Linearization in 3D Reconstruction

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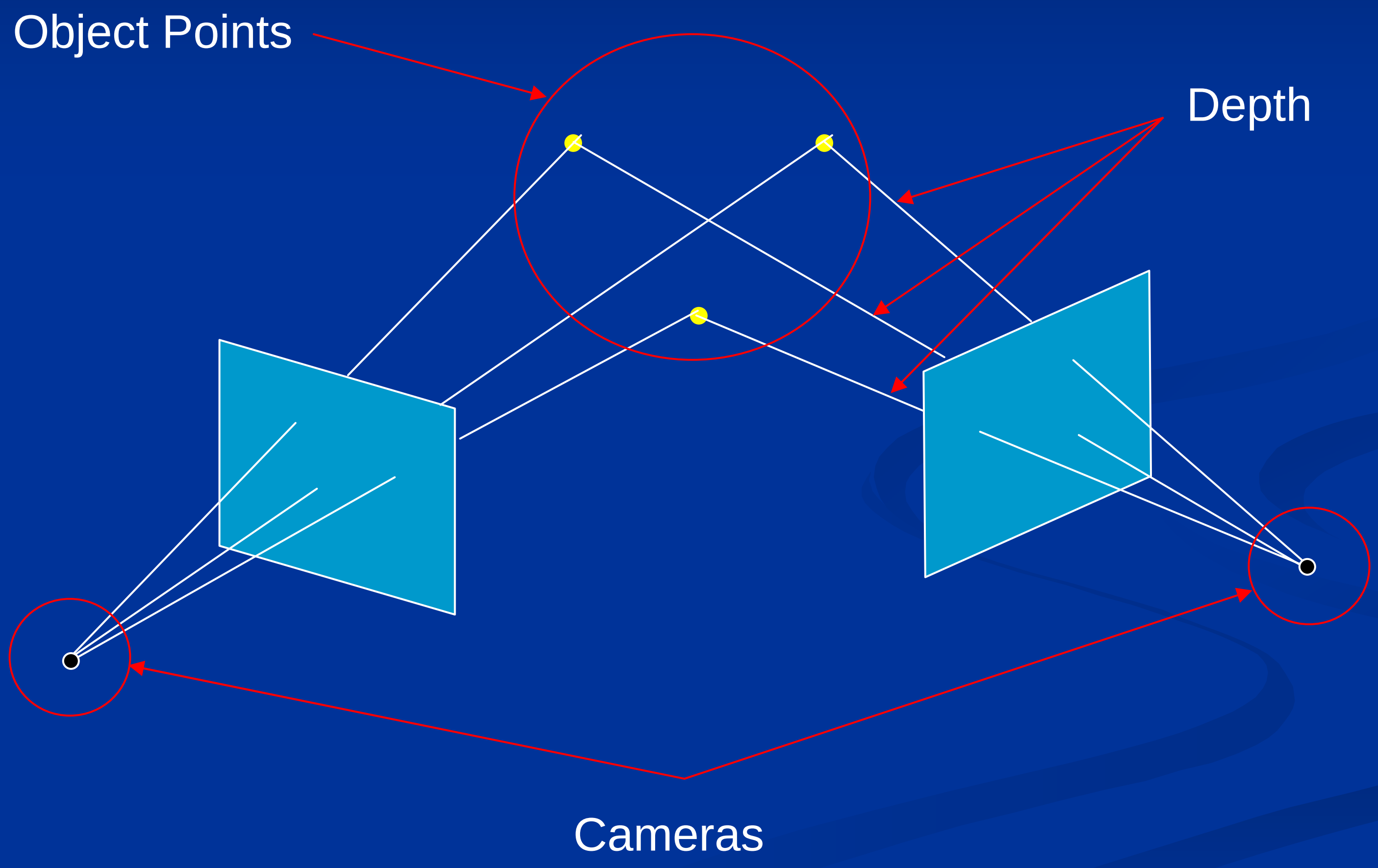
Goal

To linearize the nonlinear equations in 3D reconstruction problem

3D Reconstruction



3D Reconstruction

Pose free formula:

