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VARIABLE RADIUS BLENDING USING DUPIN CYCLIDES

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Abstract: This paper we address variable radius blending surfaces and their approximation. First, we consider how to properly specify such blends. Next, we show that Dupin cyclides serve as variable radius blends in some simple situations. Finally, we concentrate on approximating other variable radius blending surfaces by sections of Dupin cyclides.

1. INTRODUCTION

Mechanical parts can, in general, be decomposed into primary and secondary surfaces. While the primary surfaces define the part profile, the secondary surfaces are required to smoothly connect them. These secondary surfaces are referred to as *blending surfaces* or simply *blends*. It is difficult to find a mechanical part or an assembly without any blending surfaces. Fillets and rounds are the most common forms of blends. Others include fairings and sculptured surfaces. Typically, smoothness of contact is a constraint that all blending surfaces must satisfy. See also [Woodwark 86] for a survey of blending surfaces in geometric modeling.

If there are no other constraints, these blending surface are easy to derive algebraically [Hoffmann and Hopcroft 85, 86]. However, various engineering applications impose geometric constraints on the blending surfaces. The cross-sectional profile of most blends are required to be circular. As such, common practice in engineering drafting is to specify blends by an appropriate radius (e.g. *0.5R*). When circular cross-sections are required, the blending surfaces are mathematically more complex. Such surfaces are typically of high algebraic de-

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gree and are difficult to derive analytically in closed form. As a result, surface interrogations become computation intensive.

The circular cross-sections of fixed radius permit a straightforward specification. Such blends have been approximated by Rossignac and Requicha [Rossignac and Requicha 86]. An attractive feature of their approximation method is that only standard CSG primitives, in particular, cylinders and tori are used. When applications require the circular cross-section to vary in radius (e.g. in the design of molds and dies), the problem becomes more complicated. Not only do the standard CSG primitives appear to be insufficient for an approximation scheme, but even a precise analytical definition of variable radius blends does not exist. Consequently, current practice defines these surfaces ambiguously.

In this paper, we present a mathematically unambiguous procedure for defining variable radius blends. We use *Voronoi surfaces* — a generalization of the Voronoi diagrams in computational geometry — for such a definition. Next, we examine the complexity of constructing such blending surfaces explicitly. It is seen that, even in very simple cases an explicit construction is quite complex. This motivates an approximation strategy.

When approximating variable radius blends, cyclide patches seem natural approximants since their principal lines of curvature are always circles. The rich variety of geometric properties of cyclides has been reviewed in [Chandru et al. 1988]. Other authors have considered cyclides for the purpose of free-form surface design, e.g. [McLean 84], [Martin et al. 86]. However, it appears that for free-form surface design the cyclide is not sufficiently flexible, at least not when patching along principal lines of curvature. In contrast, using cyclides for blends is promising since all constraints are in one dimension, rather than two.

Later, it is shown in our case study that cyclides can be used, as a whole, for specific variable radius blending applications. This fact has also been discovered by others, including [Pratt 88]. However, when a cyclide cannot be used in its entirety to represent the variable radius blending surface, it remains a good candidate for an approximant. A Voronoi surface is still required for specifying the spine curve associated with the blending surface which is now approximated with cyclide pieces.

In general, the approximation of variable radius blends is complex. Technically, approximating a fixed radius blend requires interpolating a set of space curve points and associated tangents by pieces of circles and straight lines. For variable radius blends, the elements to be used in the approximation are restricted to be the conic spines of the cyclides. By using a scheme of biarc approximation combined with Liming's method, moreover, we increase flexibility. Geometric properties of the cyclide preserve the circularity of cross sections in such approximations. However, smooth contact with the primary surfaces is not preserved exactly, just as in the fixed radius approximation scheme of [Rossignac and Requicha 86].

2. EXISTING METHODS

In recent years, a fair amount of attention has been devoted to blending surfaces by researchers in computer-aided geometric design [Hoffmann and Hopcroft 85, 86, 86a] [Owen and Rockwood 86] [Rossignac and Requicha 86] [Varady et al. 88]. However, variable radius blends have not been addressed nearly as much. Pegna's work in modeling variable radius blends by sweeps seems to be the only literature available [Pegna 87].

From fixed radius blending we will use the concepts of a *moving sphere* and a *spine* to define variable radius blends. For fixed radius blends, the spine is well defined as the intersection of the offsets of the primary surfaces, i.e. the two surfaces being blended. More generally, given a suitable spine, a variable radius blend can be defined in principle. It is the surface generated by a variable sphere as its centre moves on the spine while it maintains contact with the primary surfaces.

In Pegna's approach, a user defined reference curve is the initial spine and one of the primary surfaces is used as a reference surface. Since the reference curve is not the intersection of offsets it is adjusted to assume the position of the actual spine by an iterative algorithm. However, Pegna's method for obtaining the correct spine has the following deficiencies.

- The iterative algorithm does not use the exact distance of a point on the reference curve to the reference surface.
- The iterations to obtain a point on the exact spine (i.e. one which is equidistant from both primary surfaces) may exhibit sideway drifts in a manner that cannot be controlled analytically.

One way of overcoming the above deficiencies is to consider planes through the two perpendiculars at each reference curve point. In Fig. 1, let p be a point on the reference curve C_r and F be the reference surface. Let $d_F(p)$ be the perpendicular distance of p to F.

