

# Art Gallery Theorems and Algorithms

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# Art Gallery

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- **Problem:** determine the minimum number of guards sufficient to cover the interior of an  $n$ -wall art gallery
  - Victor Klee, 1973
  - Vasek Chvatal, 1975

Main reference for this material:

Art Gallery Theorems and Algorithms, Joseph O'Rourke, Oxford University Press, 1987

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# Contents

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- Interior Visibility
    - Art Gallery Problem
      - Overview
      - Fisk's Proof
      - Reflex Vertices
      - Convex Partitioning
      - Orthogonal Polygons
    - Mobile Guards
    - Miscellaneous Shapes
      - Star, Spiral, Monotone
  - Exterior Visibility
    - Fortress Problem
    - Prison Yard Problem
  - Minimal Guards
-

# Contents

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    - Art Gallery Problem
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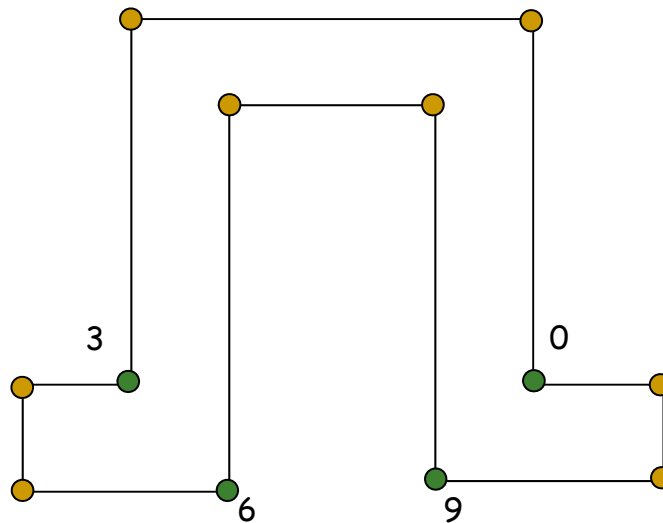
# Definitions

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- $P$  is a simple polygon (i.e., does not cross over itself)
  - Point  $x \in P$  "covers" a point  $y \in P$  if  $xy \subseteq P$
  - Let  $G(P)$  be the minimum number  $k$  of points of  $P$ , such that for any  $y \in P$ , some  $x = x_1 \dots x_k$  covers  $y$
  - Let  $g(n)$  be the  $\max(G(P))$  over all polygons of  $n$  vertices
    - Thus,  $g(n)$  guards are occasionally necessary and always sufficient
-

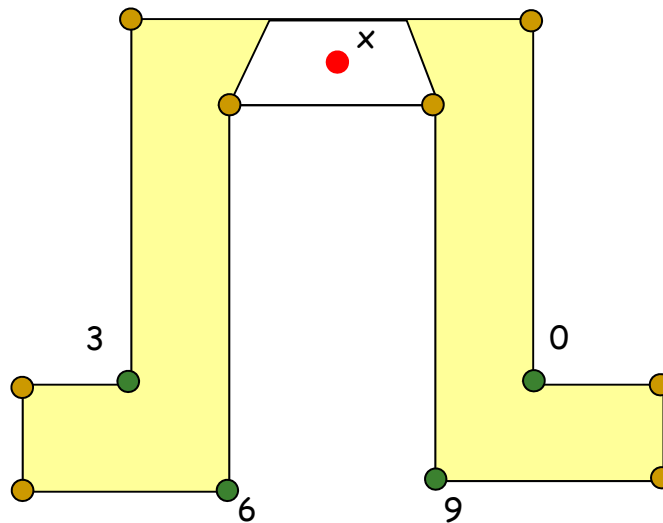
# Guard Placement

- 1. Can we just place one guard on every 3<sup>rd</sup> vertex?



# Guard Placement

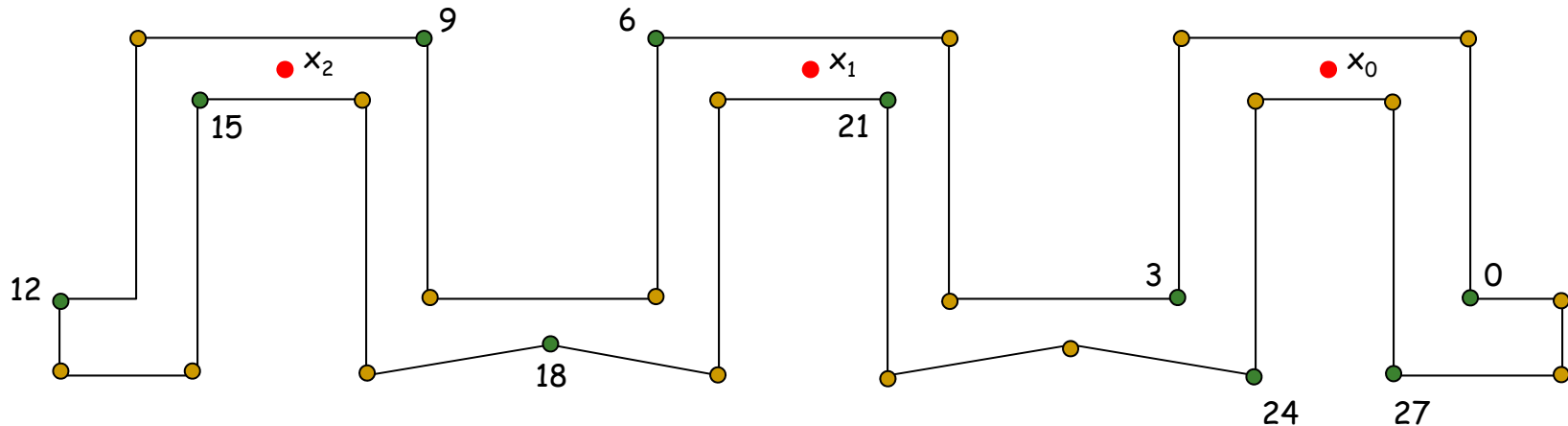
- 1. Can we just place one guard on every 3<sup>rd</sup> vertex? - No!



x is not visible

# Guard Placement

- 1. Can we just place one guard on every 3<sup>rd</sup> vertex? - No!

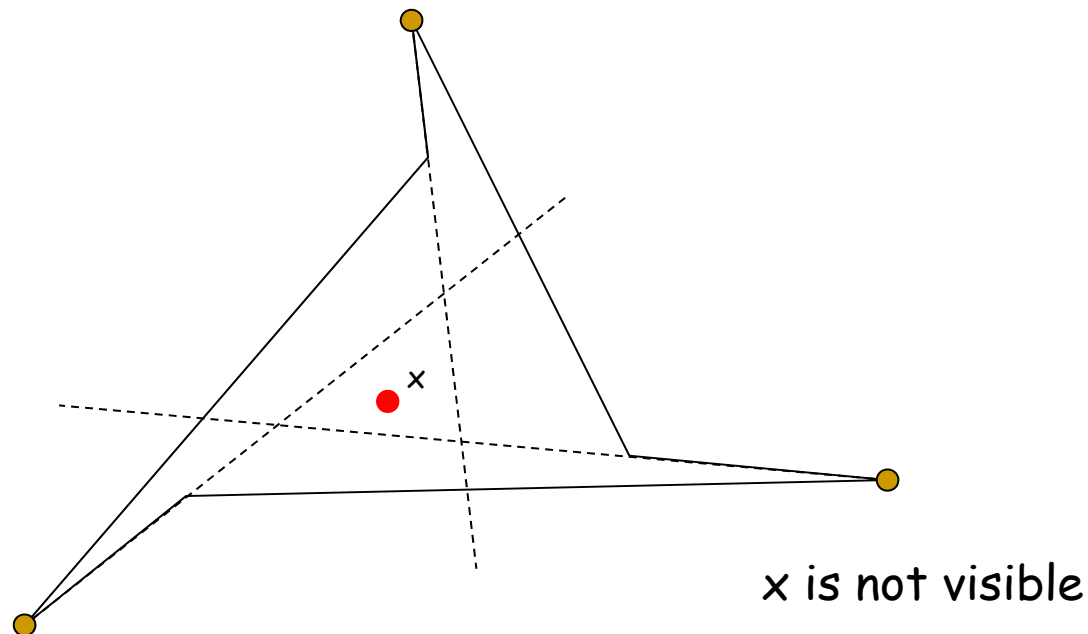


one of  $x_i$  is not visible

# Guard Placement

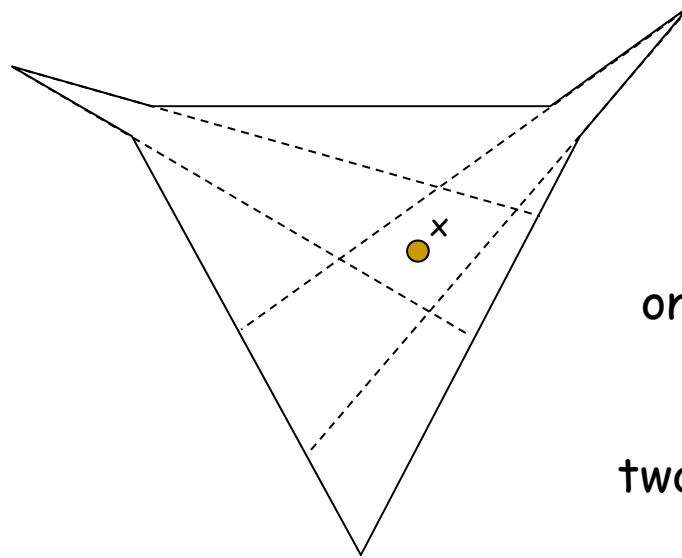
- 2. If guards placed so they can see all the walls, does that imply they can see all the interior?