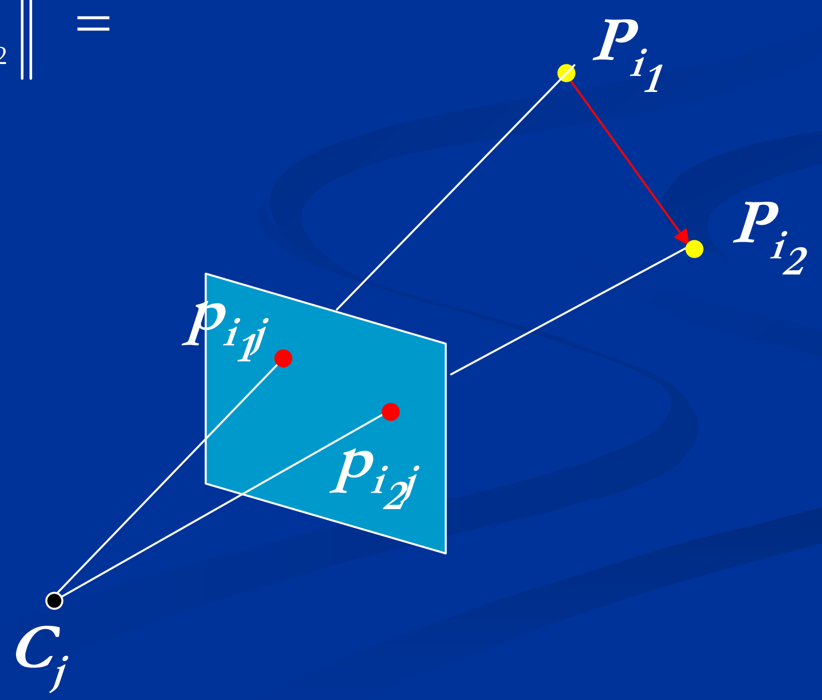
$$\gamma_{i_1 j}^2 \left\| \overrightarrow{C_j p_{i_1 j}} \right\|^2 - 2\gamma_{i_1 j}\gamma_{i_2 j} \overrightarrow{C_j p_{i_1 j}} \cdot \overrightarrow{C_j p_{i_2 j}} + \gamma_{i_2 j}^2 \left\| \overrightarrow{C_j p_{i_2 j}} \right\|^2 =$$

$$\left\| \overrightarrow{C_j P_{i_1}} \right\|^2 - 2\overrightarrow{C_j P_{i_1}} \cdot \overrightarrow{C_j P_{i_2}} + \left\| \overrightarrow{C_j P_{i_2}} \right\|^2 =$$

$$\left\| \overrightarrow{P_{i_1} P_{i_2}} \right\|^2$$

$$\gamma_{i_1 j} = \left\| \overrightarrow{C_j P_{i_1}} \right\| / \left\| \overrightarrow{C_j p_{i_1 j}} \right\|$$

$$\gamma_{i_2 j} = \left\| \overrightarrow{C_j P_{i_2}} \right\| / \left\| \overrightarrow{C_j p_{i_2 j}} \right\|$$



Traditional Method

Least Square Optimization

$$E=\sum_{i_1,i_2=1}^{N}\sum_{j=1}^{J}\left(\gamma_{i_1 j}^2\left\|\overrightarrow{C_j p_{i_1 j}}\right\|^2-2\gamma_{i_1 j}\gamma_{i_2 j}\overrightarrow{C_j p_{i_1 j}}\cdot\overrightarrow{C_j p_{i_2 j}}+\gamma_{i_2 j}^2\left\|\overrightarrow{C_j p_{i_2 j}}\right\|^2-\left\|\overrightarrow{P_{i_1} P_{i_2}}\right\|^2\right)^2$$

Need good initial guess

Our Approach

- Linearize the equations s.t. we can solve them directly
  - Partial linearization is also acceptable