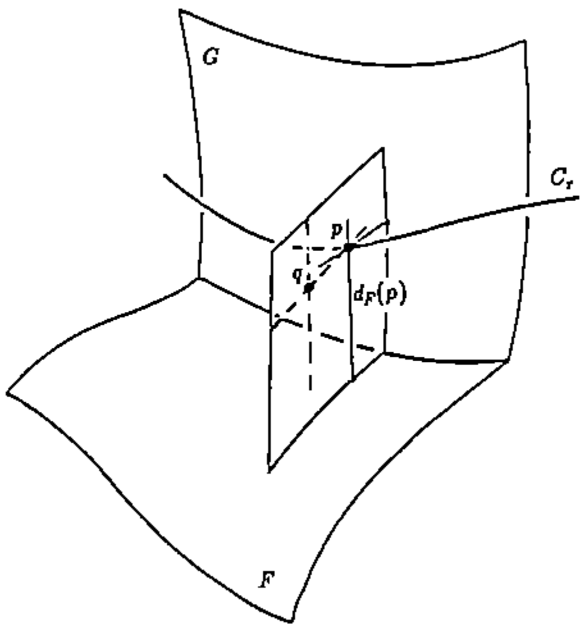


FIGURE 1

Now construct offsets of the primary surfaces F and G, by the distance $d_F(p)$. Let H denote the plane defined by the perpendiculars from p to the primary surfaces. The point q on the exact spine curve, corresponding to p on C_r, is given by the intersection of the two offset surfaces and H.

The above method to compute the exact spine curve from a given reference curve is analytically precise. However, if on the reference curve there exist points which have more than one minimum length perpendicular to the reference surface, this method would be insufficient.

Although computing the exact spine by the above method is clean and precise, we believe it is overly complicated to merit actual usage. Moreover, it does not provide any mathematical insights on the surfaces being considered that might be beneficially exploited. Thus, we do not explore this method any further, but use the concept of points equidistant from two surfaces to introduce Voronoi surfaces next.

3. VORONOI SURFACES

In computational geometry Voronoi diagrams are widely used in the study of proximity problems. In essence, a Voronoi diagram is the tessellation of a plane by polygons, each containing a point p of a given set S. The basis for construction of the polygon W associated with point p is that W is the locus of all points on the plane that are closer to p than any other point of S. Each point of S has a unique polygon containing it and the Voronoi diagram of the set S is the collection of the polygons. (See Fig. 2).

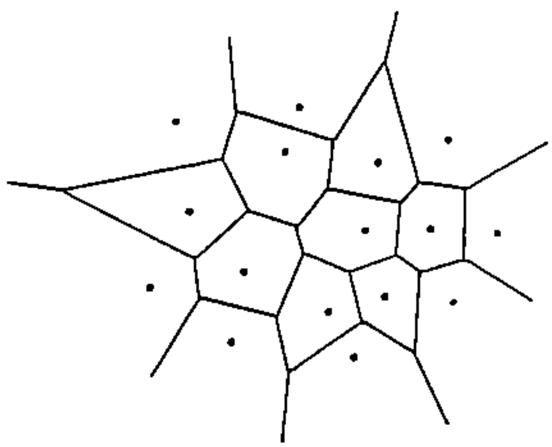


FIGURE 2

Voronoi digrams have interesting properties and for a detailed discussion the reader is referred to [Preparata and Shamos 85]. It is easy to see that each edge of a Voronoi polygon consists of points that are equidistant from points p_i and p_j, of the given set S, that it separates. Voronoi diagrams can be defined for any dimensions, in a manner analogous to the planar case.

In the previous section, we were interested in the spine curve for variable radius blends. We noted that all points of the spine curve had to be equidistant from the two primary surfaces being blended. Thus, a collection of such equidistant points, from two given surfaces, can be conceptualized as a Voronoi surface — a 3D analogue of the Voronoi diagram on the plane. The Voronoi surface we are considering, is more complicated than simply a 3D version of the planar Voronoi diagram since, the given set of point S, in 3D, is now replaced by the two primary surfaces. If there are three given surfaces, the Voronoi surfaces would be reduced to equidistant curves that are the intersections of several Voronoi surfaces. In the literature, Voronoi surfaces have also been referred to as *skeletal surfaces* and *medial axis surfaces* [Nackman 82].

3.1 Derivation for General Situation

Let us consider two primary surfaces, F and G. The Voronoi surface $V(F,G)$ is then defined to be the locus of all points equidistant from F and G. Thus mathematically,

$$V(F,G) = [\ p \in \mathbb{R}^3 \mid d_F(p) = d_G(p)\ ]$$

where $d_F(p)$ and $d_G(p)$ are the perpendicular distances of point p from F and G respectively.

