. The result is the side of the little square pieces to be cut away.
>
> Let us apply this rule to a sheet of zinc of the given dimensions, eight feet by three feet. (1) The sum of the squares of these two numbers is $64 + 9 = 73$, from which deduct $8 \times 3 = 24$, and we get 49. (2) The square root of 49 is 7. (3) Deduct 7 from $8 + 3$ and we have 4. (4) Now, if we divide four feet by 6, we get eight inches as the side of the square pieces.
>
> This is the correct answer that we want. The intelligent working man is supposed in these days to know that a number multiplied by itself is a square, and that this number is called the "root" of such a square. Even if he does not know how to find the square root of any number, there are always table books available. I therefore think it best to give the exact method instead of one of the many approximations that have been suggested.

Try the rule in the cases of sheets measuring 8 feet by 5 feet, 16 by 6, 16 by 10, and 21 by 16, and you will find that the answers work out 1, $1\frac{1}{2}$, 2, and 3 respectively. Of course, it will not always come out exact (on account of that square root), but you can get it as near as you like with decimals.

The prize has been awarded to Mr. W. Robins, Wanstead Cottage, New Wanstead, Essex. Although the majority of competitors were considerably out in their calculations, 34 correct answers were received. Some of these came from persons who admittedly merely found by trial that it was "somewhere near eight inches;" and then ventured a guess that it was eight inches exactly. Others may have done the same, so there will be no honourable mention on this occasion.

It is simple to solve this problem using calculus: Let the dimensions of the rectangular sheet be a and b. Let x be the length of each square piece. First form the volume of the cistern in terms of a, b, and x, which is $(a - 2x)(b - 2x)x = 4x^3 - 2ax^2 - 2bx^2 + abx$. Take the first derivative of the volume with respect to x, giving $12x^2 - 4ax - 4bx + ab$. Set the result to zero, and apply the quadratic formula, choosing as x the smaller of the two roots, to ensure that both $a - 2x > 0$ and $b - 2x > 0$. This gives

$$x = \frac{a + b - \sqrt{a^2 + b^2 - ab}}{6}$$

The method that Dudeney describes in steps (1)–(4) corresponds exactly to the formula that we have derived. (In fact, in the solution for the revised version of the puzzle, Dudeney presented the method by giving this formula.) However, instead of discussing how one would discover and justify the method, Dudeney simply asserted that intelligent people ought to know what a square root is!

Figure 2 illustrates the method on a sheet of metal that is 3×4. In this case $x = (7 - \sqrt{13})/6 \approx .5657$. As directed, we cut out the four square corners in Figure 2. We

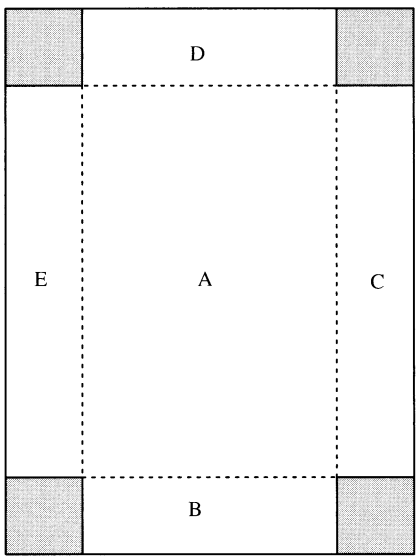

Figure 2. Traditional cuts and folds

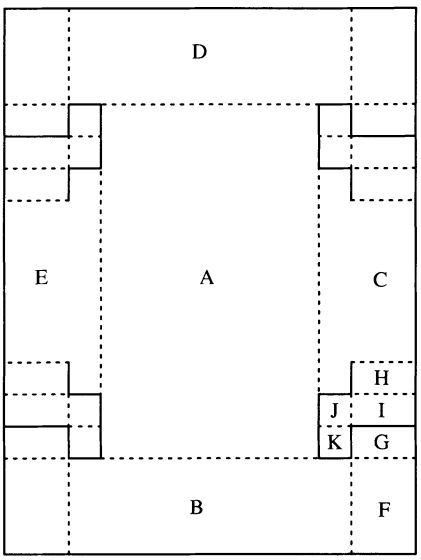

Figure 3. New millennium cuts, folds

then use rectangle A as the base of the cistern and fold rectangles B, C, D, and E up for the sides. The volume is $(35 + 13\sqrt{13})/27 \approx 3.0323$.