$$a_{11}x^2 + a_{12}y^2 = c_1$$
$$a_{21}x^2 + a_{22}y^2 = c_2$$

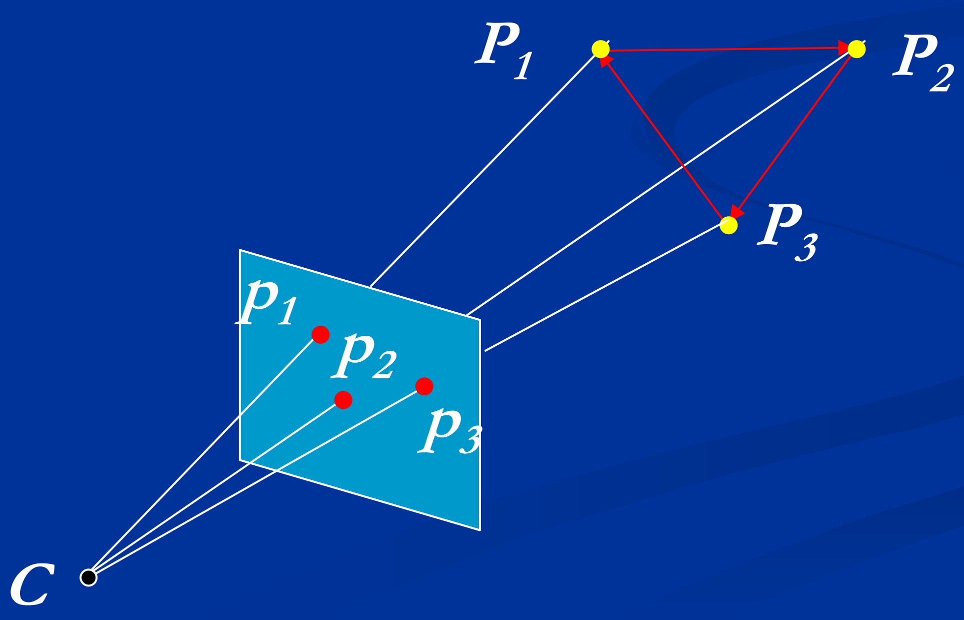
Linearization

$$\gamma_1^2\left\|\overrightarrow{Cp_1}\right\|^2 - 2\gamma_1\gamma_2\overrightarrow{Cp_1}\cdot\overrightarrow{Cp_2} + \gamma_2^2\left\|\overrightarrow{Cp_2}\right\|^2 = \left\|\overrightarrow{P_1P_2}\right\|^2$$

$$\gamma_2^2\left\|\overrightarrow{Cp_2}\right\|^2 - 2\gamma_2\gamma_3\overrightarrow{Cp_2}\cdot\overrightarrow{Cp_3} + \gamma_3^2\left\|\overrightarrow{Cp_3}\right\|^2 = \left\|\overrightarrow{P_2P_3}\right\|^2$$

$$\gamma_3^2\left\|\overrightarrow{Cp_3}\right\|^2 - 2\gamma_3\gamma_1\overrightarrow{Cp_3}\cdot\overrightarrow{Cp_1} + \gamma_1^2\left\|\overrightarrow{Cp_1}\right\|^2 = \left\|\overrightarrow{P_3P_1}\right\|^2$$

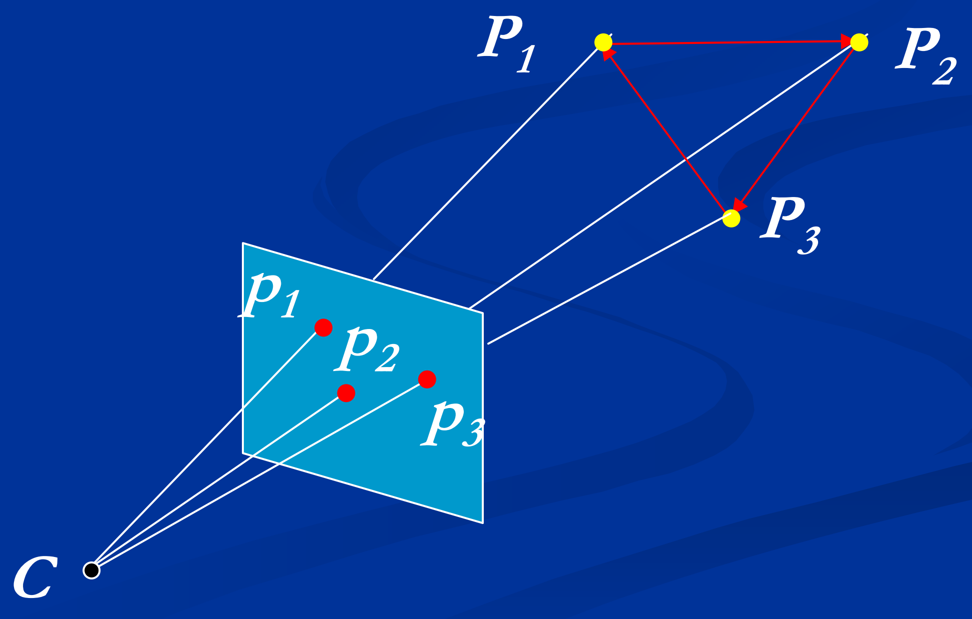
ratios



Linearization

$$-\gamma_1 \overrightarrow{Cp_1} + \gamma_2 \overrightarrow{Cp_2} = \overrightarrow{P_1P_2}$$
$$-\gamma_2 \overrightarrow{Cp_2} + \gamma_3 \overrightarrow{Cp_3} = \overrightarrow{P_2P_3}$$
$$-\gamma_3 \overrightarrow{Cp_3} + \gamma_1 \overrightarrow{Cp_1} = \overrightarrow{P_3P_1}$$