To compute the Voronoi surface $V(F,G)$ a general method can be described as follows. Consider two spheres S_F and S_G, each of radius r, with centres on F and G respectively. Letting r be the offset distance, the offset surfaces of F and G, are given by the envelopes generated by S_F and S_G as they move on F and G. This is equivalent to displacing the centres (u_1, u_2, u_3) of S_F and (v_1, v_2, v_3) of S_G, as they move on F and G, by the radius of the spheres. Denoting the pair of linearly independent tangent directions to F at any point on it by t_1 and t_2 and similarly to G by t_1' and t_2' we get the following ten equations.

$$F: \quad f(u_1, u_2, u_3) = 0$$

$$G: \quad g(v_1, v_2, v_3) = 0$$

$$S_F: \ (x-u_1)^2 + (y-u_2)^2 + (z-u_3)^2 - r^2 = 0$$

$$S_G: \ (x-v_1)^2 + (y-v_2)^2 + (z-v_3)^2 - r^2 = 0$$

$$(\ \nabla S_F \ . \ t_1\ ) = 0$$

$$(\ \nabla S_F \ . \ t_2\ ) = 0$$

$$(\ \nabla S_G \ . \ t_1'\ ) = 0$$

$$(\ \nabla S_G \ . \ t_1'\ ) = 0$$

Thus, we have eight algebraic equations in 10 unknowns. By eliminating the variables $[u_1, u_2, u_3, v_1, v_2, v_3, r]$ we can obtain the equation of the Voronoi surface $V(x,y,z)=0$ associated with F and G.

We illustrate the above procedure by an example. Let F be a cylinder of unit radius, parallel to the z-axis and having the line $(x=0,\ y=2)$ as its axis. Let G be another cylinder of unit radius, parallel to the x-axis and having line $(z=0,\ y=-2)$ as its axis. Thus we have the equations

$$F:\quad x^2 + (y-2)^2 - 1 = 0$$

$$G:\quad z^2 + (y+2)^2 - 1 = 0$$

Let r be the radius of the spheres S_F and S_G. At points (u_1, u_2, u_3) and (v_1, v_2, v_3) on F and G respectively, we have the equations

$$\hat{F}:\quad u_1^2 + (u_2-2)^2 - 1 = 0$$

$$(x-u_1)^2 + (y-u_2)^2 + (z-u_3)^2 - r^2 = 0$$

$$\hat{G}:\quad v_3^2 + (v_2+2)^2 - 1 = 0$$

$$(x-v_1)^2 + (y-v_2)^2 + (z-v_3)^2 - r^2 = 0$$

The gradients $\nabla\hat{F}$ and $\nabla\hat{G}$ are given by $[2u_1,\ 2u_2-4,\ 0]$ and $[0,\ 2v_2+4,\ 2v_3]$. The two independent tangent vectors at (u_1, u_2, u_3) on F, are (0,0,1) and $(2-u_2, u_1, 0)$ and at (v_1, v_2, v_3) on G are, (1,0,0) and $(0, -v_3, v_2+2)$. Thus, the final system of eight equations is given by

$$u_1^2 + (u_2-2)^2 - 1 = 0$$

$$(x-u_1)^2 + (y-u_2)^2 + (z-u_3) - r^2 = 0$$

$$v_3^2 + (v_2+2)^2 - 1 = 0$$

$$(x-v_1)^2 + (y-v_2)^2 + (z-v_3) - r^2 = 0$$

$$(x-u_1)(2-u_2) + (y-u_2)u_1 = 0$$

$$x - v_1 = 0$$

$$-(y-v_2)v_3 + (z-v_3)(v_2+2) = 0$$

After elimination, the following equation is obtained for the Voronoi surface.

$$x^2 - 8y - z^2 = 0$$

This equation represents a hyperbolic paraboloid, as shown in Fig. 3. An analogous construction of Voronoi surfaces can be given for sculptured surfaces [Hoffmann 88].

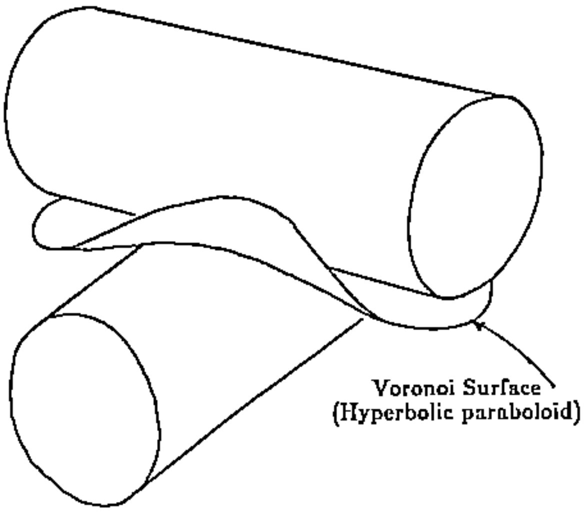


FIGURE 3

3.2 Derivation for Specialized Situations

Simpler methods for computing Voronoi surfaces are possible when the primary surfaces are simple quadrics with the property that their offset surfaces are also quadrics, i.e., combinations of cylinder, sphere and the cone. For example, we illustrate an alternative to the method described in § 3.1, for computing the Voronoi surface between two cylinders. Furthermore, we assume the cylinders have distinct radii, e.g., r_1 and r_2, respectively.