- No!



# Guard Placement

- 3. If we restrict guards to vertices, is  $g_v(n) = g(n)$ ?
  - In general, yes, equal for  $g(n) = \max(G(P))$



one point guard x  
or  
two vertex guards?

# Art Gallery

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- **Theorem:**  $\lfloor n/3 \rfloor$  guards are occasionally necessary and always sufficient to cover a polygon of  $n$  vertices
    - “Chvatal’s Art Gallery Theorem”
    - “Watchman Theorem”
-

# Fisk's Proof

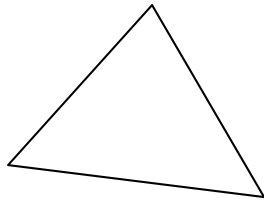
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- $g(n) = \text{floor}(n/3)$ 
    - Published in 1978 (three years after Chvatal's original proof, but it is much more compact)
  - Necessity
    - $g(n) \geq \text{floor}(n/3)$  are sometimes necessary
  - Sufficiency
    - $g(n) \leq \text{floor}(n/3)$  are always sufficient
-

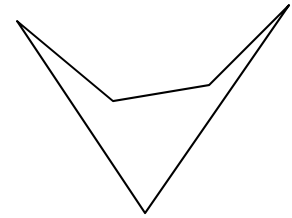
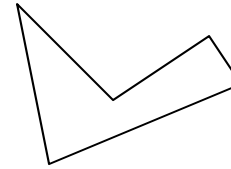
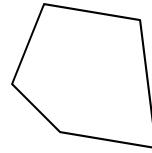
# Necessity: Base Cases

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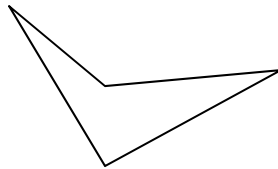
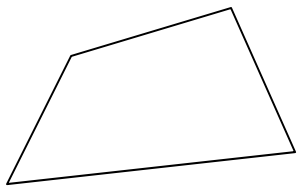
■  $n=3$



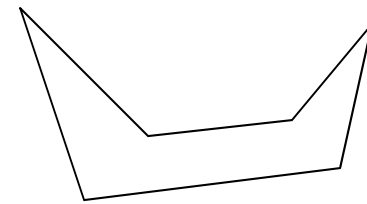
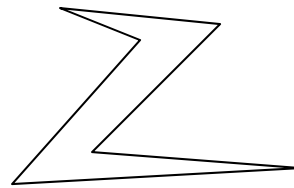
■  $n=5$



■  $n=4$



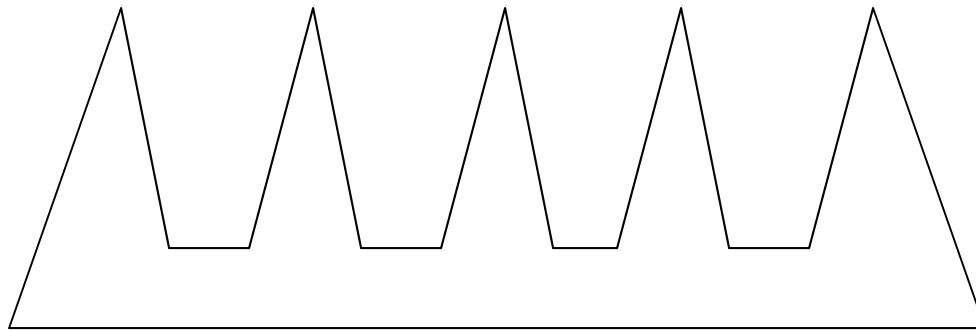
■  $n=6$



# Necessity: Base Cases

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- $n \geq 6$



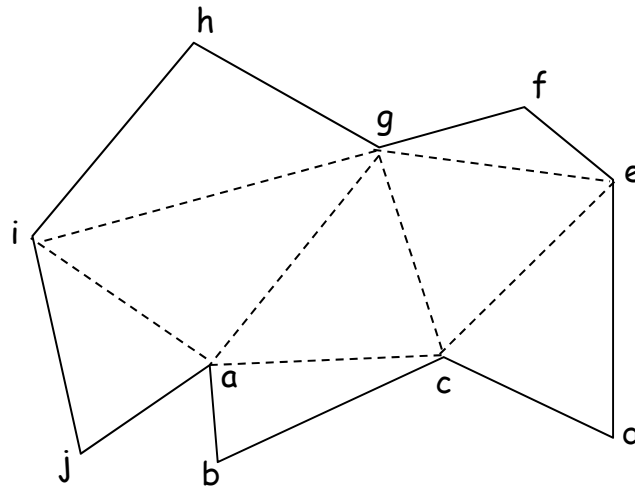
$$g(n) \geq \text{floor}(n/3)$$

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# Sufficiency: Fisk's Proof

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- Step 1 of 3
  - Triangulate the polygon  $P$  by adding only internal diagonals



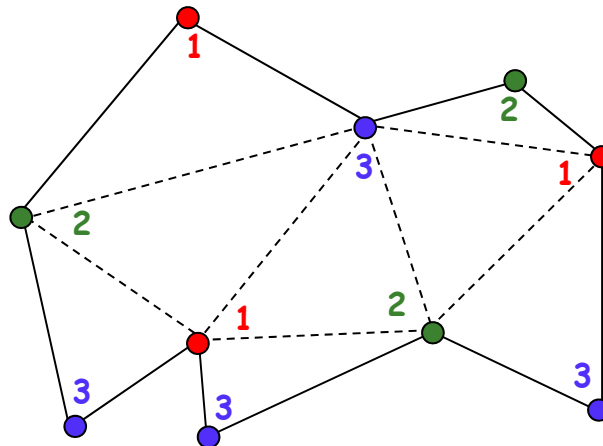
# Triangulation Theorem

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- A polygon of  $n$ -vertices may be partitioned into  $n-2$  triangles by the addition of  $n-3$  internal diagonals
-

# Sufficiency: Fisk's Proof

- Step 2 of 3
  - Perform a 3-coloring of the triangulation graph
    - Using three colors, no two adjacent nodes have same color



# Four Color Theorem

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- Problem stated in 1852 by Francis Guthrie and Augustus De Morgan
    - "Given a map on a flat plane, what is the minimum number of colors needed to color the different regions of the map in such a way that no two adjacent regions have the same color."
-

# Four Color Theorem

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- Several attempted proofs and algorithms
    - Kempe (1879), Tait (1880), Birkhoff (1922), ...
  - Appel and Haken - first complete proof (1976)
  - Robertson, Sanders, Seymour, and Thomas - second more compact proof (1994)
-

# Four Color Theorem

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- The proof creates a large number of cases (~1700 for Appel-Haken and ~600 for Robertson et al.)
  - A computer is used to rigorously check the cases
  - Solution is (still) controversial because of the use of a computer
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# Sufficiency: Fisk's Proof

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## ■ Step 3 of 3

□ Note that one of three colors must be used no more than  $\text{floor}(1/3)$  of the time

■ Let  $a, b, c$  be # of nodes of each color

■  $a \leq b \leq c$  and  $n = a + b + c$

■ If  $a > n/3$ , then  $(a+b+c) \geq n$

■ Thus  $a \leq \text{floor}(n/3)$

■ Since each triangle is a complete graph, each triangle has a node of color 'a'

■ Since each triangle is convex and the triangles partition all of  $P$ , at most 'a' guards are necessary!

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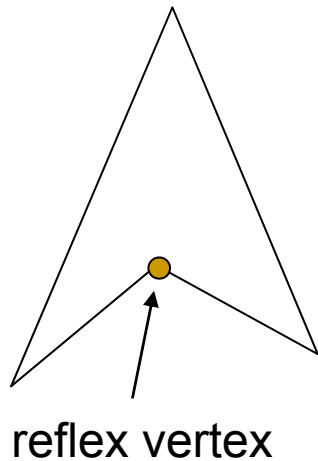
# Fisk's Proof

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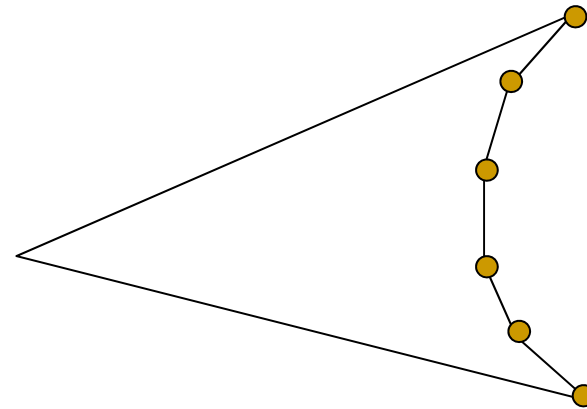
- Necessity
    - $g(n) \geq \text{floor}(n/3)$  are sometimes necessary
  - Sufficiency
    - $g(n) \leq \text{floor}(n/3)$  are always sufficient
  - Thus,  $g(n) = \text{floor}(n/3)$
  - $O(n \log n)$  overall algorithm
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# Reflex Vertices

- We wish to investigate the art gallery question as a function of  $r$  (the number of reflex vertices of a polygon)



$$r \leq (n-3)$$

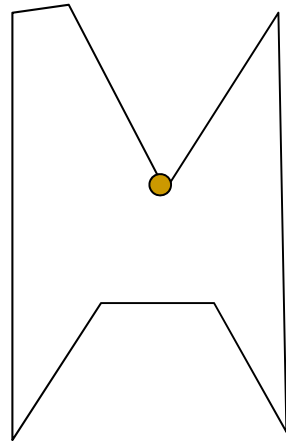


# Reflex Vertices

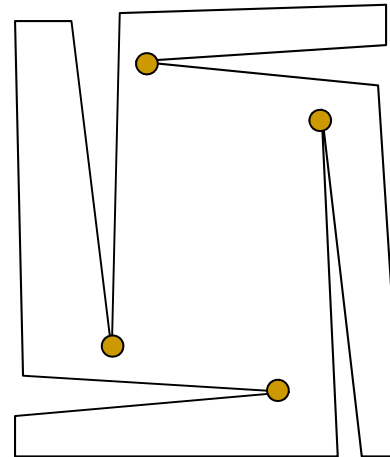
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- Necessity

- How many reflex-vertex guards are necessary?



