Variable Elimination for 3D from 2D

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Objective

To generate more compact equation sets for 3D reconstruction problem by eliminating variables.

Inspiration

- We want to use symbolic method instead of numerical method to solve nonlinear equations.
- Comparison.
  - Numerical: popular, iterative, need initial guess.
  - Symbolic: don't need initial guess, high computational complexity.
    - Resultant method: up to 10 variables

Previous Work

Traditional Equations

$$p_{ij} = c_{ij} F_j \begin{pmatrix} P_i \\ 1 \end{pmatrix},$$

$$\text{for } i = 1, \ldots, N \text{ and } j = 1, \ldots, J.$$

$$F_j = (R_j, T_j) \quad p_{ij} = (x_{ij}, y_{ij}, 1)$$

where $\boldsymbol{p_{ij}}$ represents the 2D coordinates of the 3D feature point $\boldsymbol{P_i}$ observed on picture j, $\boldsymbol{c_{ij}}$ is a depth variable, and $\boldsymbol{F_j}$ is a 3-by-4 matrix containing the camera parameters corresponding to picture j.

Previous Work

- Epipolar geometry.
  - Eliminate structure and depth (solve camera pose from images).
  - Camera pose estimation is an ill-conditioned problem.

Our Solution

Eliminate camera orientation

↓

Eliminate camera position

↓

Eliminate structure

Eliminate Camera Orientation

Invariant Based Elimination

$$\gamma_{ij}\gamma_{1j}p_{ij}\cdot p_{1j}=(P_i-C_j)\cdot(P_1-C_j),\ \text{for } i=1,\ldots,N\ (1)$$

$$\gamma_{ij}\gamma_{2j}p_{ij}\cdot p_{2j}=(P_i-C_j)\cdot(P_2-C_j),\ \text{for } i=2,\ldots,N\ (2)$$

$$\gamma_{ij}\gamma_{1j}\gamma_{2j}p_{ij}\cdot p_{1j}\times p_{2j}=(P_i-C_j)\cdot(P_1-C_j)\times(P_2-C_j),\ \text{for } i=3,\ldots,N\ (3)$$

for all $j=1,\ldots,J$

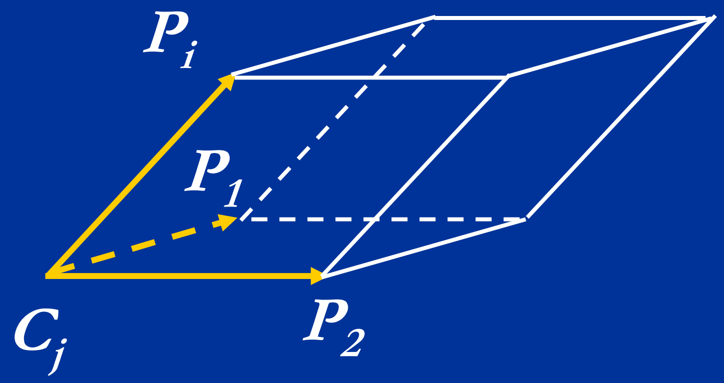
This is a basis equation set.

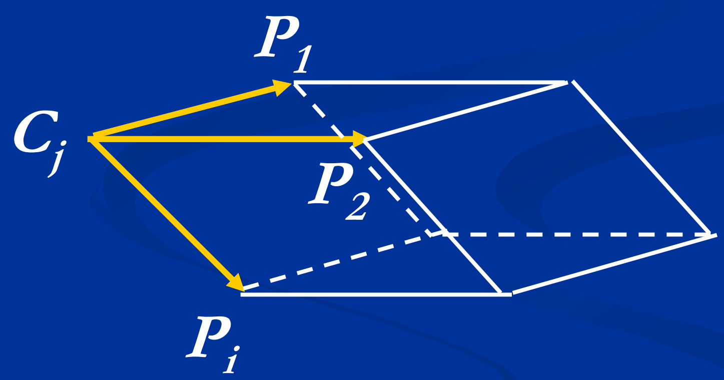
The (3) equation group is equivalent to

$$\gamma_{ij}\gamma_{ij}p_{ij}\cdot p_{ij}=(P_i-C_j)\cdot(P_i-C_j),\ \text{for } i=3,\ldots,N$$

except for an sign information.