First we determine the offset of the two cylinders. With a suitable choice of coordinates, the equation of the first cylinder is

$$x^2 + y^2 - r_1^2 = 0.$$

A similar choice of coordinates, different from the first, yields

$$\xi^2 + \eta^2 - r_2^2 = 0$$

as the equation for the second cylinder. The two coordinate systems are related by a suitable linear transformation of the form

$$\xi = u_1 x + u_2 y + u_3 z + u_4$$

$$\eta = v_1 x + v_2 y + v_3 z + v_4$$

Letting

$$f(x, y, z) = (u_1 x + u_2 y + u_3 z + u_4)^2 + (v_1 x + v_2 y + v_3 z + v_4)^2$$

the equation of the second cylinder has the form

$$f(x,y,z) - r_2^2 = 0.$$

Therefore, the offset surfaces of the two cylinders, by a distance d, are

$$x^2 + y^2 - (r_1 + d)^2 = 0$$

and

$$f(x,y,z) - (r_2 + d)^2 = 0$$

respectively. Simplification yields

$$-d^2 - 2dr_1 - r_1^2 + x^2 + y^2 = 0 \qquad (i)$$

$$-d^2 - 2dr_2 - r_2^2 + f(x,y,z) = 0 \qquad (ii)$$

Subtracting (ii) from (i), we obtain

$$-2d(r_1 - r_2) - (r_1^2 - r_2^2) + x^2 + y^2 - f(x,y,z) = 0.$$

Since we assumed unequal radii, we may divide by $(r_1 - r_2)$ and obtain $d = g(x,y,z)$, where $g(x,y,z)$ is a quadric. Now, substituting $d = g(x,y,z)$ into the equation of the first cylinder yields a quartic Voronoi surface for the pair.

However, if $r_1 = r_2$, then d is eliminated while subtracting (ii) from (i), directly yielding a quadric Voronoi surface, viz., a hyperbolic paraboloid, as was obtained by the general method for computing Voronoi surfaces described in § 3.1.

4. VARIABLE RADIUS BLENDS

We can now give a precise definition of variable radius blends. Conceptually, the variable radius blending surface can be thought of as being generated by a moving sphere of varying radius. The envelope of spheres of constant or varying radius have been referred to, in the classical literature, as canal surfaces [Hilbert and Cohn-Vossen 52]. Thus, the variable radius blend is the portion of a canal surface, bounded by the curves of tangency with the primary surfaces that the variable sphere maintains contact with, during its motion.

Mathematically, a variable radius blend $\mathcal{C}(F,G)$ smoothly connecting surfaces F and G at their curve of intersection, can be stated as

$$\mathcal{C}(F,G) = (\mathfrak{S}, \mathfrak{r})$$

where, $\mathfrak{S}$ is a spine curve and $\mathfrak{r}$ is a radius variation function. Further, the spine curve is defined

$$\mathfrak{S} = V(F,G) \cap [RS]$$

as the intersection of the Voronoi surface of F and G and a given reference surface $[RS]$. The choice of a reference surface will depend upon the particular surfaces being blended. When either F or G is a plane, $[RS]$ can be chosen to be a plane and the spine $\mathfrak{S}$ will be given by

$$\mathfrak{S}: \quad V(x_1, y_1, z_1) \bigcap [ax_1 + by_1 + cz_1 + d]$$

The radius variation function $\mathfrak{F}$ defines the law by which the radius of the moving sphere varies. Since all points of the Voronoi surface have equal distance from both surfaces to be blended, every curve $\mathfrak{S}$ on the Voronoi surface $V(F, G)$ implicitly defines a radius variation function as follows. Let p be a point on the curve $\mathfrak{S}$. Let $d(p, F)$ be the distance of p to the surface F, and $d(p, G)$ the distance of p to the surface G. Since $\mathfrak{S}$ is on $V(F, G)$, we have

$$d(p, F) = d(p, G)$$

So, at p, we use the radius $d(p, F) = d(p, G)$.

In the case of blending a cylinder with an inclined plane, the distance $d(p, F)$ can be expressed as

$$y_1 Sin(\alpha) - z_1 Cos(\alpha)$$

where $(x_1,\ y_1,\ z_1)$ is any point on the spine $\mathfrak{S}$ and α is the angle subtended by $[RS]$ with the vertical plane (see Fig. 4). The variable moving sphere $\mathfrak{F}$ is given by

$$\mathfrak{F}: \quad (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - (y_1 Sin(\alpha) - z_1 Cos(\alpha))^2$$

Thus, the variable radius blending surface can be obtained by eliminating $(x_1,\ y_1,\ z_1)$ from $\mathfrak{F}$ and its partial derivative $\mathfrak{F}'$ by α. Note that the above definition for a variable radius blend depends only on the ability to specify $V(F, G)$ and "measure" the distance from a point p on $V(F, G)$ to either F or G. When formulated as in § 3, both can be done algorithmically whether F and G are given implicitly or parametrically.

5. COMPLEXITY OF EXPLICIT VARIABLE RADIUS BLENDS

The above method for defining and generating variable radius blends is computationally very intensive. The explicit derivations of the Voronoi surface, spine curve and subsequently the envelope of the variable sphere, require excessive symbolic computations and were not possible given the computational resources available to us. While the rigour of specification is an attractive feature, the required computations render such a process impractical, except as a preprocessing step. This motivates alternative strategies. For the problem at hand, a three part alternative can be outlined as follows.

- Use geometric insights, if possible, to predetermine the form of the Voronoi surface.
- Determine the spine curve numerically only, by tracing it in higher dimensions.
- Approximate the variable radius blend by an appropriate lower order surface.

We remark on each alternative below.