$\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_3P_1}$ are unknown in camera space



Linearization

$$\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_1} = 0$$

$\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \overrightarrow{P_3P_1}$ are in camera space

$$\overrightarrow{Cp_1} \times \overrightarrow{Cp_2} \cdot \overrightarrow{P_1P_2} = 0$$

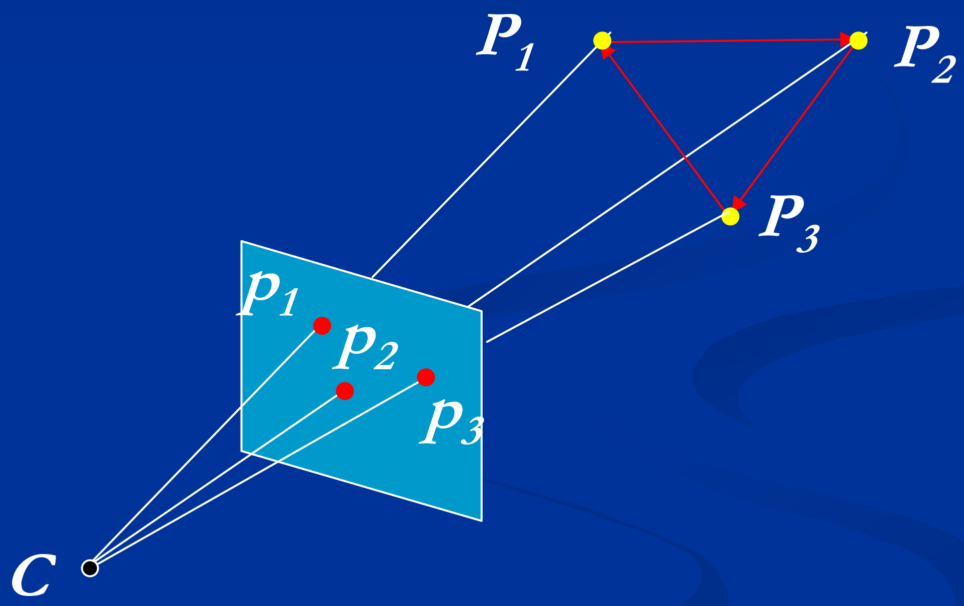
$$\overrightarrow{Cp_2} \times \overrightarrow{Cp_3} \cdot \overrightarrow{P_2P_3} = 0$$

$$\overrightarrow{Cp_3} \times \overrightarrow{Cp_1} \cdot \overrightarrow{P_3P_1} = 0$$

$$\left\|\overrightarrow{P_1P_2}\right\|^2 = A_1$$

$$\left\|\overrightarrow{P_2P_3}\right\|^2 = A_2$$

$$\left\|\overrightarrow{P_3P_1}\right\|^2 = A_3$$



Linearization

$$\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_1} = 0$$

$$\overrightarrow{Cp_1} \times \overrightarrow{Cp_2} \cdot \overrightarrow{P_1P_2} = 0$$

$$\overrightarrow{Cp_2} \times \overrightarrow{Cp_3} \cdot \overrightarrow{P_2P_3} = 0$$

$$\overrightarrow{Cp_3} \times \overrightarrow{Cp_1} \cdot \overrightarrow{P_3P_1} = 0$$

$$\begin{pmatrix} \overrightarrow{P_1P_2} \\ \overrightarrow{P_2P_3} \\ \overrightarrow{P_3P_1} \end{pmatrix} = k_1 \begin{pmatrix} \overrightarrow{v_{11}} \\ \overrightarrow{v_{12}} \\ \overrightarrow{v_{13}} \end{pmatrix} + k_2 \begin{pmatrix} \overrightarrow{v_{21}} \\ \overrightarrow{v_{22}} \\ \overrightarrow{v_{23}} \end{pmatrix} + k_3 \begin{pmatrix} \overrightarrow{v_{31}} \\ \overrightarrow{v_{32}} \\ \overrightarrow{v_{33}} \end{pmatrix}$$