1 needed



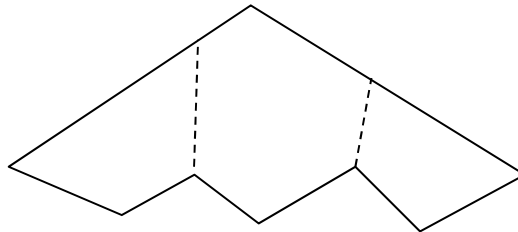
r needed

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# Reflex Vertices

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- Necessity
  - $r$  guards are sometimes necessary
- Sufficiency
  - Place 1 guard at each reflex vertex
    - Proved via a convex partitioning of the polygon  $P$
    - Any polygon  $P$  can be partitioned into at most  $r+1$  convex pieces



# Reflex Vertices

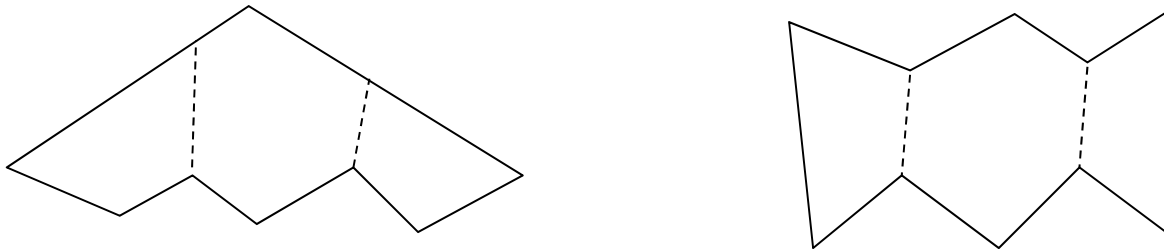
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- Necessity
    - $r$  guards are sometimes necessary
  - Sufficiency
    - Place 1 guard at each reflex vertex
  - Theorem
    - $r$  guards are occasionally necessary and always sufficient to see the interior of a  $n$ -gon of  $r \geq 1$  reflex vertices
-

# Convex Partitioning

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- Naive Algorithm (Chazelle 1980)



- Because at most two reflex vertices can be resolved by a single cut, the minimum number of pieces is  $m = \text{ceil}(r/2) + 1$
  - This approach achieves no more than  $r + 1 \leq 2m$  in  $O(rn) = O(n^2)$  time
-

# Convex Partitioning

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- A fast algorithm:  $O(n \log \log n)$ 
    - Any triangulation can be divided into  $2r+1$  convex pieces by removing diagonals
-

# Convex Partitioning

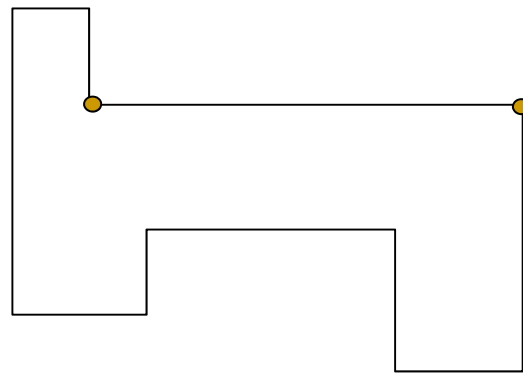
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- Chazelle 1980
    - $O(n^3)$  optimal algorithm using dynamic programming
    - (description is 97 pages long)
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# Orthogonal Polygons

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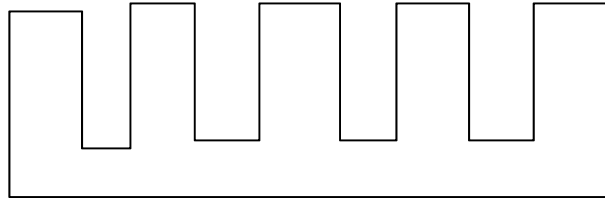
- Kahn, Klawe, Kleitman 1980
  - $\lfloor n/4 \rfloor$  guards are occasionally necessary and always sufficient
  - Based on convex quadrilateralization
    - Any orthogonal polygon  $P$  is convexly quadrilateralizable (theorem)



# Orthogonal Polygons

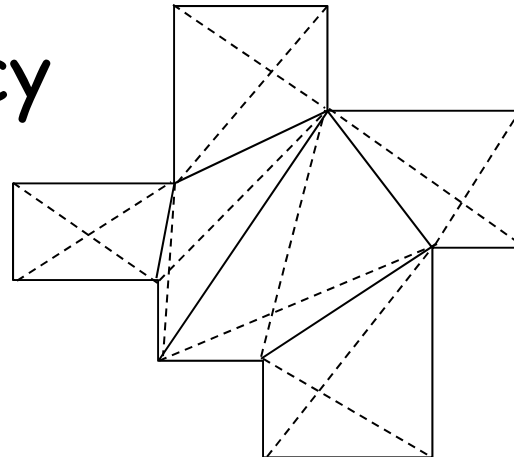
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- Necessity



$$g(n) \geq \text{floor}(n/4)$$

- Sufficiency



Four-colorable, and thus:

$$g(n) \leq \text{floor}(n/4)$$

- Theorem:  $g(n) = \text{floor}(n/4)$

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# Orthogonal Polygons

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- In an orthogonal polygon
    - $n$  vertices
    - $c$  interval vertices with  $\pi/2$
    - $r$  interval vertices with  $3\pi/2$
    - $n = c + r$
    - sum of internal angles  $(n-2)\pi$
    - yields  $n=2r+4$
  - Theorem restated as  $g(n)=\text{floor}(r/2)+1$
-

# Quadrilateralization

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- Sacks's Algorithm
    - $O(n \log n)$
  - Lubiw's Algorithm
    - $O(n \log n)$
-

# Mobile Guards

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- Theorem

| Shape      | Stationary          | Mobile                    |
|------------|---------------------|---------------------------|
| General    | $\text{floor}(n/3)$ | $\text{floor}(n/4)$       |
| Orthogonal | $\text{floor}(n/4)$ | $\text{floor}((3n+4)/16)$ |

- In general, only  $\frac{3}{4}$  as many mobile guards are needed as stationary guards
-

# Mobile Guards

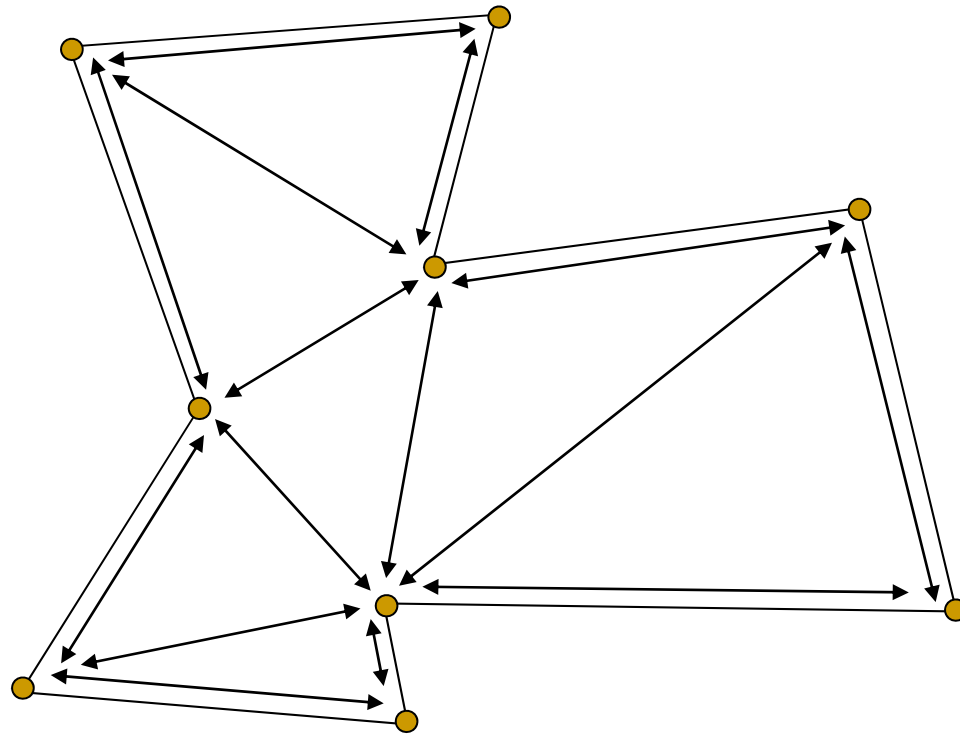
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## ■ General Polygons