The general method outlined for computing the Voronoi surface is intractable for practical purposes (i.e. lack of swap space in our computer). However, geometric properties can sometimes be exploited to overcome the computational difficulties. For example, if one of the primary surfaces is a plane and the other a cylinder or a cone, the associated Voronoi surface is always a cone. Derivations similar to the ones shown in § 3.2, yield a paraboloid of rotation as the Voronoi surface of two spheres, and a sphere and plane. The Voronoi surface of a cylinder and a sphere has degree 4, unless the cylinder and sphere have equal radius, in which case a hyperboloid is obtained as the Voronoi surface.

The evaluation of space curves defined as the intersection of two surfaces (parametric or implicit) is of prime importance in CAGD. To overcome the excessive computations necessary in order to determine such curves explicitly numerical tracing in higher dimensions is an attractive alternative [Hoffmann 89]. Merits and procedures for tracing algebraic curves can also be found in [Bajaj et al. 88], [Hoffmann 88]. Using concepts from classical algebraic geometry such numerical procedures can be made robust to overcome difficulties associated with curve singularities. In our problem, tracing the spine curve numerically in 10-dimensional space serves a dual purpose. Not only does it simplify the computation, but also yields the common distance and footpoints of the perpendiculars from each curve point to both the primary surfaces.

Finally, the high degree of the exact blending surface, (in general degree 16 or higher) and difficult computations associated with such surfaces make approximations an attractive alternative. In the case of fixed radius blends, a method exists due to Rossignac and Requicha [Rossignac and Requicha 86]. Their method approximates the fixed radius blends by smoothly joined pieces of cylinders and tori. For variable radius blends, cyclides appear to be a natural choice for the approximant.

6. CASE STUDY: CYLINDER AND INCLINED PLANE

In this section we will consider an example to demonstrate the concepts developed so far. We have chosen the example with a view to keep computations tractable but one which does not compromise on the general problem characteristics. In particular, we consider the problem of variable radius blending of a circular cylinder and an inclined plane. This problem permits us the choice of a simplest possible reference surface — a plane.

6.1 Voronoi Surface Computation

Let C be the cylinder of radius R with its axis coincident with the z-axis. Let L be the inclined plane making an angle α with the z-axis (see Fig. 4).

To obtain the Voronoi surface $V(C,R)$ we can apply the proposed method. The cylinder offset surface by a distance d is given by

$$x^2 + y^2 - (R+d)^2 = 0$$

The offset of the plane by the same distance d is given by

$$Cos(\alpha)\ z - Sin(\alpha)\ y + d = 0$$

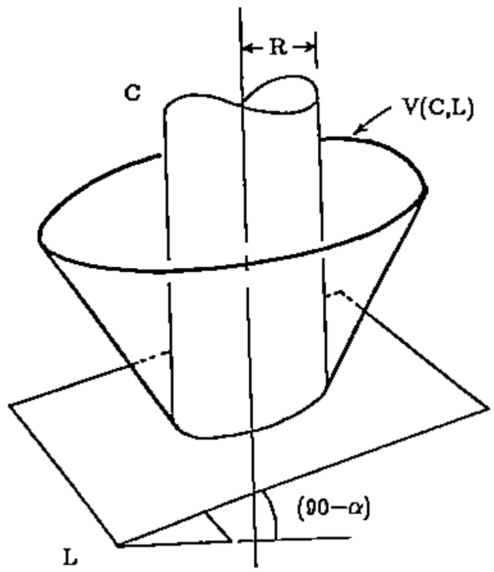


FIGURE 4

The Voronoi surface $V(C,L)$ being the locus of all points equidistant from C and L is now obtained by eliminating the distance parameter d between the above two equations. Doing so, we obtain

$$x^2 + y^2 - [R + (y\,Sin(\alpha) - zCos(\alpha))]^2$$

which is a cone having for its base an ellipse on the plane L.

6.2 Cyclides and Special Reference Planes

We consider the simplest possible reference surface, a plane. So, the spine curve which is defined as the intersection of the reference surface and the Voronoi surface, is now a conic and can be easily computed.

Adhering to the morphology of cyclides presented in [Chandru et al. 88], we know that a *central cyclide,* with an ellipse and a hyperbola as spines, has the general equation

$$(x^2 + y^2 + z^2 - r^2)^2 - 2(x^2 + r^2)(a^2 + f^2) - 2(y^2 - z^2)(a^2 - f^2)$$
$$+ 8afrx + (a^2 - f^2)^2 = 0.$$

Associated with the central cyclide, in the above equation, are three parameters, viz., a, f and r. Here, a and f are the semi-major axis and focal lengths, of the ellipse respectively, and r is a constant. When $f < r < a$, the form of the central cyclide resembles a squashed torus and is referred to as a ring central cyclide. Elementary geometric properties of the ring central cyclide readily imply the following.

Observation: The variable radius blend is a ring central cyclide iff $[RS]$ is orthogonal to the cylinder axis.