The box problem is a nice application of calculus and leads to a variety of interesting related problems. Dick Stanley [**18**] and Wally Dodge and Steve Viktora [**4**] observed that the optimal solution has an intriguing property, namely that for any shape of rectangle, the total area of the sides of the box will equal the area of the bottom. They also proved the same property for a corresponding approach applied to sheets of metal whose boundaries are polygons. James Duemmel [**8**], Al Cuoco [**2**], and Philip Hotchkiss [**12**] characterized pairs of integral dimensions for which x is a rational number. Richard St. André [**17**] and Kay Dundas [**9**] identified variations in which the box is self-bracing.

Several people have suggested that the problem is a bit silly from a practical point of view. John Friedlander and John Wilker [**11**], Kay Dundas [**9**], and Donna Marie Pirich [**15**] pointed out that the corners are wasted. Friedlander and Wilker used this as an opportunity to apply the same technique recursively to the resulting squares, producing an infinite succession of boxes whose total volume is to be maximized.

Although it might be nice to have a large collection of boxes and at the same time to avoid waste, it might be nicer to have just one container to hold water. Who says that a cistern must be in the shape of a rectangular solid? Thus I ask for a container of any shape, open on the top, formed from a rectangular sheet of metal, using tin snips and solder, that maximizes the volume. To retain the emphasis on folding that is present in the original version of the problem, let's require that all material to be used in the container remain connected after the cutting. To make matters simpler, let's consider here only containers whose surfaces are parallel to the faces of a cube and are of uniformly single thickness. The latter condition rules out the convoluted, solder-intensive approach of Racine Carré [**1**].

We quickly discover that we can do better than soldering together the infinite number of boxes produced by Friedlander and Wilker. We need not form the sides that would be soldered together. In fact, for each small square of side x we can fashion a corresponding "bulge" in the cistern of volume $8x^3/27$. In Figure 3, I cut a sheet of the same shape as before, using solid edges to denote cuts and dotted edges to show folds. Suitably folded, this produces our cistern for the new millennium in Figure 4, with the edges that we solder in bold and the folds shown with edges of normal thickness. We can easily see that panels J and K are small squares, and panels G, H, and I are twice as long as they are wide.

Since we no longer have small squares to discard, let's now use x to denote the height of the resulting cistern. The volume will be $(a - 2x)(b - 2x)x + 4(8x^3/27) = 140x^3/27 - 2ax^2 - 2bx^2 + abx$. Taking the first derivative of the volume with respect to x gives $140x^2/9 - 4ax - 4bx + ab$. The optimizing value will be

$$x = (9/70)(a + b - \sqrt{a^2 + b^2 - 17ab/9}).$$

For $a = 3$ and $b = 4$, we get $x = (9/10)(1 - \sqrt{1/21}) \approx .7036$, and a volume of $(9/25)(9 + \sqrt{1/21}) \approx 3.3185$.

To compare with Friedlander and Wilker's construction, let's also consider the case that $a = b = 1$ (a square sheet of metal). They optimize with $x = \sin 10°$ and a volume of $(2/3)(1 - 2\sin 10°)\sin 10° \approx .07556$. By contrast, the textbook method, which Dudeney described, would have $x = 1/6$ and a volume of $2/27 \approx .07407$. My approach has $x = 3/14$ and a volume of $4/49 \approx .08163$, which beats Friedlander and Wilker by 8% and Dudeney by 10%.