Vertex guards

Edge guards

Diagonal guards



# Mobile Guards

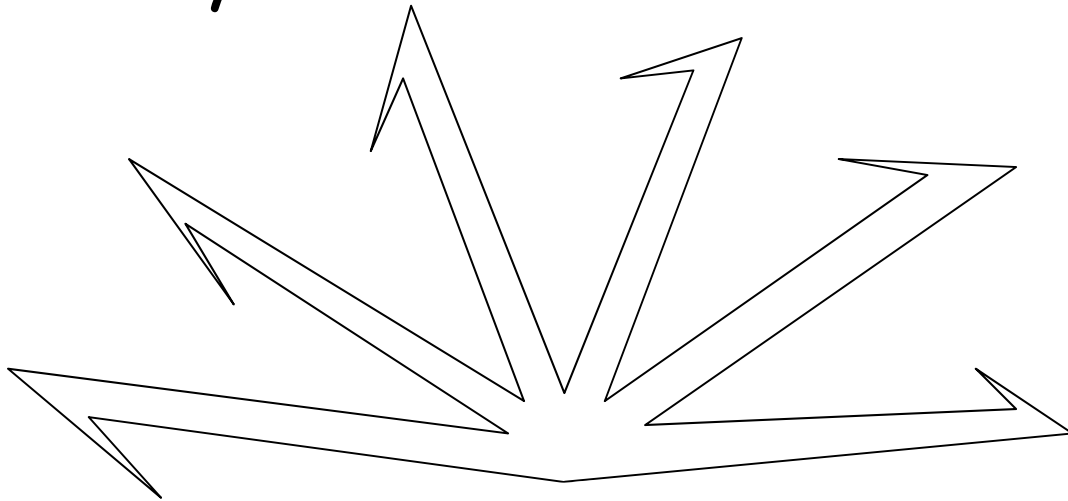
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- Goal of the proof
    - Given a triangulation graph  $T$ 
      - Vertex guard = node
      - Edge guard = adjacent arc
      - Diagonal guard = any arc
    - The analog of covering is domination
    - A collection of guards  $C = \{g_1, \dots, g_k\}$  dominates triangulation graph  $T$  if every face has at least one of its three nodes in some  $g_i \in C$ .
-

# Mobile Guards

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- Necessity



Polygon that requires  $\text{floor}(n/4)$  edge, diagonal (or line) guards

- Sufficiency: a little more complicated...

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# Miscellaneous Shapes

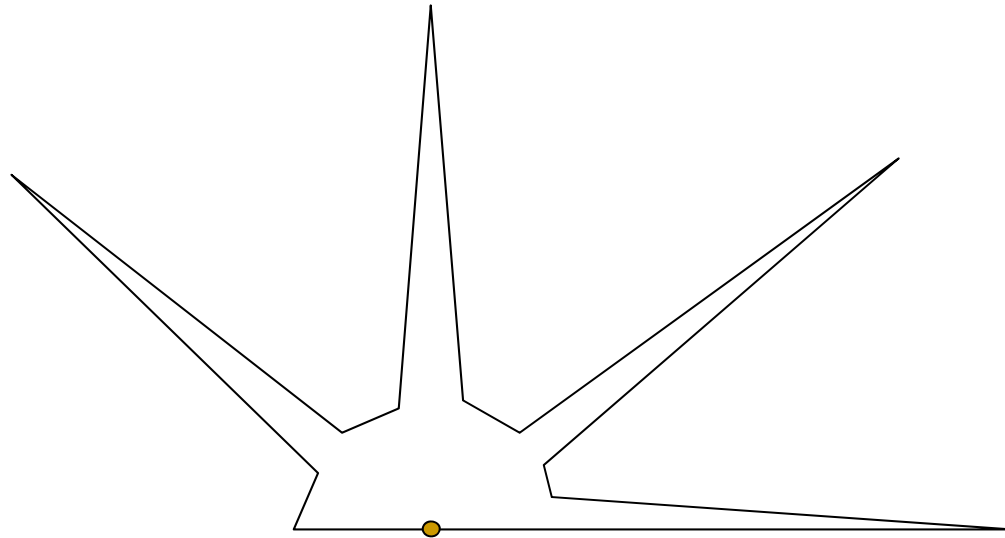
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- (General polygon, convex, orthogonal)
  - Star, spiral, monotone
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# Star Shape

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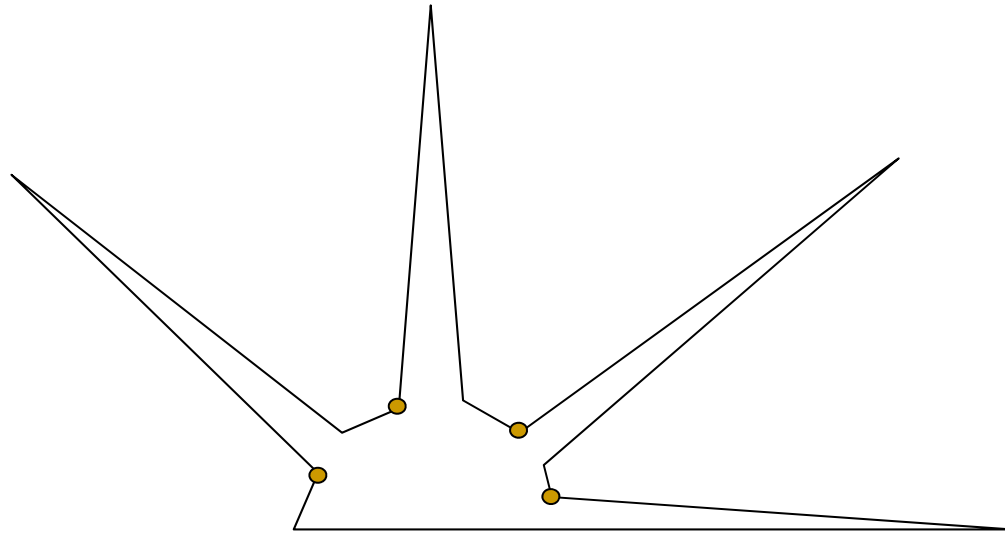
- A star polygon  $P$  is a polygon that may be covered by a single point guard



# Star Shape

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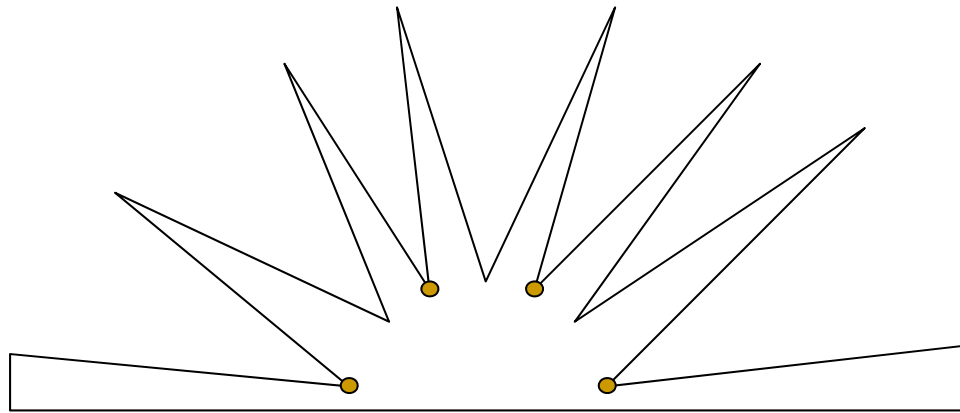
- Toussaint's Theorem
  - A star polygon  $P$  requires  $\text{floor}(n/3)$  vertex guards



# Star Shape

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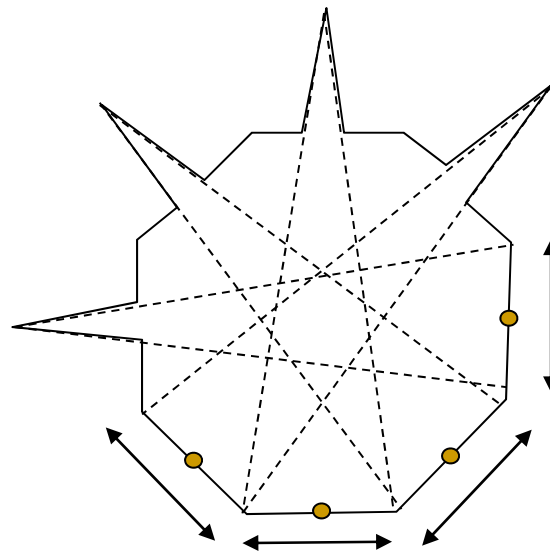
- Toussaint's Theorem
  - A star polygon  $P$  requires  $\text{floor}(r/2)+1$  reflex guards