This observation was also made by Pratt [Pratt 88]. When $[RS]$ is orthogonal to the cylinder axis, the cross-sectional profile of the blended joint, on the XZ-plane, is as shown in Fig. 5. Thus, the radius variation function can be specified as a maximal radii (e.g. r_1 in Fig. 5). The cyclide parameters a, r and f are then determined as follows.

$$a = R + r_1$$

$$r = R\ Sin(\alpha) + r_1$$

$$f = (R + r_1)\ Sin(\alpha)$$

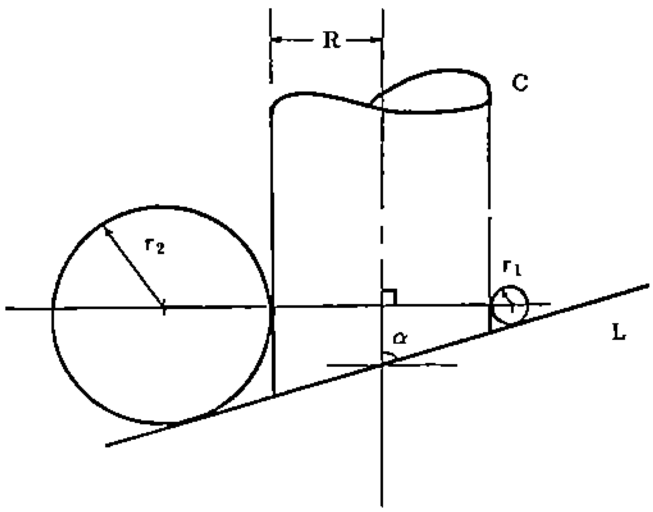


FIGURE 5

The cyclide surface and hence the required blend, can be constructed by algorithms discussed in [Chandru et al. 88]. In our case study, i.e., for the special position of $[RS]$, the circular cylinder is tangent to the cyclide at its inner extreme circle on the plane of the elliptic spine. Thus, the blending surface is actually a quarter of the cyclide surface bounded by two special latitudinal lines of curvature, which are, the inner extreme circle on the plane of the elliptic spine, and the circle at which its lower tangent plane touches the cyclide.

6.3 General Reference Planes

When the reference plane $[RS]$ is in a general position (i.e. not orthogonal to the cylinder axis), a conic is still obtained for the spine curve of the blending surface. However, a ring central cyclide can no longer be used in its entirety to blend the cylinder and inclined plane at their intersection. In general, we obtain the following set of equations describing the blending surface.

1. Spine: [Voronoi Surface] $\bigcap$ $[RS]$.

$$[V(u,v,w)] \bigcap [mu + nv + pw + q]$$

2. Moving sphere S centered at (u, v, w).

$$(x-u)^2 + (y-v)^2 + (z-w)^2 - (v\ Sin(\alpha) - w\ Cos(\alpha))^2$$

3. Directional derivative at (u, v, w).

$$(S_u, S_v, S_w) \cdot (m, n, 1) \times (V_u, V_v, V_w)$$

Here, $[(m, n, 1) \times (V_u, V_v, V_w)]$ is the spine tangent obtained as the cross product of the intersecting surface normals. The equation of the variable radius blending surface can be obtained by eliminating the variables u, v and w from the above set of equations. However, even when $m = 0$ and specific values are given for α, n, p and q, extensive computation is required for determining the variable radius blending surface. For $m = 0$, the exact blending surface is of degree 16.

The computational difficulties motivate approximating the blending surface. For fixed radius blends, a method exists due to Rossignac and Requicha [Rossignac and Requicha 86]. In their method, the blending surface is approximated by piecewise cylinders and tori, implying that the spine curve is approximated by circles and straight lines. Using cyclide pieces instead of cylinders and tori requires that the spine curve of the variable radius blend now be approximated by the elliptic spines of the cyclides.

7. APPROXIMATION WITH CYCLIDES

Insights into the global geometry of cyclides are helpful in outlining a procedure for approximating blends with cyclides [Chandru et al. 88]. Central to this issue is a methodology for space curve approximation by cyclide spines. The following steps outline the procedure to join cyclides.

1. Define the spine to be approximated.
2. Approximate it with piecewise cyclide spines.
3. Erect over each element of the approximated spine a cyclide piece.

Step 1 is the definition of the spine for a variable radius blend and has been dealt with in Section 4. Step 3 requires algorithms for the construction of a cyclide by its lines of curvature and has been discussed in detail in [Chandru et al. 88]. Thus, it remains to outline a procedure for Step 2. However, we note that the Steps 2 and 3 are not independent of each other in that, for conics to be cyclide spines, additional constraints are imposed. We remark on the constraints further.

Firstly, the tangents to the curve points are skewed lines in space. Thus, a single conic element cannot interpolate a pair of curve points *and* the associated tangent directions. To overcome this difficulty, we use *biarc* elements for the approximation. In essence, conic biarcs are pairs of smoothly joined conics that interpolate the given curve points and their associated tangent directions, by introducing a set of points we refer to as the *join points*. The join point Q_i is associated with the biarc $B_i(U, V)$. As shown in Fig. 6, biarc $B_i(U, V)$ interpolates space curve points (P_i, P_{i+1}) and tangents (T_i, T_{i+1}). The join point Q_i lies on a line R_i that can be thought of as a *connecting rod* resting on the tangents T_i and T_{i+1}.