Eliminate Camera Orientation





Eliminate Camera Position

Observe

$$(P_i - P_1)\cdot(P_i - P_1)$$
$$= ((P_i - C_j) - (P_1 - C_j))\cdot((P_i - C_j) - (P_1 - C_j))$$
$$= \underbrace{(P_i - C_j)\cdot(P_i - C_j)}_{(3)} - 2\underbrace{(P_i - C_j)\cdot(P_1 - C_j)}_{(1)} + \underbrace{(P_1 - C_j)\cdot(P_1 - C_j)}_{(1)}$$

Replace (1) $\gamma_{ij}\gamma_{1j}\, p_{ij}\cdot p_{1j} = (P_i - C_j)\cdot(P_1 - C_j)$ with

$$(\gamma_{ij} p_{ij} - \gamma_{1j} p_{1j})\cdot(\gamma_{ij} p_{ij} - \gamma_{1j} p_{1j}) = (P_i - P_1)\cdot(P_i - P_1)$$

Eliminate Camera Position

Similarly, we can replace (2) and (3)

$$\gamma_{ij}\gamma_{2j}p_{ij}\cdot p_{2j}=(P_i-C_j)\cdot(P_2-C_j),\ \text{for } i=2,\ldots,N$$

$$\gamma_{ij}\gamma_{1j}\gamma_{2j}p_{ij}\cdot p_{1j}\times p_{2j}=(P_i-C_j)\cdot(P_1-C_j)\times(P_2-C_j),\ \text{for } i=3,\ldots,N\ (3)$$

with

$$(\gamma_{ij}p_{ij}-\gamma_{2j}p_{2j})\cdot(\gamma_{ij}p_{ij}-\gamma_{2j}p_{2j})=(P_i-P_2)\cdot(P_i-P_2),\ \text{for } i=3,\ldots,N$$

$$(\gamma_{ij}p_{ij}-\gamma_{3j}p_{3j})\cdot(\gamma_{1j}p_{1j}-\gamma_{3j}p_{3j})\times(\gamma_{2j}p_{2j}-\gamma_{3j}p_{3j})=(P_i-P_3)\cdot(P_1-P_3)\times(P_2-P_3)$$

$$\text{for } i=4,\ldots,N$$

Eliminate Camera Position

A new camera pose free equation set

$$(\gamma_{ij} p_{ij} - \gamma_{1j} p_{1j}) \cdot (\gamma_{ij} p_{ij} - \gamma_{1j} p_{1j}) = (P_i - P_1) \cdot (P_i - P_1), \text{ for } i = 2,\ldots,N$$

$$(\gamma_{ij} p_{ij} - \gamma_{2j} p_{2j}) \cdot (\gamma_{ij} p_{ij} - \gamma_{2j} p_{2j}) = (P_i - P_2) \cdot (P_i - P_2), \text{ for } i = 3,\ldots,N$$

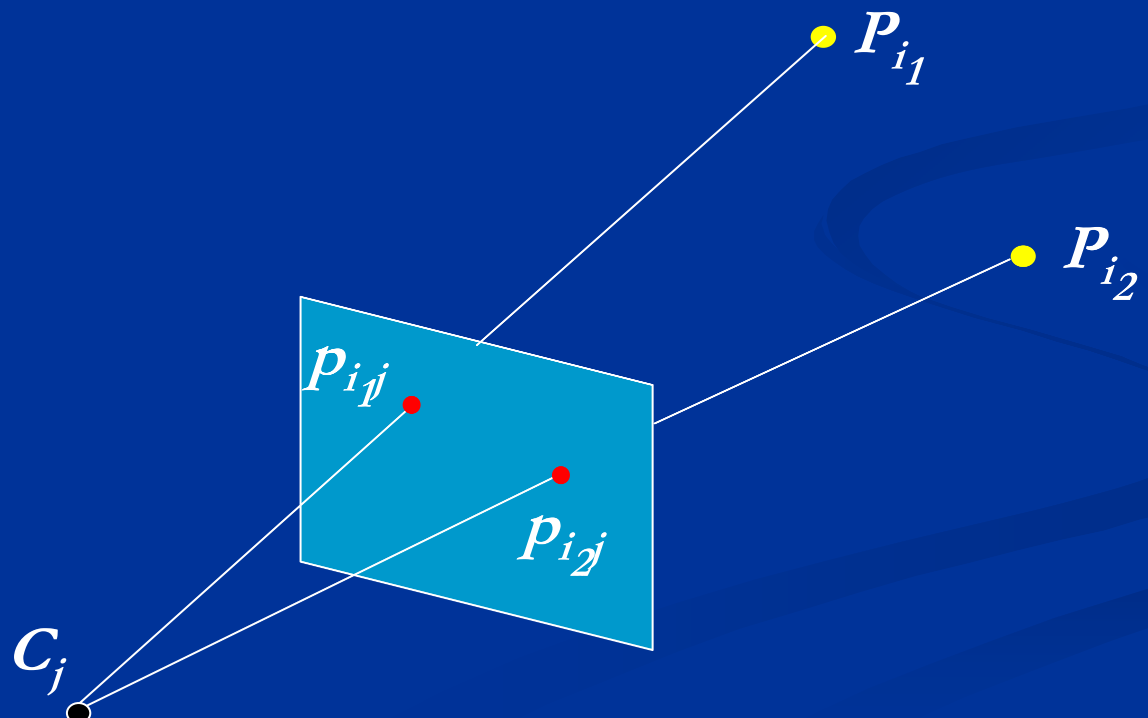
$$(\gamma_{ij} p_{ij} - \gamma_{3j} p_{3j}) \cdot (\gamma_{1j} p_{1j} - \gamma_{3j} p_{3j}) \times (\gamma_{2j} p_{2j} - \gamma_{3j} p_{3j}) = (P_i - P_3) \cdot (P_1 - P_3) \times (P_2 - P_3)$$