# Star Shape

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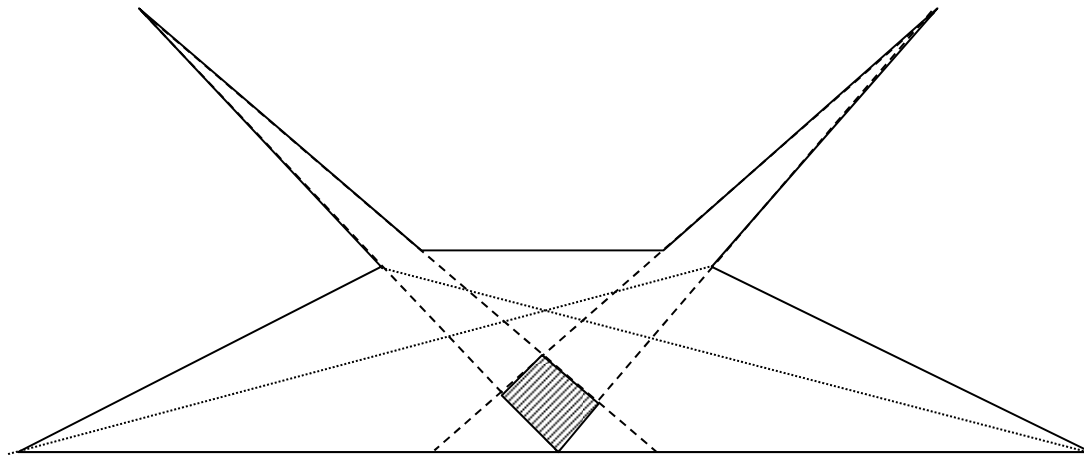
- Toussaint's Theorem
  - A star polygon  $P$  requires at least  $\text{floor}(n/5)$  edge guards



# Star Shape

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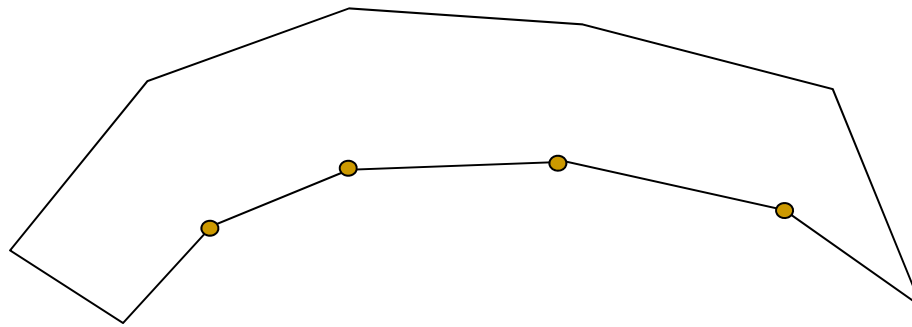
- Toussaint's Theorem
  - For a star polygon  $P$ 
    - Unrestricted patrol, one line guard is needed
    - Restricted to diagonal lines, two are needed



# Spiral Polygon

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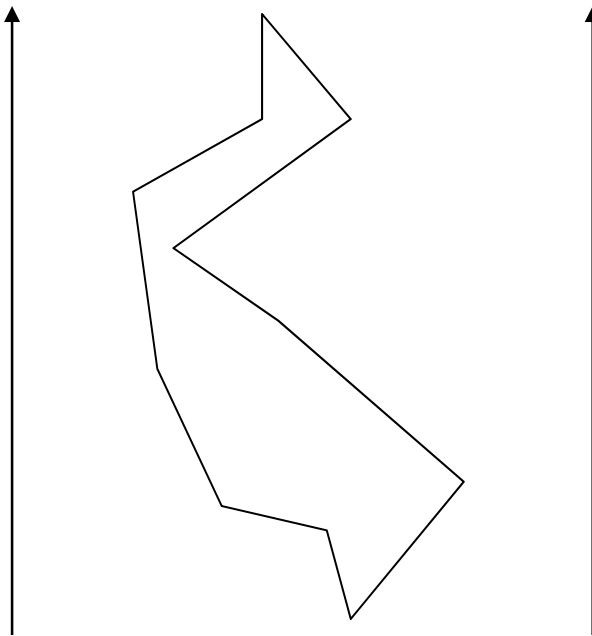
- A spiral polygon is a polygon with at most one chain of reflex vertices



# Monotone Polygon

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- A polygon with no "doubling back" with respect to a line



# Spiral and Monotone Polygons

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- Aggarwal's Theorem
    - $\lfloor n/3 \rfloor$  vertex guards are needed
    - $\lfloor r/2 \rfloor + 1$  reflex-vertex guards are needed
    - $\lfloor (n+2)/5 \rfloor$  diagonals guards are needed
-

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-

# Exterior Visibility

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- "Fortress Problem"
- "Prison Yard Problem"

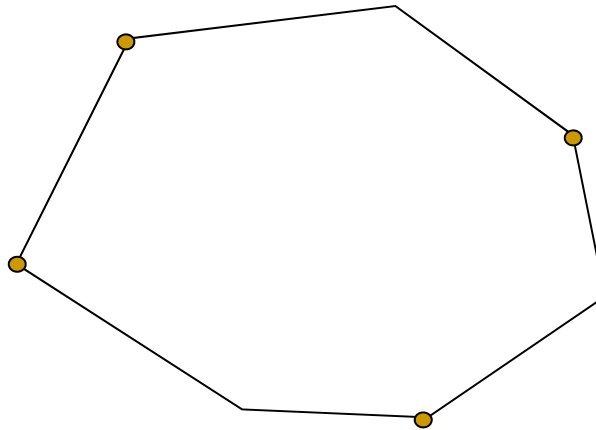
(independently stated by Derick Wood and Joseph Malkelvitch, early 1980s)

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# Fortress Problem

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- How many vertex guards are needed to see the exterior of a polygon  $P$ ?
- Simplex convex polygon



$\text{ceil}(n/2)$  vertex guards

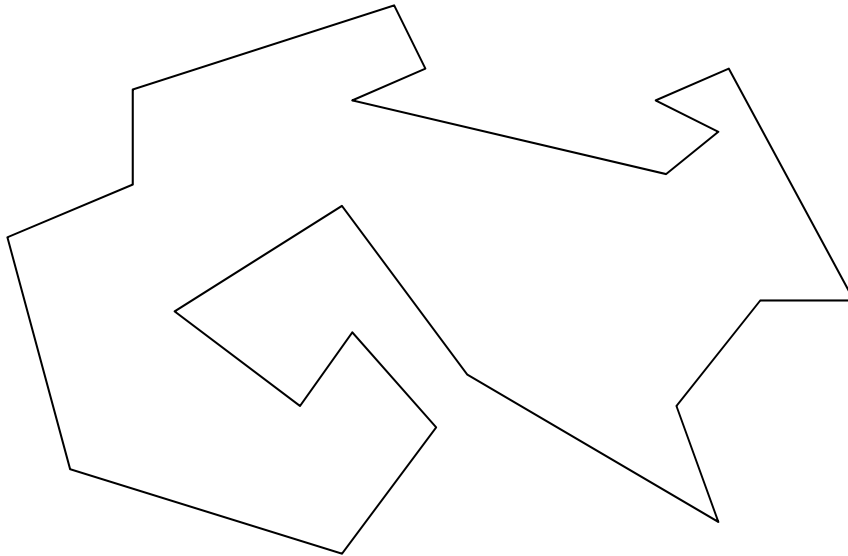
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# Fortress Problem

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- Arbitrary polygon

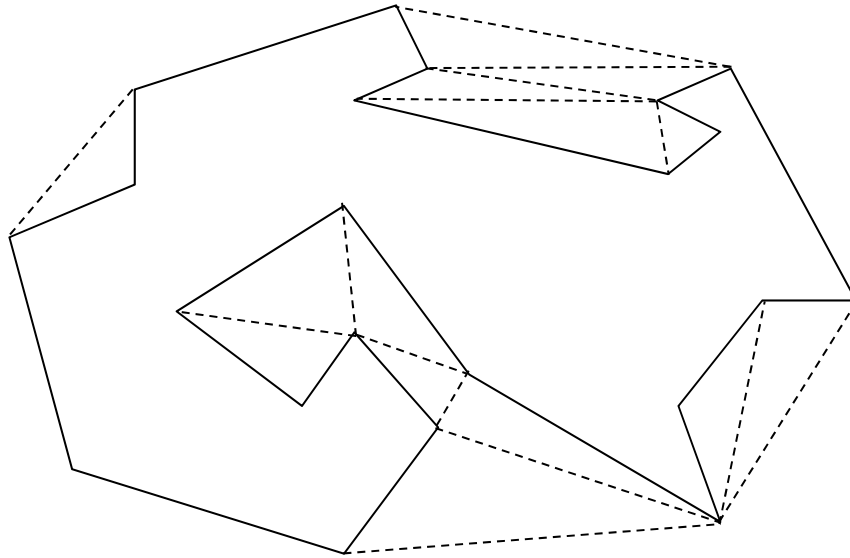
$\text{ceil}(n/2)$  vertex guards?