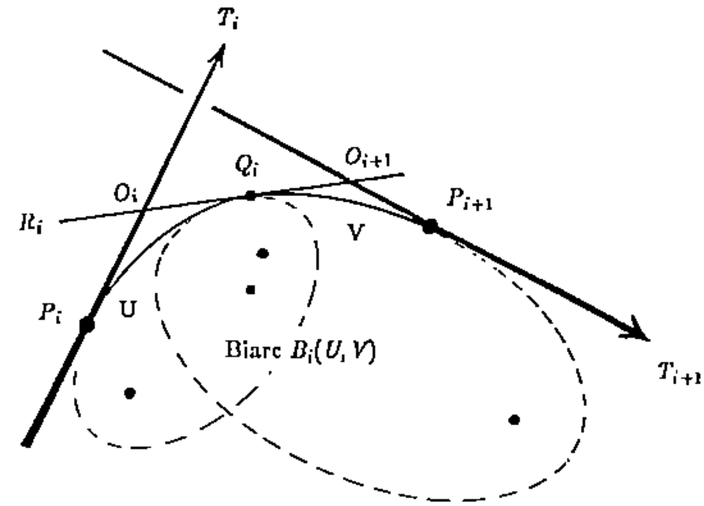


FIGURE 6

Each biarc in itself defines two control triangles. In Fig. 6, the control triangles are $\Delta P_i O_i Q_i$ and $\Delta Q_i O_{i+1} P_{i+1}$ respectively. Within every such triangle, Liming's method can be conveniently used to obtain specific conics — ellipses in our case [Faux and Pratt 79]. As shown in Fig. 7, the pencil of conics obtained by Liming's method pass through the vertices A and B, of ΔABC, and are tangent to the sides AC and BC. The following equation describes the pencil of conics

$$(\lambda)\, l_1 l_2 + (1-\lambda)\, l_3^2 = 0$$

The choice of a shoulder point on the median line CD, predetermines the specific type of conic — ellipse, parabola, or hyperbola — obtained within the control triangle ΔABC [Dutta 89].

The collection of surface normals at every longitudinal line of curvature on the cyclide forms a right circular cone [Chandru et al. 88]. The elliptic spine of a cyclide is the locus of vertices of all such cones. Thus, to obtain a C^1 continuity between the surfaces of adjacent cyclide pieces, it is necessary for the cones (of surface normals) of adjoining cyclides to coincide. Illustrated in Fig. 8, this constraint implies the following:

- Angles at the vertices of the two cones must be equal.
- Perpendicular distances of the bases of the two cones, from their common vertex, must agree in magnitude and direction.

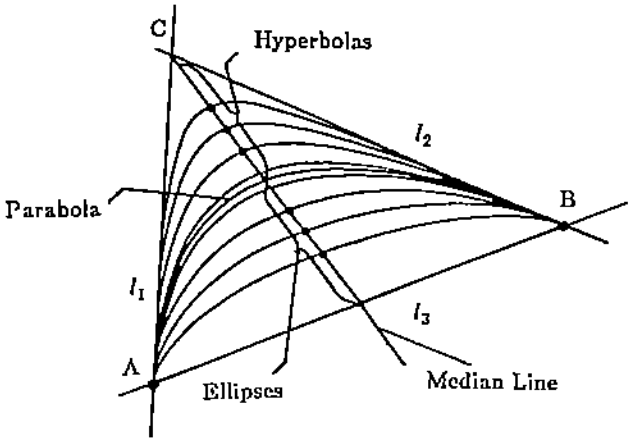
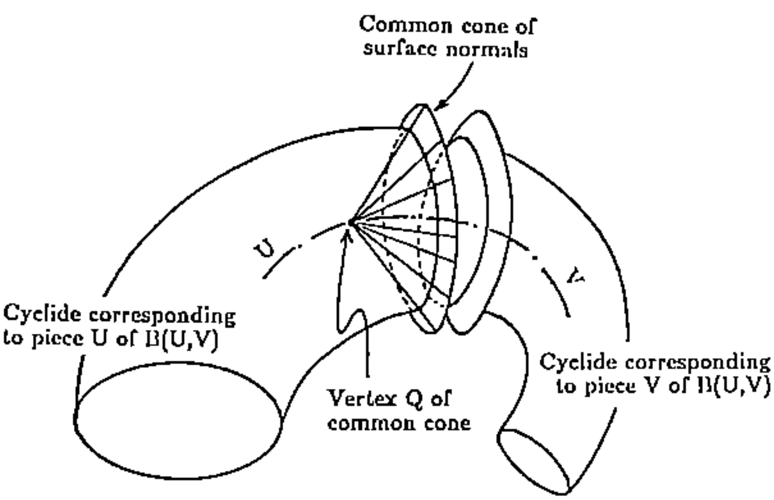


FIGURE 7

The approximation procedure is detailed in [Dutta 89]. A simplifying fact is that, since right circular cones are being matched, appropriate rotations about their respective axes has the effect of *untwisting* the space curve approximation procedure into one plane. In particular, we consider the problem on the plane containing the elliptic spines of adjacent cyclide pieces. Thus, the aforementioned constraints associated with the common cone, in 3D space, can now be related to a pair of intersecting (generating) lines that represent the cone on the plane of axial cross-section.