$$\left\| \overrightarrow{P_1P_2} \right\|^2 = A_1 \qquad \left\| k_1\overrightarrow{v_{11}} + k_2\overrightarrow{v_{21}} + k_3\overrightarrow{v_{31}} \right\|^2 = A_1$$

$$\left\| \overrightarrow{P_2P_3} \right\|^2 = A_2 \qquad \left\| k_1\overrightarrow{v_{12}} + k_2\overrightarrow{v_{22}} + k_3\overrightarrow{v_{32}} \right\|^2 = A_2$$

$$\left\| \overrightarrow{P_3P_1} \right\|^2 = A_3 \qquad \left\| k_1\overrightarrow{v_{13}} + k_2\overrightarrow{v_{23}} + k_3\overrightarrow{v_{33}} \right\|^2 = A_3$$

Linearization

$$\gamma_1^2\left\|\overrightarrow{Cp_1}\right\|^2 - 2\gamma_1\gamma_2\overrightarrow{Cp_1}\cdot\overrightarrow{Cp_2} + \gamma_2^2\left\|\overrightarrow{Cp_2}\right\|^2 = \left\|\overrightarrow{P_1P_2}\right\|^2$$

$$\gamma_2^2\left\|\overrightarrow{Cp_2}\right\|^2 - 2\gamma_2\gamma_3\overrightarrow{Cp_2}\cdot\overrightarrow{Cp_3} + \gamma_3^2\left\|\overrightarrow{Cp_3}\right\|^2 = \left\|\overrightarrow{P_2P_3}\right\|^2$$

$$\gamma_3^2\left\|\overrightarrow{Cp_3}\right\|^2 - 2\gamma_3\gamma_1\overrightarrow{Cp_3}\cdot\overrightarrow{Cp_1} + \gamma_1^2\left\|\overrightarrow{Cp_1}\right\|^2 = \left\|\overrightarrow{P_3P_1}\right\|^2$$

$$\begin{pmatrix}\overrightarrow{v_{11}}\\ \overrightarrow{v_{12}}\\ \overrightarrow{v_{13}}\end{pmatrix} = \begin{pmatrix}\overrightarrow{Cp_1}\\ 0\\ -\overrightarrow{Cp_1}\end{pmatrix}$$

$$\begin{pmatrix}\overrightarrow{v_{21}}\\ \overrightarrow{v_{22}}\\ \overrightarrow{v_{23}}\end{pmatrix} = \begin{pmatrix}-\overrightarrow{Cp_2}\\ \overrightarrow{Cp_2}\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\overrightarrow{v_{31}}\\ \overrightarrow{v_{32}}\\ \overrightarrow{v_{33}}\end{pmatrix} = \begin{pmatrix}0\\ -\overrightarrow{Cp_3}\\ \overrightarrow{Cp_3}\end{pmatrix}$$

$$\left\|\gamma_1\overrightarrow{v_{11}} + \gamma_2\overrightarrow{v_{21}} + \gamma_3\overrightarrow{v_{31}}\right\|^2 = \left\|\overrightarrow{P_1P_2}\right\|^2 = A_1$$

$$\left\|\gamma_1\overrightarrow{v_{12}} + \gamma_2\overrightarrow{v_{22}} + \gamma_3\overrightarrow{v_{32}}\right\|^2 = \left\|\overrightarrow{P_2P_3}\right\|^2 = A_2$$

$$\left\|\gamma_1\overrightarrow{v_{13}} + \gamma_2\overrightarrow{v_{23}} + \gamma_3\overrightarrow{v_{33}}\right\|^2 = \left\|\overrightarrow{P_3P_1}\right\|^2 = A_3$$

Linearization

$$\left\| k_1 \overrightarrow{v_{11}} + k_2 \overrightarrow{v_{21}} + k_3 \overrightarrow{v_{31}} \right\|^2 = A_1 \qquad \left\| \gamma_1 \overrightarrow{v_{11}} + \gamma_2 \overrightarrow{v_{21}} + \gamma_3 \overrightarrow{v_{31}} \right\|^2 = \left\| \overrightarrow{P_1 P_2} \right\|^2 = A_1$$