# Fortress Problem

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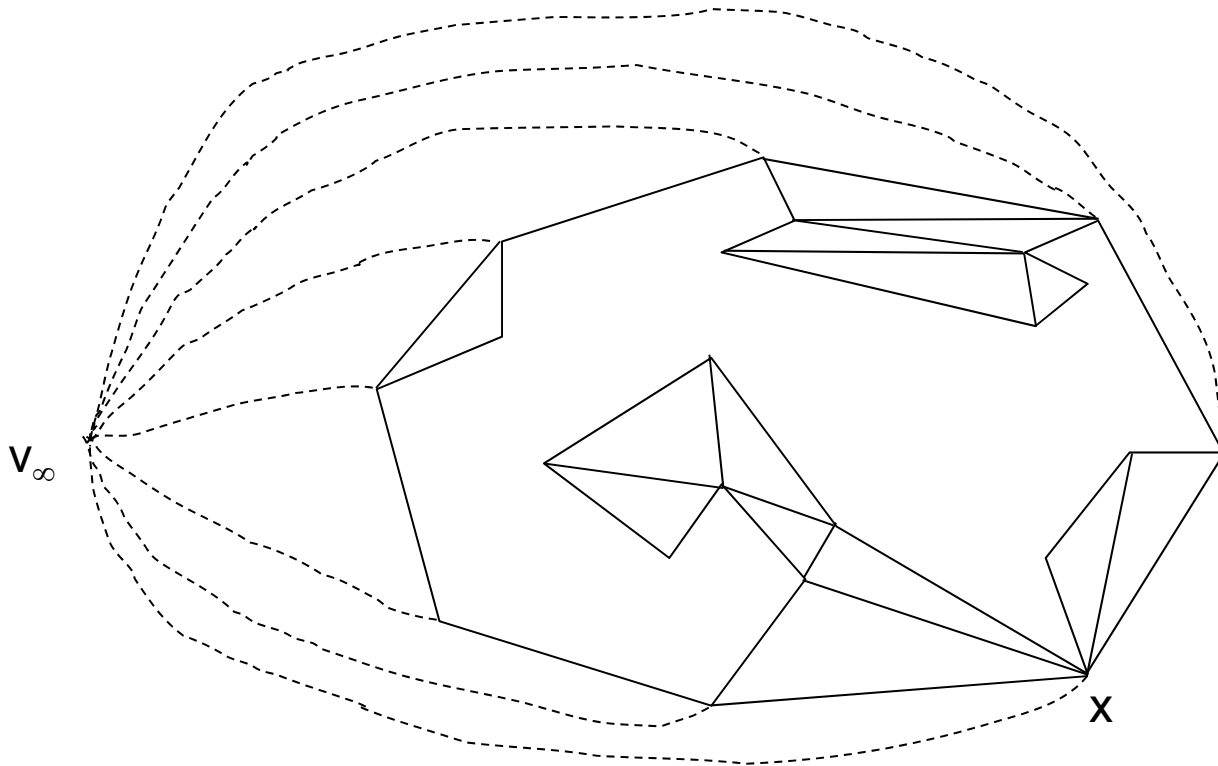
- Arbitrary polygon



# Fortress Problem

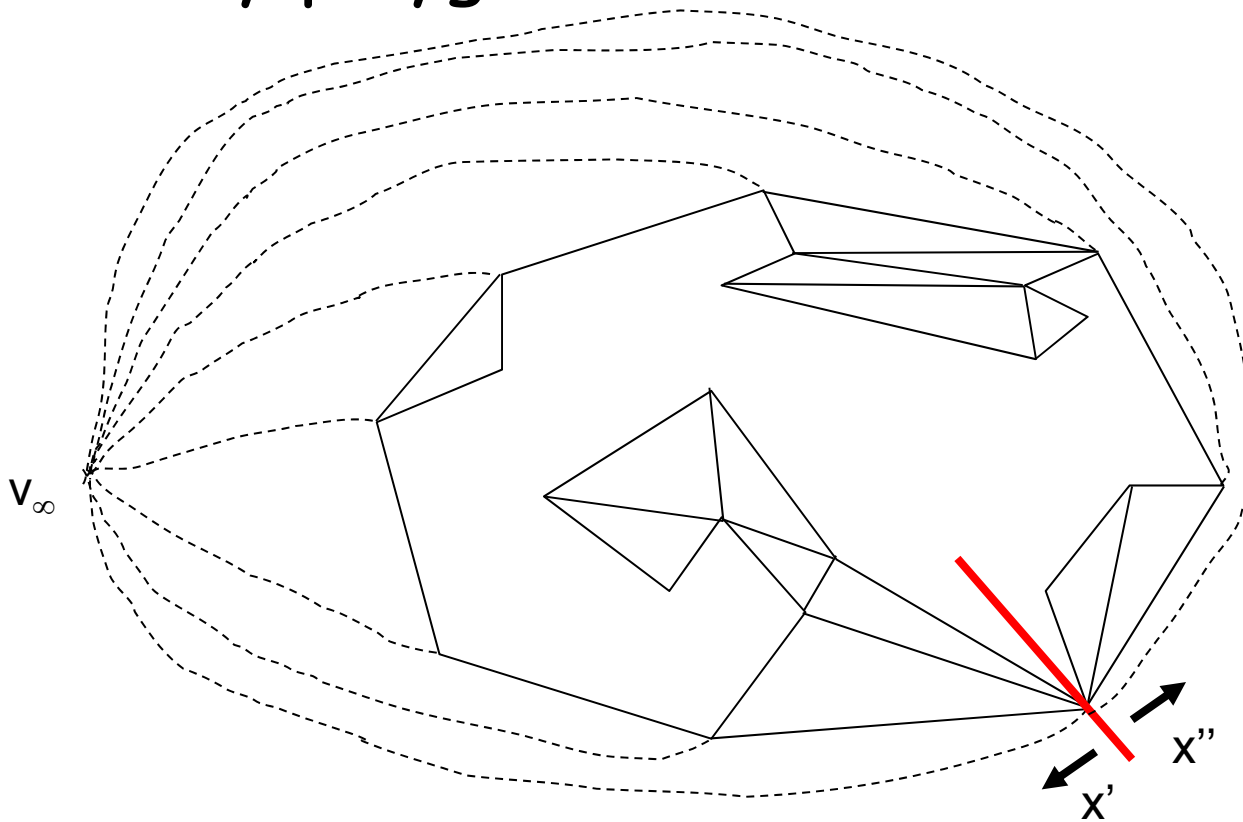
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- Arbitrary polygon



# Fortress Problem

- Arbitrary polygon



# Fortress Problem

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- Arbitrary polygon
    - Three-color the resulting triangulation graph  $T$  (of  $n+2$  nodes)
-

# Fortress Problem

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- Arbitrary polygon
    - If least frequently used color is red and  $v_\infty$  is **not** red then,
      - $\text{floor}((n+2)/3)$  vertex guards are needed
-

# Fortress Problem

---

- Arbitrary polygon

- If least frequently used color is red and  $v_\infty$  is red then,

- No guard can be placed at  $v_\infty$  because it's not part of original polygon
  - Thus, place guards at second least frequently used color
  - $a \leq b \leq c$  and  $a + b + c = n + 2$
  - $a \geq 1$  and  $b + c \leq n + 1$
  - $b \leq \text{floor}((n+1)/2) = \text{ceil}(n/2)$  vertex guards are needed
-

# Fortress Problem

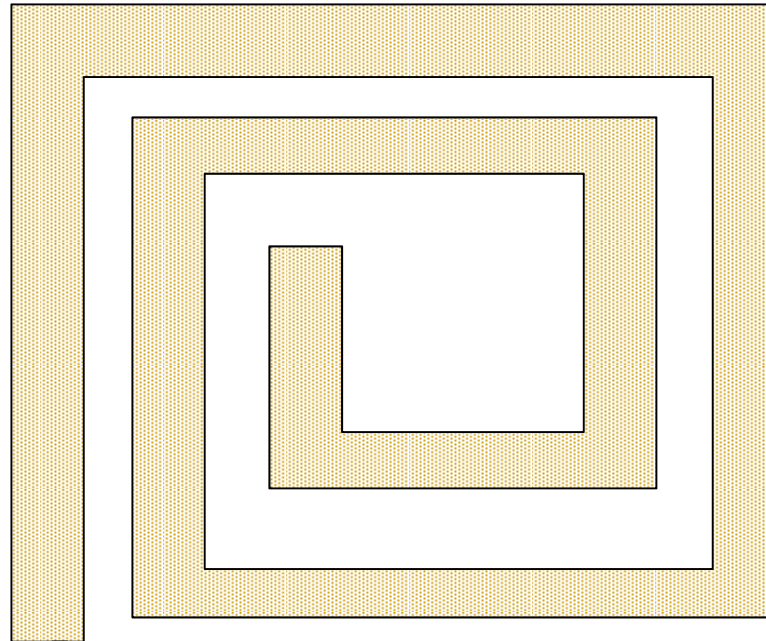
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- Arbitrary polygon (Summary)
    - 1. Triangulate the convex hull of the polygon  $P$
    - 2. Add edges from all exterior vertices to new vertex  $v_\infty$
    - 3. Split a vertex  $x$  into  $x'$  and  $x''$
    - 4. Open-up the convex hull, straighten the lines to  $v_\infty$ , and form a triangulation graph  $T$  of  $(n+2)$  nodes
    - 5. Three-color graph  $T$
    - 6. Use least or second least frequently used color
  - At most  $\text{ceil}(n/2)$  vertex guards are needed
-