Joining two cyclides

FIGURE 8

Each right cone that has its vertex on the elliptic spine of a cyclide contains the foci of the ellipse. Thus, on the plane of the ellipse, an axial cross-section of every such cone is given by the pair of intersecting lines joining the foci to that point on the ellipse, which is the vertex of the cone being considered. In view of the above fact, the vertex angle constraint for common cones between adjacent pieces, translates to common focal lines through the cone vertex, on the plane of the ellipses. Focal lines L_1 and L_2 are common to the adjacent ellipses U and V, in Fig. 9. It is shown in [Dutta 89] that Liming's method can still be used to obtain ellipses within the control triangles, that satisfy this constraint.

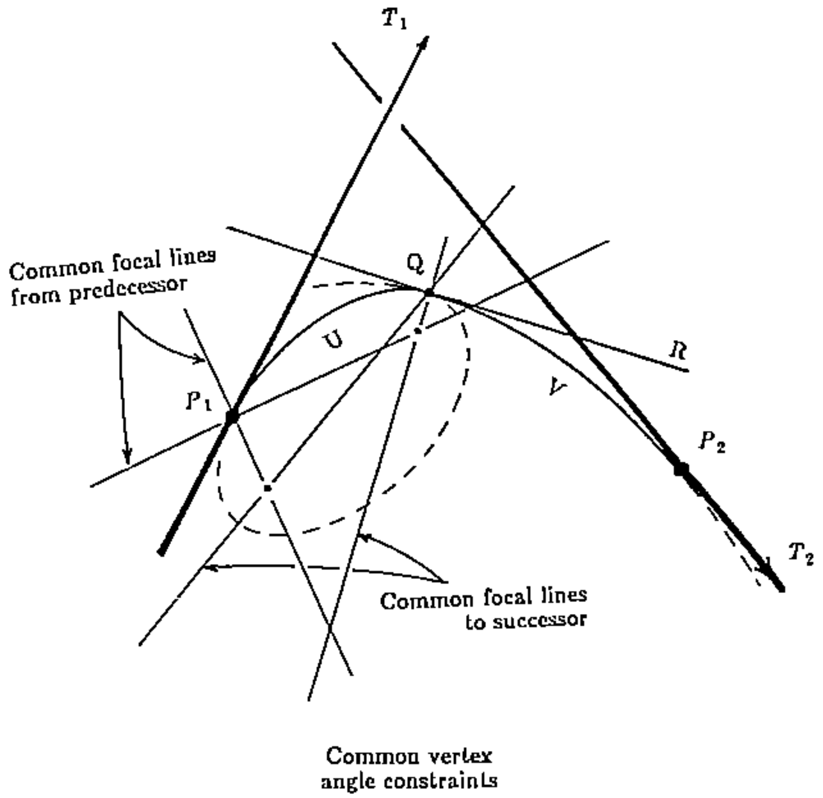
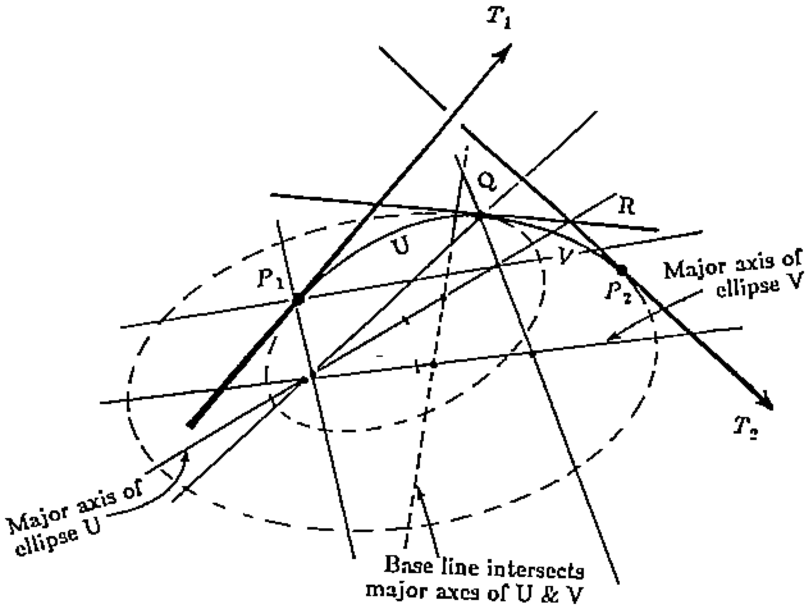


FIGURE 9

Every longitudinal circle of curvature on the cyclide, intersects the plane of its elliptic spine in two diametral end-points. The line joining these two points is the projection of the circle of curvature of the cyclide, on the plane of the ellipse. Thus, on such a plane, the constraint of a common base implies that the line through the diametral end-points is common to both ellipses (see Fig. 10). This line always intersects the major axes of the adjacent ellipses. The ratio in which the point of intersection divides each major axis determines the sub-form of the associated cyclide i.e., horned, ring, or spindle [Chandru et al. 88].



Common base constraints

FIGURE 10

8. SUMMARY

We have provided a mathematically precise definition of variable radius blends. Since deriving a closed form expression for the blending surface is prohibitive, we explored an approximation procedure using cyclides. Inherent geometric properties of the cyclide preserve circular cross sections and mandate a conic approximation of the spine curve. This approximation is simplified by "untwisting" the space curve. A limitation of our approach, however, is that only one degree of freedom is available for controlling the radius variation. We believe using generalized cyclides would yield greater degrees of freedom and merits further investigation.

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