$$\left\| k_1 \overrightarrow{v_{12}} + k_2 \overrightarrow{v_{22}} + k_3 \overrightarrow{v_{32}} \right\|^2 = A_2 \qquad \left\| \gamma_1 \overrightarrow{v_{12}} + \gamma_2 \overrightarrow{v_{22}} + \gamma_3 \overrightarrow{v_{32}} \right\|^2 = \left\| \overrightarrow{P_2 P_3} \right\|^2 = A_2$$

$$\left\| k_1 \overrightarrow{v_{13}} + k_2 \overrightarrow{v_{23}} + k_3 \overrightarrow{v_{33}} \right\|^2 = A_3 \qquad \left\| \gamma_1 \overrightarrow{v_{13}} + \gamma_2 \overrightarrow{v_{23}} + \gamma_3 \overrightarrow{v_{33}} \right\|^2 = \left\| \overrightarrow{P_3 P_1} \right\|^2 = A_3$$

Partially Orthogonal Base

Find a base $\begin{pmatrix} \overrightarrow{v_{11}} \\ \overrightarrow{v_{12}} \\ \overrightarrow{v_{12}} \end{pmatrix}, \begin{pmatrix} \overrightarrow{v_{21}} \\ \overrightarrow{v_{22}} \\ \overrightarrow{v_{22}} \end{pmatrix}, \begin{pmatrix} \overrightarrow{v_{31}} \\ \overrightarrow{v_{32}} \\ \overrightarrow{v_{32}} \end{pmatrix}$ for the solution space of $\begin{pmatrix} \overrightarrow{P_1P_2} \\ \overrightarrow{P_2P_3} \\ \overrightarrow{P_3P_1} \end{pmatrix}$

$$\left\| k_1\overrightarrow{v_{11}} + k_2\overrightarrow{v_{21}} + k_3\overrightarrow{v_{31}} \right\|^2 = A_1$$

s.t. $\left\| k_1\overrightarrow{v_{12}} + k_2\overrightarrow{v_{22}} + k_3\overrightarrow{v_{32}} \right\|^2 = A_2$ is easy to solve

$$\left\| k_1\overrightarrow{v_{13}} + k_2\overrightarrow{v_{23}} + k_3\overrightarrow{v_{33}} \right\|^2 = A_3$$

Partially Orthogonal Base

$$\left\|k_1\overrightarrow{v_{11}}+k_2\overrightarrow{v_{21}}+k_3\overrightarrow{v_{31}}\right\|^2 = A_1 \qquad k_1^2\left\|\overrightarrow{v_{11}}\right\|^2+k_2^2\left\|\overrightarrow{v_{12}}\right\|^2+k_3^2\left\|\overrightarrow{v_{13}}\right\|^2 = A_1$$

$$\left\|k_1\overrightarrow{v_{12}}+k_2\overrightarrow{v_{22}}+k_3\overrightarrow{v_{32}}\right\|^2 = A_2 \quad\Rightarrow\quad k_1^2\left\|\overrightarrow{v_{21}}\right\|^2+k_2^2\left\|\overrightarrow{v_{22}}\right\|^2+k_3^2\left\|\overrightarrow{v_{23}}\right\|^2 = A_2$$

$$\left\|k_1\overrightarrow{v_{13}}+k_2\overrightarrow{v_{23}}+k_3\overrightarrow{v_{33}}\right\|^2 = A_3 \qquad k_1^2\left\|\overrightarrow{v_{31}}\right\|^2+k_2^2\left\|\overrightarrow{v_{32}}\right\|^2+k_3^2\left\|\overrightarrow{v_{33}}\right\|^2 = A_3$$

If $\begin{pmatrix}\overrightarrow{v_{11}}\\ \overrightarrow{v_{12}}\\ \overrightarrow{v_{13}}\end{pmatrix},\begin{pmatrix}\overrightarrow{v_{21}}\\ \overrightarrow{v_{22}}\\ \overrightarrow{v_{23}}\end{pmatrix},\begin{pmatrix}\overrightarrow{v_{31}}\\ \overrightarrow{v_{32}}\\ \overrightarrow{v_{33}}\end{pmatrix}$ are partially orthogonal, which means

$\left(\overrightarrow{v_{11}}\quad\overrightarrow{v_{12}}\quad\overrightarrow{v_{13}}\right),\left(\overrightarrow{v_{21}}\quad\overrightarrow{v_{22}}\quad\overrightarrow{v_{23}}\right),\left(\overrightarrow{v_{31}}\quad\overrightarrow{v_{32}}\quad\overrightarrow{v_{33}}\right)$ are orthogonal