# Fortress Problem

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- Orthogonal polygon

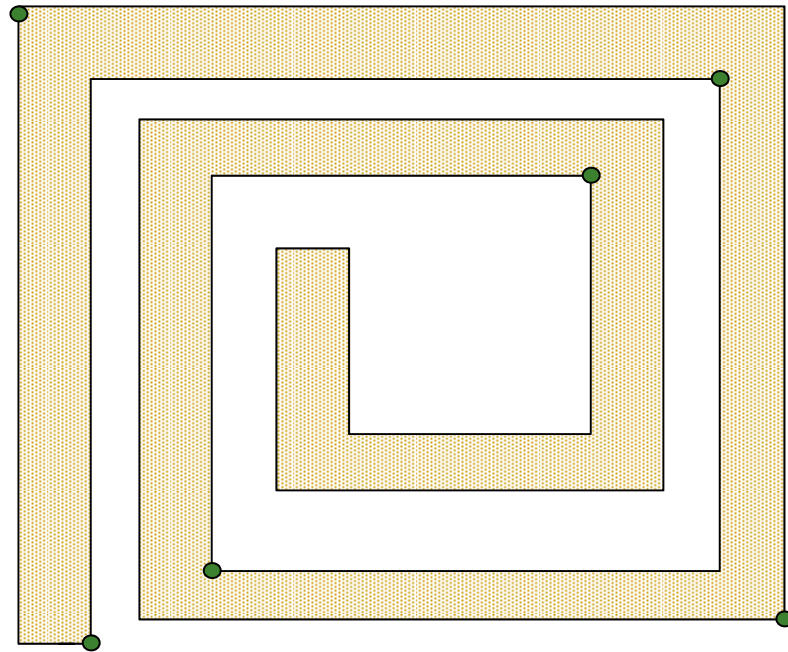




# Fortress Problem

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- Orthogonal polygon

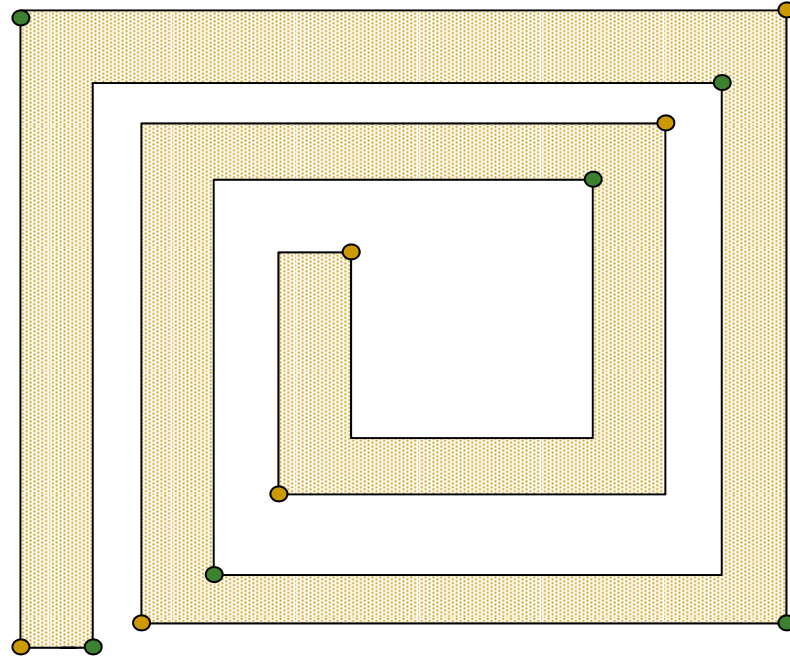


# Fortress Problem

## ■ Orthogonal polygon

- Solution A
- Solution B

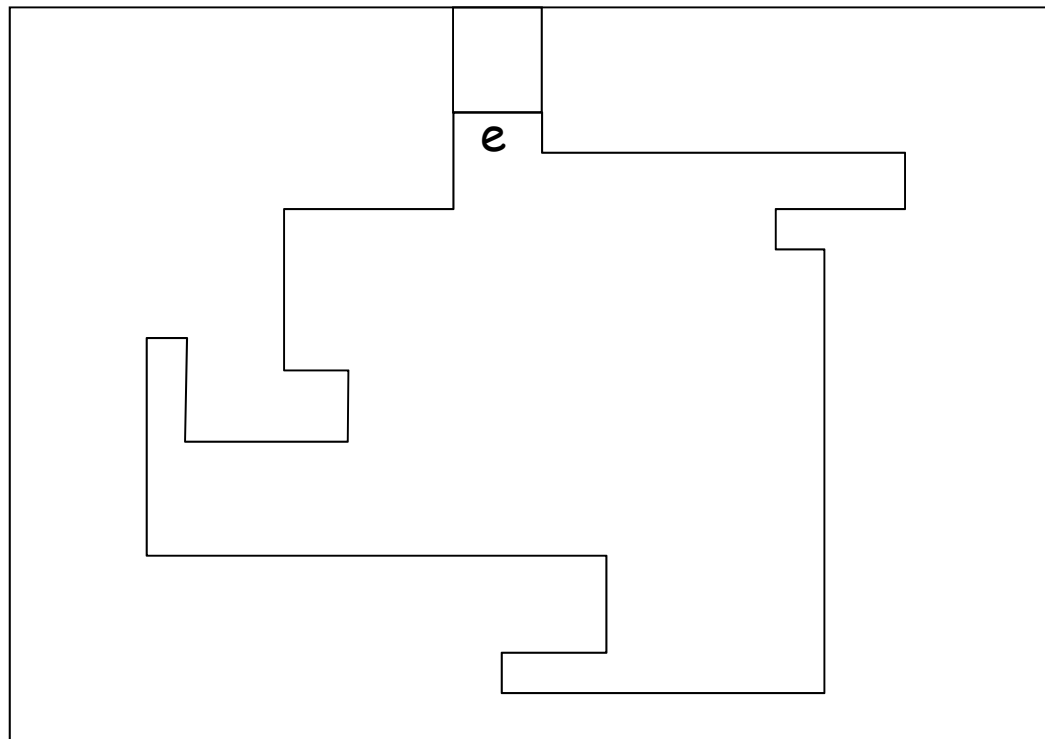
$\text{ceil}(n/4)+1$  vertex guards necessary



# Fortress Problem

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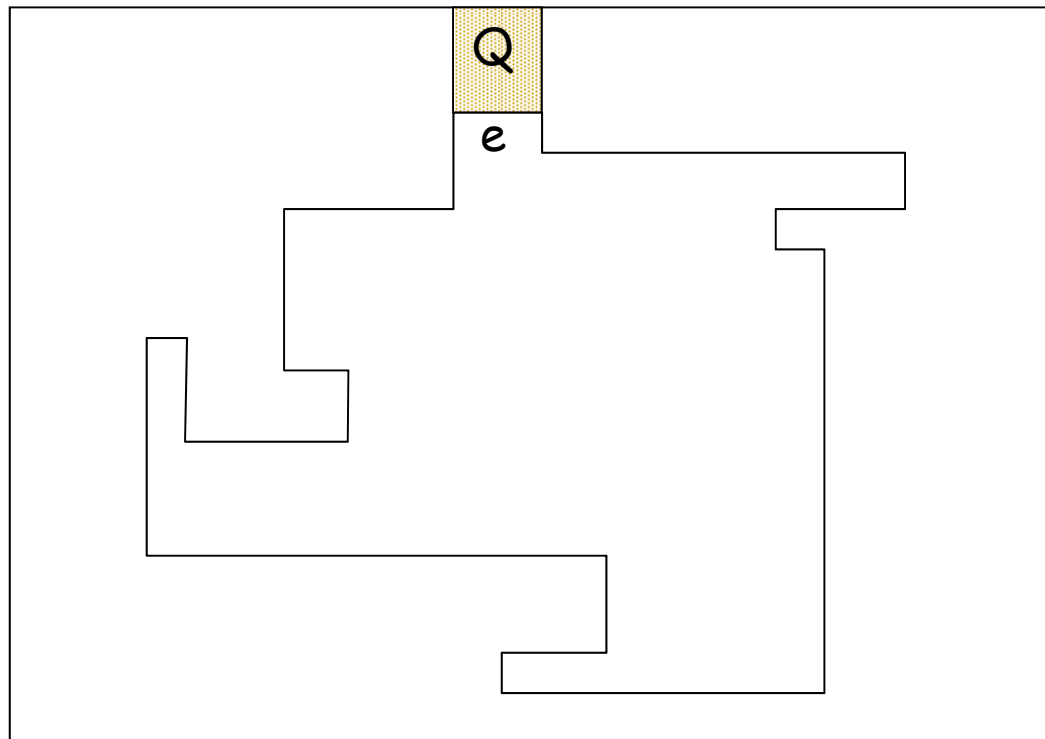
- Orthogonal polygon



# Fortress Problem

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- Orthogonal polygon



# Fortress Problem

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- Orthogonal polygon
    - Interior of new polygon  $P'$  coincides with the immediate exterior of  $P$ , except for  $Q$  which is exterior to both
-