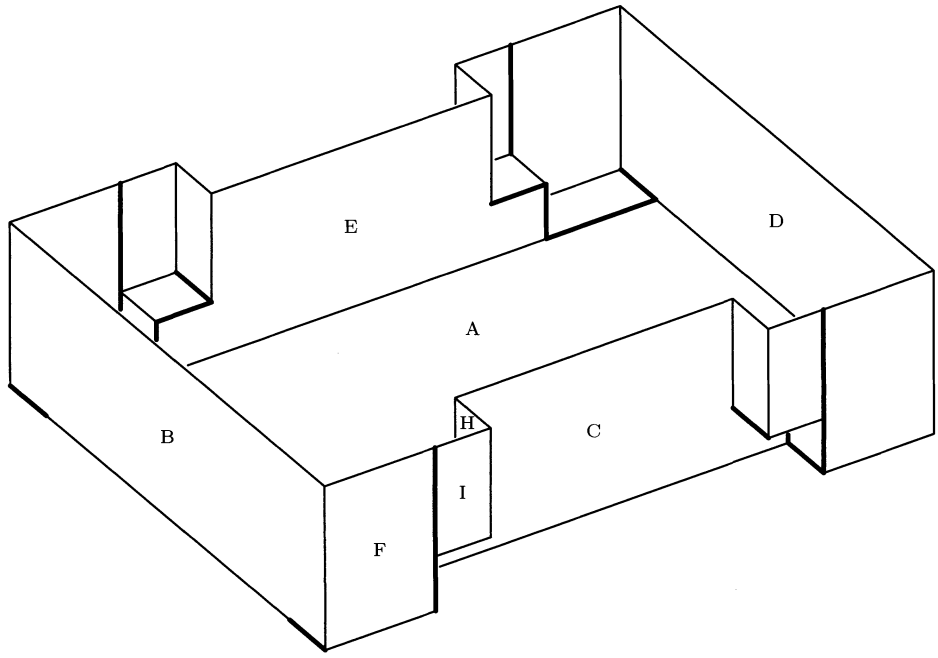

Figure 4. Perspective view of a new millennium cistern

The new approach does the best against the standard method when a and b are equal, and loses much of its advantage as these values grow apart. For the values that Dudeney chose, namely 3 and 8, the standard method has $x = 2/3$ and a volume of $200/27 \approx 7.4074$, whereas my method has $x = (3/70)(33 - \sqrt{249}) \approx .73801$ and a volume of approximately 7.8140. Even so, this a gain of over 5%. Is it too late to collect that half-guinea?

Aside from any potential economic gain, it's rewarding to discover that there is a more clever approach to cutting and folding. Of course, my solution respects the constraint that the faces be parallel to the faces of a cube. Kay Dundas [**9**], Nick Lord [**13**], Neville Reed [**16**], and Kenzi Odani [**14**] looked beyond the orthogonal world to produce even better solutions. And I have found a further wrinkle on Dundas's approach, which I describe in my book (in progress) on certain types of folding problems [**10**], from which the material in this article is adapted.

Regarding the apparent unhappiness of the two men in the illustration, perhaps they were so flustered by the square root that they cut the squares out of the wrong sheet of zinc. Clearly, the sheet in the illustration was not originally 8×3. Furthermore, they cut the squares with sidelength $x = a/4$, which cannot maximize the volume for any value of $b \geq a$. As luck would have it, $x = a/4$ is the smallest value for which this is regrettably true.

Acknowledgments. I would like to thank Margherita Barile for assistance in identifying historical references. I would also like to thank Apollonia Steele and Jim Burton of the University of Calgary library for providing me an electronic version of the illustration, and Dan Trinkle at Purdue for assistance in converting it into a suitable format. The cover photograph, and that of the author, were taken by Richard Myers-Walls. This article was supported in part by the National Science Foundation under grant CCR-9731758.

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How's That Proportion Go?

From *Brazzaville Beach* by William Boyd (Morrow, 1990, p. 112):

> Now, bear with me. I love the ring of this one, it sounds so good. Let's see what we can make of it. (I found it hard too: formulae have a narcoleptic effect on my brain, but I think I've got it right.) Take this simple formula: $x^2 + y^2 = z^2$. Make the letters numbers. Say: $3^2 + 4^2 = 5^2$. All further numbers proportional to these will fit the formula. For example: $9^2 + 12^2 = 15^2$. Or, taking the proportionality downward: $12^2 + 5^2 = 13^2$. Intriguing, no? Another example of the curious magic, the severe grace of numbers.

"Severe grace" is good, but the proportion is obscure. In his latest novel, the estimable *Any Human Heart*, Mr. Boyd includes no mathematics.