Partially Orthogonal Base

Suppose we have the partially orthogonal base

$$\begin{pmatrix} \overrightarrow{w_{11}} \\ \overrightarrow{w_{12}} \\ \overrightarrow{w_{13}} \end{pmatrix} = \begin{pmatrix} \overrightarrow{v_{11}} \\ \overrightarrow{v_{12}} \\ \overrightarrow{v_{13}} \end{pmatrix} + t_1 \begin{pmatrix} \overrightarrow{v_{21}} \\ \overrightarrow{v_{22}} \\ \overrightarrow{v_{23}} \end{pmatrix} + t_2 \begin{pmatrix} \overrightarrow{v_{31}} \\ \overrightarrow{v_{32}} \\ \overrightarrow{v_{33}} \end{pmatrix} = \begin{pmatrix} \overrightarrow{v_{11}} + t_1 \overrightarrow{v_{21}} + t_2 \overrightarrow{v_{31}} \\ \overrightarrow{v_{12}} + t_1 \overrightarrow{v_{22}} + t_2 \overrightarrow{v_{32}} \\ \overrightarrow{v_{13}} + t_1 \overrightarrow{v_{23}} + t_2 \overrightarrow{v_{33}} \end{pmatrix}$$

$$\begin{pmatrix} \overrightarrow{w_{21}} \\ \overrightarrow{w_{22}} \\ \overrightarrow{w_{23}} \end{pmatrix} = t_3 \begin{pmatrix} \overrightarrow{v_{11}} \\ \overrightarrow{v_{12}} \\ \overrightarrow{v_{13}} \end{pmatrix} + \begin{pmatrix} \overrightarrow{v_{21}} \\ \overrightarrow{v_{22}} \\ \overrightarrow{v_{23}} \end{pmatrix} + t_4 \begin{pmatrix} \overrightarrow{v_{31}} \\ \overrightarrow{v_{32}} \\ \overrightarrow{v_{33}} \end{pmatrix} = \begin{pmatrix} t_3 \overrightarrow{v_{11}} + \overrightarrow{v_{21}} + t_4 \overrightarrow{v_{31}} \\ t_3 \overrightarrow{v_{12}} + \overrightarrow{v_{22}} + t_4 \overrightarrow{v_{32}} \\ t_3 \overrightarrow{v_{13}} + \overrightarrow{v_{23}} + t_4 \overrightarrow{v_{33}} \end{pmatrix}$$

$$\begin{pmatrix} \overrightarrow{w_{31}} \\ \overrightarrow{w_{32}} \\ \overrightarrow{w_{33}} \end{pmatrix} = t_5 \begin{pmatrix} \overrightarrow{v_{11}} \\ \overrightarrow{v_{12}} \\ \overrightarrow{v_{13}} \end{pmatrix} + t_6 \begin{pmatrix} \overrightarrow{v_{21}} \\ \overrightarrow{v_{22}} \\ \overrightarrow{v_{23}} \end{pmatrix} + \begin{pmatrix} \overrightarrow{v_{31}} \\ \overrightarrow{v_{32}} \\ \overrightarrow{v_{33}} \end{pmatrix} = \begin{pmatrix} t_5 \overrightarrow{v_{11}} + t_6 \overrightarrow{v_{21}} + \overrightarrow{v_{31}} \\ t_5 \overrightarrow{v_{12}} + t_6 \overrightarrow{v_{22}} + \overrightarrow{v_{32}} \\ t_5 \overrightarrow{v_{13}} + t_6 \overrightarrow{v_{23}} + \overrightarrow{v_{33}} \end{pmatrix}$$