# Fortress Problem

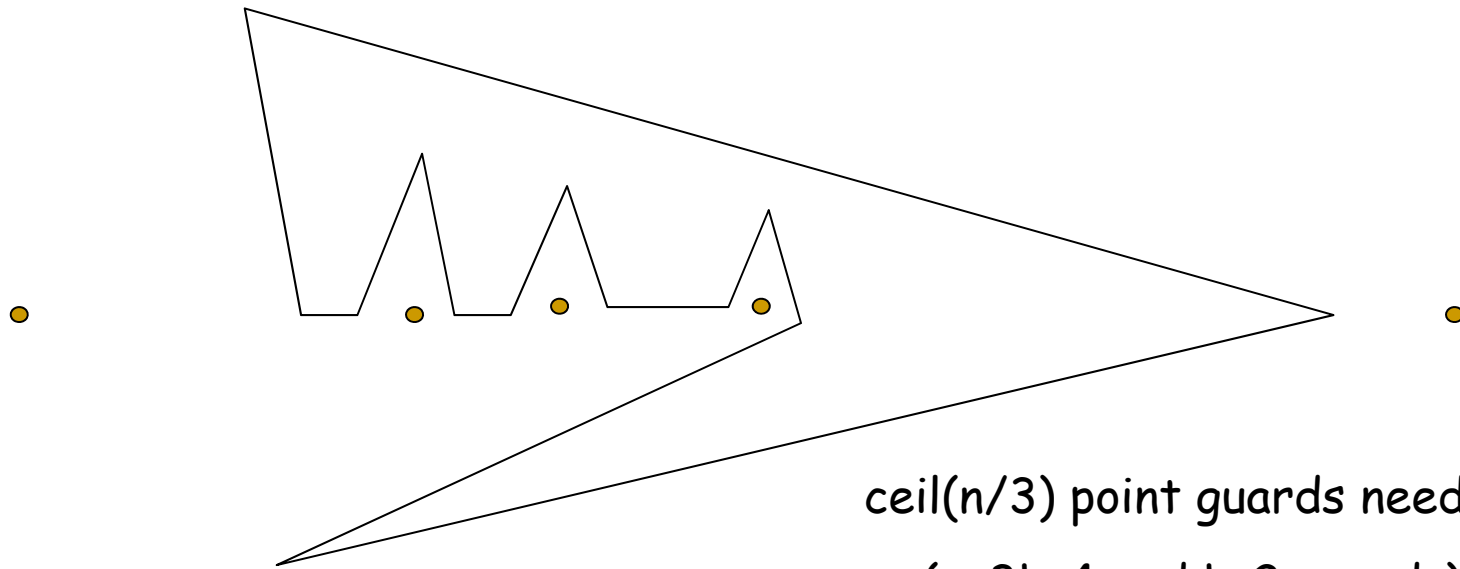
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- Orthogonal polygon
    - For  $P'$  of  $n+4$  vertices,  $\text{floor}(r/2)+1$  or  $\text{floor}((n+4)/4)$  vertex guards suffice to cover the interior
      - None of the new vertices of  $P'$  are reflex vertices
      - Need an additional one for  $Q$
    - Thus,  $\text{floor}(n/4)+2$  vertex guards are sufficient
      - For  $n \bmod 4=0$ ,  $\text{ceil}(n/4)+1$
-

# Fortress Problem

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- Guards in the plane
  - Necessity

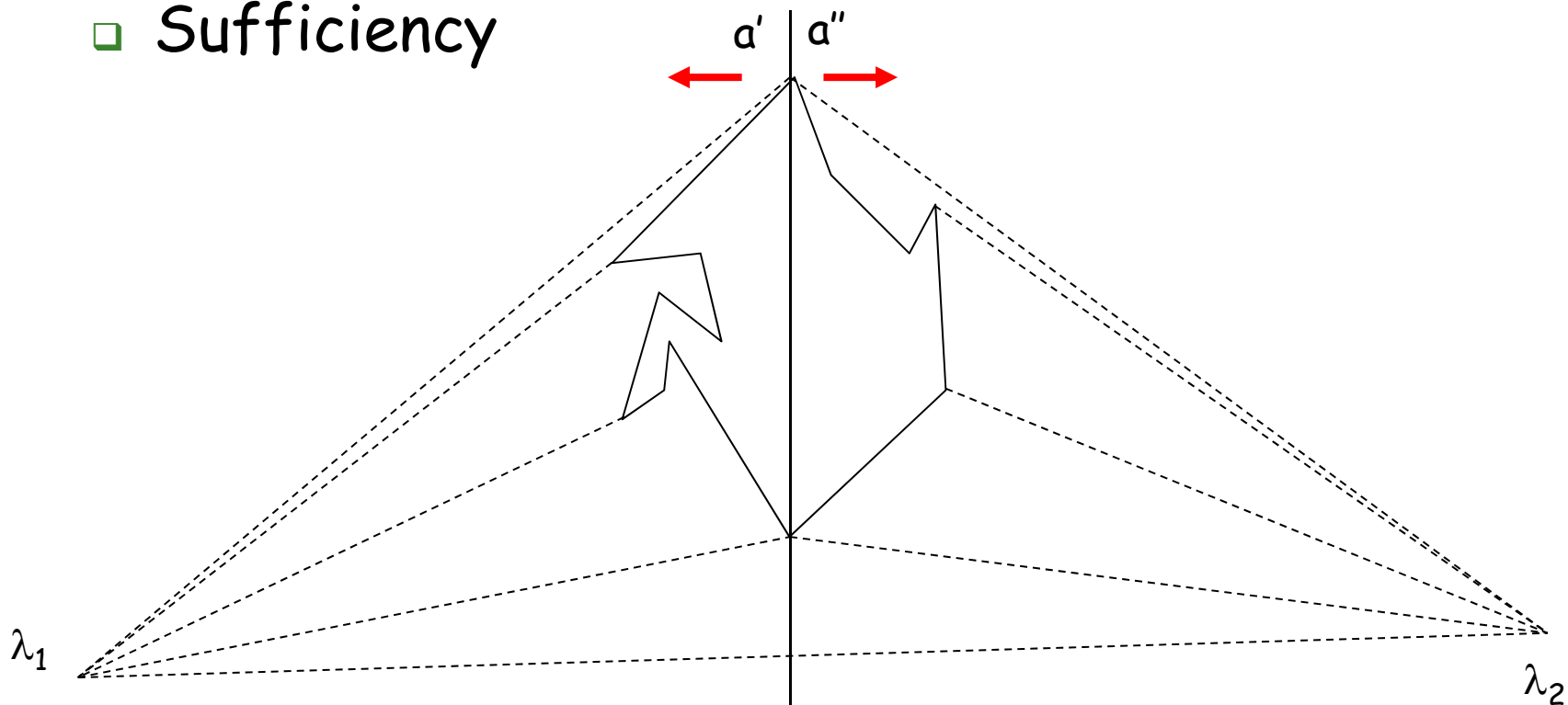


$\text{ceil}(n/3)$  point guards needed  
( $n=3k+4$  and  $k+2$  guards)

---

# Fortress Problem

- Guards in the plane
  - Sufficiency



# Fortress Problem

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- Guards in the plane
    - New triangulated polygon  $P'$  of  $n+3$  vertices
      - $\text{floor}((n+3)/3) = \text{ceil}((n+1)/3)$  point guards
-

# Fortress Problem

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- Guards in the plane
    - More lengthy proof to remove “1/3 of a guard”
      - Add only 2 guards and 3-color triangulation
      - If even hull vertices, trivial
      - If odd hull vertices, need some extra work
    - Result:  $\text{ceil}(n/3)$  point guards are necessary to cover the exterior of a polygon  $P$  of  $n$  vertices
      - Nice duality with  $\text{floor}(n/3)$  for the interior
-

# Prison Yard Problem

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- How many vertex guards are needed to simultaneously see the exterior and interior of polygon  $P$ ?
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# Prison Yard Problem

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- General Polygons

- Worst-case is a convex polygon

- $\text{ceil}(n/2)$  vertex guards needed

- Multiply-connected polygons

- $\min(\text{ceil}(n/2), \text{floor}((n+\text{ceil}(h/2))/2), \text{floor}(2n/3))$

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# Prison Yard Problem

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- Orthogonal Polygons
    - $\text{floor}((7n/16)+5)$  vertex guards are needed
-

# Fortress/Prison Yard Problem

| Problem            | Techniques                          | Guards                                 | Time     |
|--------------------|-------------------------------------|--|----------|
| <b>Fortress</b>    |                                     |  |          |
| <b>General</b>     | Triangulation, 3-coloring           | $\text{ceil}(n/2)$                     | $O(T)$   |
| <b>Orthogonal</b>  | L-shaped partition                  | $\text{ceil}(n/4)+1$                   | $O(T)$   |
| <b>Prison Yard</b> |                                     |  |          |
| <b>General</b>     | Exterior                            | $\text{ceil}(n/2)+r$                   | $O(T)$   |
|                    | Triangulation, 4-coloring           | $\text{floor}((n+\text{ceil}(h/2))/2)$ | $O(n^2)$ |
|                    | Triangulation, 4-coloring           | $\text{floor}(2n/3)$                   | $O(n^2)$ |
|                    | Exterior, triangulation, 3-coloring | $\text{floor}(2n/3+1)$                 | $O(T)$   |
| <b>Orthogonal</b>  | Exterior, quad., 4-coloring         | $\text{floor}((7n/16)+5)$              | $O(T)$   |

# Contents

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    - Miscellaneous Shapes
      - Star, Spiral, Monotone
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    - Fortress Problem
    - Prison Yard Problem
  - ⇒ ■ **Minimal Guards**
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# Minimal Guard Coverage

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- Seek the placement of a minimal number of guards that cover a polygon  $P$ 
    - In general, a NP-complete problem
-

# Minimal Guard Coverage

| Polygon             | Cover             |                    | Partition         |                    |
|---------------------|-------------------|--------------------|-------------------|--------------------|
|                     | <i>w. Steiner</i> | <i>w/o Steiner</i> | <i>w. Steiner</i> | <i>w/o Steiner</i> |
| Simple Polygons     | NP-hard           | NP-complete        | ?                 | $O(n^7 \log n)$    |
| Polygons with Holes | NP-hard           | NP-complete        | NP-hard           | NP-complete        |