$$\text{for } i = 4,\ldots,N$$

Eliminate Camera Position

Geometric meaning (representing 3D vectors in different coordinate systems)

$$(\gamma_{i_1 j} p_{i_1 j} - \gamma_{i_2 j} p_{i_2 j}) \cdot (\gamma_{i_1 j} p_{i_1 j} - \gamma_{i_2 j} p_{i_2 j}) = (P_{i_1} - P_{i_2}) \cdot (P_{i_1} - P_{i_2})$$



Eliminate Camera Position

Is it a basis equation set?

Let $\boldsymbol{Q_i} = \gamma_{ij}\boldsymbol{p_{ij}}$, then the equation set becomes

$$(Q_i - Q_1)\cdot(Q_i - Q_1) = (P_i - P_1)\cdot(P_i - P_1),\ \text{for } i = 2,\ldots,N$$
$$(Q_i - Q_2)\cdot(Q_i - Q_2) = (P_i - P_2)\cdot(P_i - P_2),\ \text{for } i = 3,\ldots,N$$
$$(Q_i - Q_3)\cdot(Q_1 - Q_3)\times(Q_2 - Q_3) = (P_i - P_3)\cdot(P_1 - P_3)\times(P_2 - P_3)$$
$$\text{for } i = 4,\ldots,N$$

Eliminate Camera Position

Is it a basis equation set?

Now we have two sets of points $\boldsymbol{P_i}$, $\boldsymbol{Q_i}$ s.t. $\left\|Q_i - Q_j\right\| = \left\|P_i - P_j\right\|$