Partially Orthogonal Base

$$\left(\overrightarrow{v_{11}}+t_1\overrightarrow{v_{21}}+t_2\overrightarrow{v_{31}}\right)\cdot\left(t_3\overrightarrow{v_{11}}+\overrightarrow{v_{21}}+t_4\overrightarrow{v_{31}}\right)=0$$
$$\left(\overrightarrow{v_{11}}+t_1\overrightarrow{v_{21}}+t_2\overrightarrow{v_{31}}\right)\cdot\left(t_5\overrightarrow{v_{11}}+t_6\overrightarrow{v_{21}}+\overrightarrow{v_{31}}\right)=0$$
$$\left(t_3\overrightarrow{v_{11}}+\overrightarrow{v_{21}}+t_4\overrightarrow{v_{31}}\right)\cdot\left(t_5\overrightarrow{v_{11}}+t_6\overrightarrow{v_{21}}+\overrightarrow{v_{31}}\right)=0$$
$$\left(\overrightarrow{v_{12}}+t_1\overrightarrow{v_{22}}+t_2\overrightarrow{v_{32}}\right)\cdot\left(t_3\overrightarrow{v_{12}}+\overrightarrow{v_{22}}+t_4\overrightarrow{v_{32}}\right)=0$$
$$\left(\overrightarrow{v_{12}}+t_1\overrightarrow{v_{22}}+t_2\overrightarrow{v_{32}}\right)\cdot\left(t_5\overrightarrow{v_{12}}+t_6\overrightarrow{v_{22}}+\overrightarrow{v_{32}}\right)=0$$
$$\left(t_3\overrightarrow{v_{12}}+\overrightarrow{v_{22}}+t_4\overrightarrow{v_{32}}\right)\cdot\left(t_5\overrightarrow{v_{12}}+t_6\overrightarrow{v_{22}}+\overrightarrow{v_{32}}\right)=0$$
$$\left(\overrightarrow{v_{13}}+t_1\overrightarrow{v_{23}}+t_2\overrightarrow{v_{33}}\right)\cdot\left(t_3\overrightarrow{v_{13}}+\overrightarrow{v_{23}}+t_4\overrightarrow{v_{33}}\right)=0$$
$$\left(\overrightarrow{v_{13}}+t_1\overrightarrow{v_{23}}+t_2\overrightarrow{v_{33}}\right)\cdot\left(t_5\overrightarrow{v_{13}}+t_6\overrightarrow{v_{23}}+\overrightarrow{v_{33}}\right)=0$$
$$\left(t_3\overrightarrow{v_{13}}+\overrightarrow{v_{23}}+t_4\overrightarrow{v_{33}}\right)\cdot\left(t_5\overrightarrow{v_{13}}+t_6\overrightarrow{v_{23}}+\overrightarrow{v_{33}}\right)=0$$

9 equations

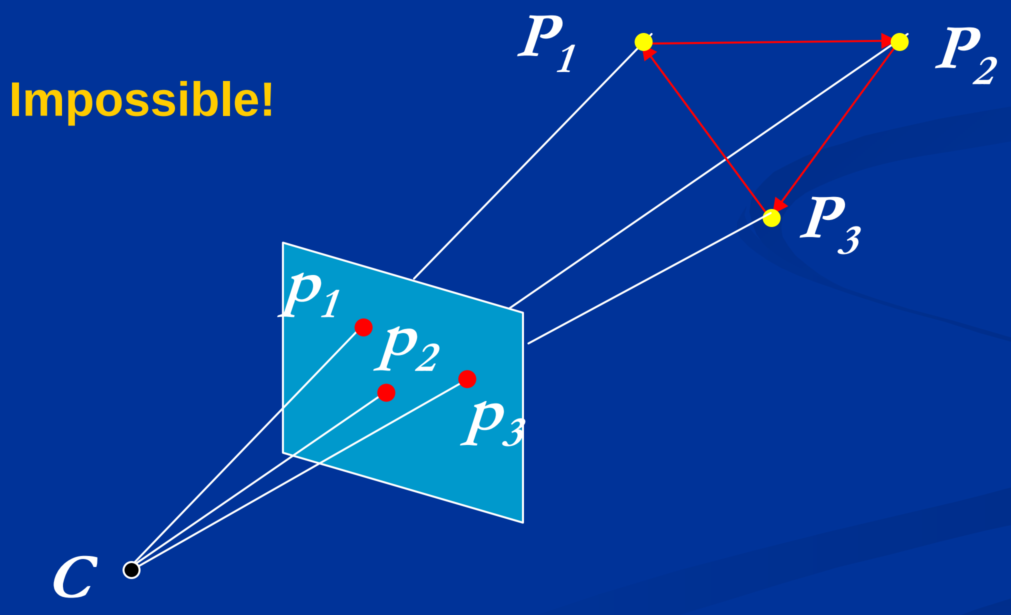
6 variables

Is there any solution or not?

Partially Orthogonal Base

$\overrightarrow{w_{11}}, \overrightarrow{w_{21}}, \overrightarrow{w_{31}}$ are solutions for $\overrightarrow{P_1P_2}$

which means they lie in the plane subtended by $\overrightarrow{Cp_1}, \overrightarrow{Cp_2}$



Partially Orthogonal Base

- Future Work
  - Maybe some other forms of base will help