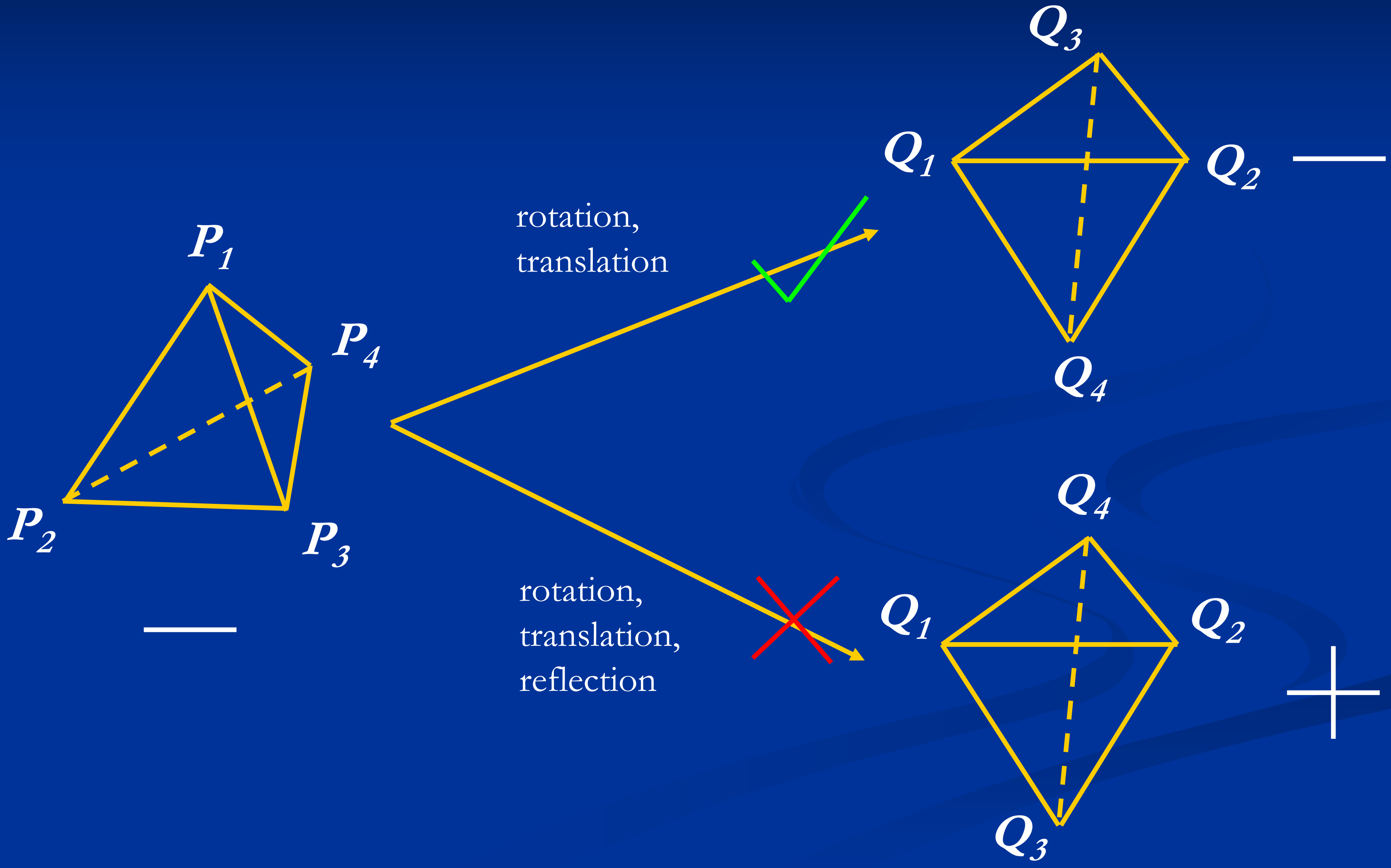
then there exist an orthogonal matrix $\boldsymbol{A}$ and a translation vector $\boldsymbol{T}$ s.t. $\boldsymbol{Q_i = AP_i + T}$. And the (3) equation set

$$(Q_i - Q_3)\cdot(Q_1 - Q_3)\times(Q_2 - Q_3) = (P_i - P_3)\cdot(P_1 - P_3)\times(P_2 - P_3)$$

prevent the reflect operation, so the determinant of $\boldsymbol{A}$ can not be negative. Therefore $\boldsymbol{A}$ is a rotation matrix.

$$\gamma_{ij}\, p_{ij} = AP_i + T$$

Eliminate Camera Position



Eliminate Structure

Observe the pose free equation set, here $Q_{ij} = \gamma_{ij} p_{ij}$

$$(Q_{ij} - Q_{1j}) \cdot (Q_{ij} - Q_{1j}) = (P_i - P_1) \cdot (P_i - P_1), \text{ for } i = 2, \ldots, N$$

$$(Q_{ij} - Q_{2j}) \cdot (Q_{ij} - Q_{2j}) = (P_i - P_2) \cdot (P_i - P_2), \text{ for } i = 3, \ldots, N$$

$$(Q_{ij} - Q_{3j}) \cdot (Q_{1j} - Q_{3j}) \times (Q_{2j} - Q_{3j}) = (P_i - P_3) \cdot (P_1 - P_3) \times (P_2 - P_3)$$

$$\text{for } i = 4, \ldots, N$$

Eliminate Structure

A new depth from motion equation set

$$(Q_{ij_1}-Q_{1j_1})\cdot(Q_{ij_1}-Q_{1j_1})=(Q_{ij_2}-Q_{1j_2})\cdot(Q_{ij_2}-Q_{1j_2}),\ \text{for } i=2,\ldots,N$$

$$(Q_{ij_1}-Q_{2j_1})\cdot(Q_{ij_1}-Q_{2j_1})=(Q_{ij_2}-Q_{2j_2})\cdot(Q_{ij_2}-Q_{2j_2}),\ \text{for } i=3,\ldots,N$$

$$(Q_{ij_1}-Q_{3j_1})\cdot(Q_{1j_1}-Q_{3j_1})\times(Q_{2j_1}-Q_{3j_1})=(Q_{ij_2}-Q_{3j_2})\cdot(Q_{1j_2}-Q_{3j_2})\times(Q_{2j_2}-Q_{3j_2})$$

$$\text{for } i=4,\ldots,N$$

This is a basis equation set. For ***N*** points and ***J*** images, we have ***NJ*** variables, and we can provide (3***N***-6)(***J***-1) equations.

Eliminate Structure

Let ***J***=2, ***N***=5, we have the 9 equations

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{2j_1}p_{2j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{2j_2}p_{2j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{3j_1}p_{3j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{3j_2}p_{3j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{4j_1}p_{4j_1}\right\|^2=\left\|\gamma_{1j_2}p_{4j_2}-\gamma_{4j_2}p_{4j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{5j_1}p_{5j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{5j_2}p_{5j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{3j_1}p_{3j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{3j_2}p_{3j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{4j_1}p_{4j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{4j_2}p_{4j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{5j_1}p_{5j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{5j_2}p_{5j_2}\right\|^2$$

$$(\gamma_{4j_1}p_{4j_1}-\gamma_{3j_1}p_{3j_1})\cdot(\gamma_{1j_1}p_{1j_1}-\gamma_{3j_1}p_{3j_1})\times(\gamma_{2j_1}p_{2j_1}-\gamma_{3j_1}p_{3j_1})=(\gamma_{4j_2}p_{4j_2}-\gamma_{3j_2}p_{3j_2})\cdot(\gamma_{1j_2}p_{1j_2}-\gamma_{3j_2}p_{3j_2})\times(\gamma_{2j_2}p_{2j_2}-\gamma_{3j_2}p_{3j_2})$$

$$(\gamma_{5j_1}p_{5j_1}-\gamma_{3j_1}p_{3j_1})\cdot(\gamma_{1j_1}p_{1j_1}-\gamma_{3j_1}p_{3j_1})\times(\gamma_{2j_1}p_{2j_1}-\gamma_{3j_1}p_{3j_1})=(\gamma_{5j_2}p_{5j_2}-\gamma_{3j_2}p_{3j_2})\cdot(\gamma_{1j_2}p_{1j_2}-\gamma_{3j_2}p_{3j_2})\times(\gamma_{2j_2}p_{2j_2}-\gamma_{3j_2}p_{3j_2})$$

Eliminate Structure

We can simplify the last two equations

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{2j_1}p_{2j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{2j_2}p_{2j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{3j_1}p_{3j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{3j_2}p_{3j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{4j_1}p_{4j_1}\right\|^2=\left\|\gamma_{1j_2}p_{4j_2}-\gamma_{4j_2}p_{4j_2}\right\|^2$$

$$\left\|\gamma_{1j_1}p_{1j_1}-\gamma_{5j_1}p_{5j_1}\right\|^2=\left\|\gamma_{1j_2}p_{1j_2}-\gamma_{5j_2}p_{5j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{3j_1}p_{3j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{3j_2}p_{3j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{4j_1}p_{4j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{4j_2}p_{4j_2}\right\|^2$$

$$\left\|\gamma_{2j_1}p_{2j_1}-\gamma_{5j_1}p_{5j_1}\right\|^2=\left\|\gamma_{2j_2}p_{2j_2}-\gamma_{5j_2}p_{5j_2}\right\|^2$$

$$\left\|\gamma_{3j_1}p_{3j_1}-\gamma_{4j_1}p_{4j_1}\right\|^2=\left\|\gamma_{3j_2}p_{3j_2}-\gamma_{4j_2}p_{4j_2}\right\|^2$$

$$\left\|\gamma_{4j_1}p_{4j_1}-\gamma_{5j_1}p_{5j_1}\right\|^2=\left\|\gamma_{4j_2}p_{4j_2}-\gamma_{5j_2}p_{5j_2}\right\|^2$$

Eliminate Structure

Use resultant method to solve the system. According to the complexity, the problem is solvable. But we haven't got the answer yet because of some technique problems.

Thank you!